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On Slater's criterion for the breakup of invariant curves

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HIGHLIGHTS

- We investigate Slater's theorem in the context of area-preserving maps.
- The breakup diagram of the nontwist map was obtained using Slater's criterion.
- Slater's criterion can be implemented to determine the last invariant curve.
- To the standard map our heuristic Slater's criterion was $K_c = 0.9716394$.
- Our result is very close to the widely accepted Greene's result, $K_c = 0.971635$.

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ABSTRACT

We numerically explore Slater's theorem in the context of dynamical systems to study the breakup of invariant curves. Slater's theorem states that an irrational translation over a circle returns to an arbitrary interval in at most three different recurrence times expressible by the continued fraction expansion of the related irrational number. The hypothesis considered in this paper is that Slater's theorem can be also verified in the dynamics of invariant curves. Hence, we use Slater's theorem to develop a qualitative and quantitative numerical approach to determine the breakup of invariant curves in the phase space of area-preserving maps.

1. Introduction

In the late of 1940s, N.B. Slater proved that an irrational translation over a unity circle can take at most three different return values to a connected interval of size $\epsilon < 1$. In addition, these three recurrence times are expressible by the continued fraction expansion of the irrational number used to the translation [1]. This remarkable result has an immediate connection with two dimension dynamical systems, because regular solutions of such systems are constituted by a set of quasi-periodic orbits named invariant curves, whose rotation in the phase space is also irrational and can be related to a motion over the circle.

Our goal with the present paper is to relate the recurrent behavior of invariant curves according to Slater's theorem develop a procedure for determining the breakup of such invariants in the phase space of area-preserving maps. In physics, it is important to predict breakup of invariant curves because, in two degrees of freedom, such curves represent absolute barriers in the phase space, playing a crucial role to the confinement and transport of orbits. This property is valid only in 2D-phase space because the dimension of the invariant surface differs by one unit from the space. Otherwise, invariant surfaces in high-dimensional systems do not split the phase space and, thus, do not represent barriers for the chaotic trajectories, leading to the phenomenon of Arnold diffusion [2].

Different methods have been proposed to study the persistence of invariant curves under perturbation [3–5]. It is worth mentioning the pioneer quantitative method, known as Greene's residue criterion [6–8], that relates the existence of an invariant curve to the stability of a family of periodic orbits nearby. On the other hand, the observation of at most three recurrence times in invariant curves have shown a useful method to treat set of parameters in area-preserving maps with special symmetries [9,10]. However, Slater's theorem is quite robust to be limited to qualitative analysis. In the present paper we use Slater's theorem to estimate the breakup of invariant curves by a qualitative and quantitative numerical approach. Hence, we initiate our paper introducing Slater's theorem (Section 2); next, we apply the qualitative technique based only on the three recurrence times to determine the breakup diagram of the standard nontwist map (Section 3). Finally, we step forward considering evidences and hypotheses to

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develop a procedure able to indicate with accuracy, the breakup of the last invariant curve in the standard map (Section 4).

2. Slater's criterion

Let us consider a circle of unit circumference and an irrational $\{\theta\}$ (where $\{x\}$ means the fractional part of x) such that $\{N\theta\}$ with N integer, partition the circle into segments. Surprisingly, no matter which θ and the number of steps N we take, there will be at most three distinct sizes to these segments. Furthermore, according to Slater [1], as a consequence of the three segments, if we consider the time (iterations) between the exit and the first return to a connected interval $\epsilon < 1$, i.e., $\{N\theta\} < \epsilon$, at most three different recurrence times are expected and expressible by the continued fraction expansion of the irrational θ,

$$\theta = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \frac{1}{\cdots}}}}. \tag{1}$$

From (1), notice that the continued fraction of an irrational number is infinite and a convergent P_s/Q_s represents a rational approximation of order s, e.g., $[a_1, a_2, \ldots, a_s] = P_s/Q_s$. The problem of the distribution of the sequence $\{N\theta\} < \epsilon$ was extensively explored by mathematicians [1,11,12]. The steps to such a solution have been done by Slater in Ref. [1]. According to Slater any ϵ between 0 and 1 can be expressed uniquely in the form,

$$\epsilon = (n+1)\eta_s + \eta_{s+1} + \Psi \quad (0 < \Psi \le \eta_s), \tag{2}$$

with n integer and η_s given by the decreasing sequence [12],

$$\eta_s = (-1)^{s-1}(\theta Q_{s-1} - P_{s-1}), \tag{3}$$

with $\eta_s > 0$, $\eta_1 = \theta$ and $\eta_0 = 1$. Indeed, given some irrational number θ and the interval ϵ, there is a unique pair (n, s) that satisfies (2).

According to Slater, the three recurrences, such that $\{N\theta\} < \epsilon$, where θ is irrational, are given by:

$$\begin{aligned} \sigma_1 &= Q_{s-1}, \\ \sigma_2 &= Q_s - nQ_{s-1}, \\ \sigma_3 &= Q_s - (n+1)Q_{s-1}, \end{aligned} \tag{4}$$

where n and s are found by solving (2) and (3).

Thus, the distribution of the sequence $\{N\theta\} < \epsilon$ presents at most three recurrence times expressible by the denominators Q_s of the continued fraction expansion of the number θ. Furthermore, note that $\sigma_2 = \sigma_1 + \sigma_3$, i.e., one of the recurrence times is always the sum of the other two. From Slater's theorem (4) and Eqs. (2) and (3) it is possible to verify that the recurrences times σ_1, σ_2 and σ_3 depend on ϵ and, the third recurrence, σ_3, appears only if $\Psi > 0$ as a consequence of (2).

As an example, let us consider the irrational $1/\gamma = 0.618033988\ldots$ (inverse of the golden mean) whose continued fraction expansion is shown in Table 1. Taking $\epsilon = 0.05$, it is found that the unique solution of (2) is given by $n = 0$ and $s = 8$. So, according to (4), for $\{N(1/\gamma)\} < 0.05$ we have $(Q_7, Q_8, Q_8 - Q_7) = (21, 34, 13)$ no matter how big is the integer N. It means that the constant translation of $1/\gamma$ over a unity circle returns to a connected interval of size 0.05 only after 13, 21 or 34 iterations. For more details see Ref. [1].

The above result has an immediate connection with dynamical systems, since the regular part of phase spaces of area-preserving maps presents a set of quasi-periodic orbits named invariant curves. These invariants have irrational rotation on the phase space and are persistent under small perturbation. Moreover, the quasi-periodic motion of invariant curves is related with a simple rotation of a circle since such invariant curves are graphs [13], or can be represented by giving some parametrization in which the motion becomes a rotation [14]. Therefore, we expect to verify three recurrence times within an interval of size ϵ for the quasi-periodic orbits in the phase space of area-preserving maps. However, (2)–(4) are no longer valid because the rotation of the points that compose invariant curves in the phase space are not uniformly distributed as the rotation over the circle previously described. This conjecture is numerically confirmed in the following sections.

Table 1
A set of convergents for the irrational $\frac{1}{\gamma}$ (inverse of the golden mean) obtained by the truncation of the continued fraction expansion.

s	a_s	P_s/Q_s
0	0	$P_0 = 0, Q_0 = 1$
1	1	1
2	1	1/2
3	1	2/3
4	1	3/5
5	1	5/8
6	1	8/13
7	1	13/21
8	1	21/34
⋮	⋮	⋮
24	1	46 368/75 025
25	1	75 025/121 393
26	1	121 393/196 418
27	1	196 418/317 811
⋮	⋮	⋮

3. Breakup diagram of standard nontwist map

In Ref. [9] the authors used Slater's three recurrence times to determine, qualitatively, the breakup of a shearless curve in a parameter space of a non Hamiltonian nontwist system. In this section we apply the technique to the standard nontwist map (SNM).

The SNM is given by,

$$(x', y') \to (x + a(1 - y'^2),\ y - b\sin(2\pi x)), \tag{5}$$

where x is mod 1 and (a, b) are parameters. The map (5) is area-preserving and violates the twist condition, i.e., $(\partial x'/\partial y) = 0$ along the curve $y = b\sin(2\pi x)$. The violation of the twist condition asserts at least one maximum or minimum point in the rotation number profile which leads the nontwist maps generally have orbits with the same rotation number. The invariant curve in the phase space whose rotation number is maximum or minimum point of the rotation number profile is called *shearless*.

The particular interest on the shearless curve comes from the fact that along it the shear $\partial x'/\partial y$ vanishes, so analysis about its stability is outside of the range of KAM theory. Around the shearless curve some nontwist phenomena are observed with different scenarios, e.g., separatrix reconnection and island chains collisions [15–19]. Furthermore, the shearless curve possesses remarkable stability owing to small resonances widths nearby [20]. It implies that, usually, the shearless curve is the last to be broken as the parameters are modified and, therefore, its breakup is related to the onset of global chaos.

The reversing symmetry group of the SNM [21] gives a set of fixed points that lies on the shearless curve whenever it is not broken. These set of fixed points, sometimes denoted as indicator points (**IP**), are: [22],

$$\left(\frac{n}{2} - \frac{1}{4}, (-1)^{n+1}\frac{b}{2}\right) \quad \text{and} \quad \left(\frac{a}{2} + \frac{n}{2} - \frac{1}{4}, 0\right). \tag{6}$$

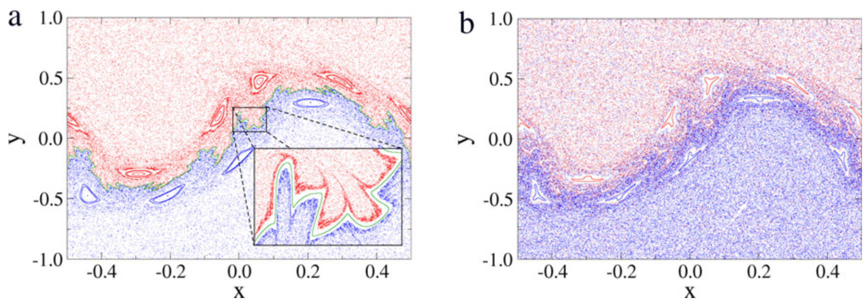

Fig. 1. Phase space of the nontwist map with $a = 0.455$ and: (a) $b = 0.800$, emphasizing the robustness of the shearless curve (green curve). (b) $b = 0.847$, showing the breakup of the shearless curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

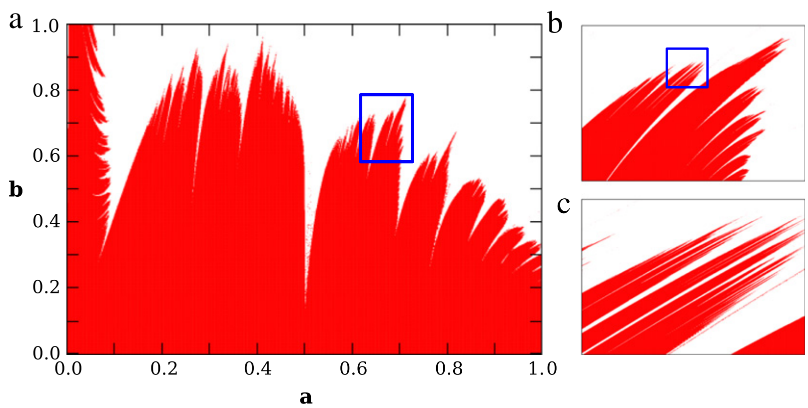

Fig. 2. Parameter space of SNM showing the breakup boundary for the central shearless curve. The red color indicates the set of parameters (a, b) in which the shearless curve exists in the phase space. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In Fig. 1, we show a typical phase space of the SNM in different stages. In Fig. 1(a) with parameters $a = 0.455$ and $b = 0.800$ the shearless curve (obtained by **IP**: $(x, y) = (1/4, b/2)$) remains in the phase space separating the phase space in two isolated regions. Fig. 1(b) shows that the shearless curve does not exist anymore and the two chaotic regions are unified, allowing the transport through the y-coordinate [23].

As previously mentioned the stability of the shearless curve depends on the parameters (a, b), and its destruction allows the full transport through the phase space. Thus, the existence of the shearless curve should be verified in the parameter space (a, b)-breakup diagram.

In order to study the breakup diagram, we applied Slater's recurrence as made in Ref. [9]. The method consists in counting the number of different recurrence times of the iterates of some **IP** inside an arbitrary region of size ϵ. In this case, the shearless curve is considered broken if the number of recurrence times exceeds three or one of them is not the sum of the other two as the theorem (4) states. Note that, a priori, the cases for one or two recurrences are not conclusive and may be modified as we increase the number of iteration. Even that, in our numerical procedure we considered both cases as an indicative that the shearless curve is not broken. The implementation of the procedure is straightforward since the reversing symmetry group of the SNM provides **IP**s. Using the **IP**: $(1/4; b/2)$ as initial condition, and choosing a square of size ϵ around it, we can evaluate the recurrence times in order to point out the existence or not of the shearless curve according to Slater's theorem. In Fig. 2 we show the breakup diagram using $\epsilon = 0.02$ and $N = 1 \times 10^6$.

The red color in Fig. 2 represents the set of parameters for which the shearless curves exist in the phase space. Thus, the limit boundary between red and white colors represents the threshold of the shearless curve breakup. Fig. 2(b) and (c) shows two successive amplification about threshold regions.

Different procedures were previously employed to study the breakup diagram of SNM. In Ref. [22], the authors obtained a rough estimate for the breakup of shearless curves by investigating for a range of parameter values whether iterates of one of the **IP** remain bounded. A different strategy was presented in Ref. [17] where the authors analyzed the fluctuations of the rotation number of some **IP**, basing on the fact that a trajectory in a periodic or quasi-periodic motion leads to a converging rotation number. Both procedures are computationally expensive and the results may be masked by stuck trajectories that spend very long time around regular islands. We stress that the procedure based on Slater's recurrence does not require a large CPU-time, because whenever the fourth recurrence appears we consider the curve as broken, and we moved on to the next pair of parameters (a, b). For Fig. 2, for example, our simulation took less than half an hour, using a single core of 2.5 GHz.

Another method relies on Greene's residue criterion [6–8], usually indicated when high accuracy is needed. This method can be used, for example, to determine with precision the critical parameters (a_c, b_c) that correspond to specific shearless curves breakups [16,24,25], however, it may not be suited to explore a large sets of parameters. On the other hand, the counting method of recurrence times, based on Slater's theorem (4), may be efficient to scan sets of parameters because we can control the size of ϵ and the number of iterations N in detriment of the accuracy. Consequently, the procedure requires a shorter computational time to determine critical curves in parameter space.

In the next section, we will show a procedure able to define, with accuracy, the breakup of a single invariant curve.

4. The last invariant curve of the standard map

In order to study the breakup of a single invariant curve with high precision, let us introduce the standard map (SM), also knows

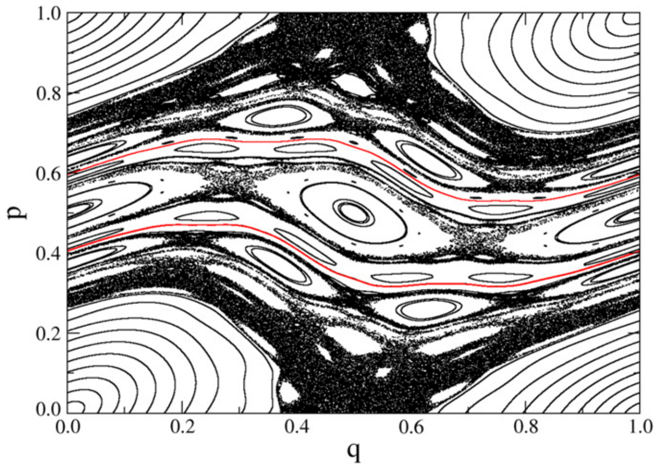

Fig. 3. Phase space of the SM with $K = 0.971$. The red line shows the $1/\gamma$-invariant curve estimated via Slater's theorem. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

as the Chirikov–Taylor map [26]:

$$(q', p') \to \left(q + p',\ p - \frac{K}{2\pi}\sin(2\pi q)\right), \tag{7}$$

where both coordinates are mod 1. For $K = 0$ the map is integrable and only periodic or quasi-periodic orbits are possible. The chaotic dynamic is achieved by increasing the parameter K, with appropriate initial conditions.

In contrast to the SNM, the SM is a twist map since $(\partial q'/\partial p) \neq 0$. Hence, the destruction of invariants curves and the formation of regular islands, as we perturb the map, are in the scope of KAM and Poincaré–Birkhoff theorems, respectively. As a result of both theorems the phase space presents the following picture. Small values of K do not destroy the most of invariant curves. As we increase the K parameter we should observe a set of invariant curves dividing regular island formations that, usually, may contain chaotic orbits around them. In this case the invariant curves divide the phase space in the sense that chaotic orbits are formed by different initial conditions, i.e., they are distinct and disconnected, see Fig. 3 with $K = 0.971$. The invariant curves are destroyed as we increase K and, at some critical value, K_c, the last invariant disappears leading to a spreading of the chaotic region.

Some different methods were proposed to find the critical parameter K_c for the SM. Among analytical [5] and numerical results [3,4,6], the great accuracy relies again on Greene's method [6] where the criticality was estimated as $K_c = 0.971635$. In the present paper we propose an alternative numerical procedure to calculate the critical parameter K_c through the three recurrence times stated by Slater's theorem. It is worth to note that the present case is essentially different from the map of the previous section, because the symmetry of the SM does not provide the called indicator points.

Firstly, it is well established from KAM and renormalization theory [27,28] that the last invariant curve to be broken in the standard map Eq. (7) is the one whose rotation number is the most irrational as possible. By continued fraction expansion (1), a given number is more irrational than another if its approximation by rational numbers (see the convergent in Table 1) is slower. In terms of (1) it means that the most irrational number possible is one that presents $a_{is} = 1$, resulting in the so called golden mean: $\gamma = (\sqrt{5} + 1)/2$. As the SM is mod 1, we have in the phase space the inverse of the golden mean curve, $1/\gamma$.

As discussed before, we do not expect to verify (2) and (3) for the perturbed system. It means that we are not able to predict the recurrence times from the size ϵ. However, since we know that invariant curves can be reduced to a simple rotation of a circle, it seems reasonable to assume Slater's theorem here, i.e., we are

Table 2
Estimated value p_e for the $1/\gamma$-curve in the SM with $K = 0.971$. For this procedure we record the three recurrence times $\sigma_{i=1,2,3}$ according to Slater's theorem and we vary the size ϵ, the number of iteration N and the step of initial conditions.

ϵ	$\sigma_{i=1/2/3}$	N	Step	p_e
1×10^{-5}	10 946/17 711/28 657	1×10^{6}	10^{-09}	0.66472043300
5×10^{-6}	28 657/75 025/46 368	5×10^{6}	10^{-10}	0.66472043430
1×10^{-6}	121 393/196 418/317 811	1×10^{7}	10^{-11}	0.66472043445

assuming that the three recurrence times for perturbed invariant curve remain to be expressible by continued fraction expansion of its rotation number. Thus, to investigate the breakup of the last curve in SM we can use this assumption to find the $1/\gamma$-curve in the phase space and determine the critical parameter K_c when the recurrence times to the interval ϵ do not satisfy Slater's criterion anymore.

To start our procedure we need to determine the location of $1/\gamma$-curve in the phase space. For that, we provide initial conditions over the line $q = 0.5$. Each initial condition is placed in the middle of a square of size ϵ and iterated a large number of times until finding recurrence times that satisfy the values of three consecutive denominators of the continued fraction expansion of the irrational $1/\gamma$. In our procedure, we started from $K = 0.971$ (see Fig. 3) with $\epsilon = 10^{-5}$ iterating each initial condition 10^6 times. In this case, the first condition that meets the requirements is $p_e = 0.66472043300$, however, there are a lot of curves whose three recurrence times are also related to the convergent of the irrational $1/\gamma$, namely, $\sigma_1 = 10\,946$; $\sigma_2 = 17\,711$ and $\sigma_3 = 28\,657$. Therefore, we conclude that the real $1/\gamma$-curve is enclosed by other orbits with very close rotation number and that are not destroyed yet. In order to improve our estimate, we decrease the size of the square to $\epsilon = 5 \times 10^{-6}$ with 5×10^6 iterations and, finally, we stopped with $\epsilon = 10^{-6}$ and 2×10^7 iterations. Table 2 shows the estimated value, p_e, to the position of the $1/\gamma$-curve in the SM with $K = 0.971$. We also record the number of times N that each initial condition was iterated and the step considered to vary the initial conditions. Note that the smaller the size of the box ϵ the larger the recurrence times and, consequently, more longer the required CPU-time.

From now, to determine the critical parameter K_c for the $1/\gamma$-curve, we repeat the process discussed above changing slightly the parameter K until that the three exact recurrence times are not observed anymore. We should stress that the three recurrence times need to be, necessarily, convergent of the continued fraction expansion of the number $1/\gamma$. Following this procedure we found the critical parameter as $K_c = 0.9716394$. To confirm the result, different sizes of ϵ should have recurrence values related to the convergent of the number $1/\gamma$ without the emergence of the fourth recurrence. The recurrence times $\sigma_{1,2,3}$ are shown in order of appearance in Table 3, as well as the number of times $N_{1,2,3}$ that each one was verified after 5×10^6 iterations.

It is remarkable that even after 5×10^6 iterations we verify only three recurrence times to a limited region of size ϵ and all of them are consecutive denominators of the continued fraction expansion of the irrational number $1/\gamma$, in accordance to Slater's theorem proved to the translation over a unity circle. Furthermore, we would like to point out that our result is very close to the widely accepted result provided by Greene [14], where K_c was estimated as 0.971635.

To finalize, in Fig. 4 we show the phase space for the SM with $K_c = 0.9716394$ and successive amplifications around the estimated last invariant curve. Note that the invariant curve really seems to be unbroken, giving support to our procedure.

We would like to call attention for problems in Slater's criterion when orbits after criticality describe a cantorus. As is well known, a cantorus is a *remnant* of an invariant curve after its breakup, constituted by an infinite set of nowhere dense invariant points, i.e., a cantorus presents a countable infinity of gaps. As Slater's criterion

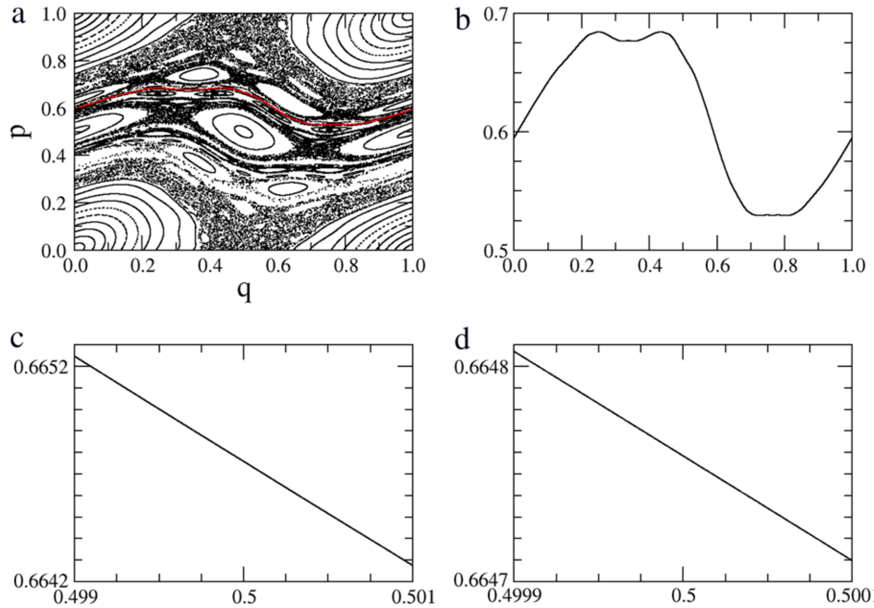

Fig. 4. (a) Phase space for the SM with $K_c = 0.9716394$. (b) The $1/\gamma$-last invariant curve. (c) and (d) are amplifications of the last invariant curve.

Table 3
(Left) The three recurrence times in order of appearance for different ϵ for the last invariant curve in SM ($K_c = 0.971639(4)$) estimated via Slater's theorem. All the recurrence values belong to the continued fraction expansion of the number $1/\gamma$. (Right) The number of times that each recurrence appeared after 5.0×10^6 iterations.

ϵ	σ_1	σ_2	σ_3	N_1	N_2	N_3
1×10^{-6}	196 418	121 393	75 025	10	10	23
5×10^{-6}	28 657	17 711	46 368	88	34	40
1×10^{-5}	17 711	10 946	28 657	128	53	75
5×10^{-5}	4 181	2 584	6 765	616	285	249
1×10^{-4}	2 584	1 597	987	1 250	1 015	150
5×10^{-4}	610	377	233	4 943	4 654	988
1×10^{-3}	377	233	144	7 594	7 934	2 003
5×10^{-3}	89	55	34	29 627	36 151	11 026
1×10^{-2}	34	21	55	93 082	52 429	13 349
5×10^{-2}	8	5	13	421 699	249 489	29 151

depends on the connected interval ϵ, an orbit on a cantorus will not satisfy the three recurrence if the interval ϵ is placed exactly in the gap. Otherwise, a cantorus may return at most three recurrence times and, therefore, Slater's criterion fails since the cantorus is not an invariant curve. In such case, orbits in cantori could eventually contribute to a slightly overestimated critical parameter values as compared with the values obtained by applying Greene's criterion.

5. Conclusion

The present paper has used Slater's theorem to study the breakup of invariant curves in phase space of area-preserving maps. Basically, we have explored the concept that the sequence $N\theta$ mod 1 returns to an arbitrary interval ϵ in at most three different recurrence times expressible by the continued fraction expansion of the irrational θ. As the quasi-periodic motion of invariant curves in the phase space is related with a simple rotation of a circle by parametrization, we have shown that the three recurrence times can be an indicator to determine if such invariants are broken or not.

The procedure based on Slater's theorem (three recurrence times) is indicated to verify the existence of invariant curves in a system with a set of parameters, as done in Refs. [9,10] and in the present paper for the nontwist map. Nevertheless, we have shown that the motion of invariant curves in the phase space has at most three recurrence times also expressible by continued fraction expansion of their rotation number. As a consequence, an approaching procedure based on the observation of these three recurrence times was developed to determine the breakup of invariant curves in the standard map, where we obtained $K_c = 0.9716394$. Since the accuracy depends critically on choosing an optimum relation among the return region of size ϵ, the number of iteration and, the initial conditions steps, our result for K_c can be improved, although it is likely that the exact value is about our $K_c = 0.9716394$ and Greene's result, $K_c = 0.971635$.

The procedure considered here offers an alternative method for determining criticality of invariant curves in bi-dimensional phase spaces, moreover its applicability is straightforward, adaptable and, fast to simulate.

We have observed that a relation should exist between the return region of size ϵ and the three recurrence times $\sigma_{1,2,3}$ for area-preserving maps, likewise Slater's theorem states for the translation over a unity circle. No mathematical proof has been obtained in this paper, however, we believe that our numerical solution can be useful to estimate invariant curves breakup of other dynamical systems.

Apart from area-preserving maps, it could be interesting to extend the procedure based on Slater's theorem for high dimensional maps (e.g., volume preserving maps [29]) and also continuous systems. However, applications of the procedure used in the present paper to determine critical parameters in other area preserving maps and especially for those with higher dimension may require to overcome specific difficulties.

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