Seminário de Caos

IFUSP - USP - SP

Transition from normal to super diffusion in a one-dimensional impact system

André L. P. Livorati

Departamento de Física (IGCE) - UNESP - Rio Claro - SP - Brazil

São Paulo, December 6^{th} 2017.

Collaborators

- Edson Denis Leonel (UNESP - Rio Claro - SP - Brazil)
- Iberê Luiz Caldas (IFUSP - São Paulo - SP - Brazil)
- Carl P. Dettmann (School of Mathematics - University of Bristol - UK)
- Tiago Kroetz (UFTPR - Pato Branco - PR - Brazil)

Outlook

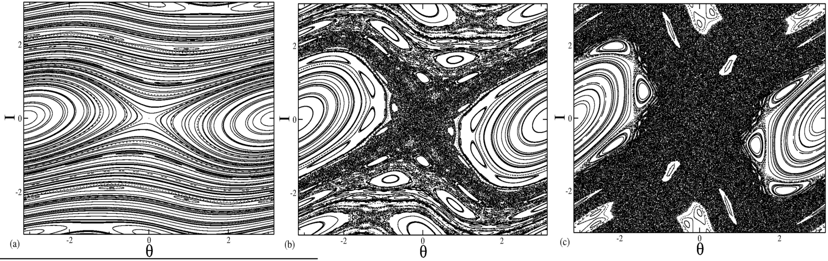
Dynamical Scenario

- Describing the dynamics via Hamiltonian formalism, each pair of coordinates (q_i, p_i), with $i = 1, 2, 3, \ldots$ denotes a degree of freedom of the system. So,

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} . \quad (1)$$

- The phase space is defined as the set of the whole possible orbits. Depending on the control parameters and initial conditions, one may find mainly three important dynamical behaviour *(i)* integrable (stable), *(ii)* ergodic (chaotic) or *(iii)* mixed.

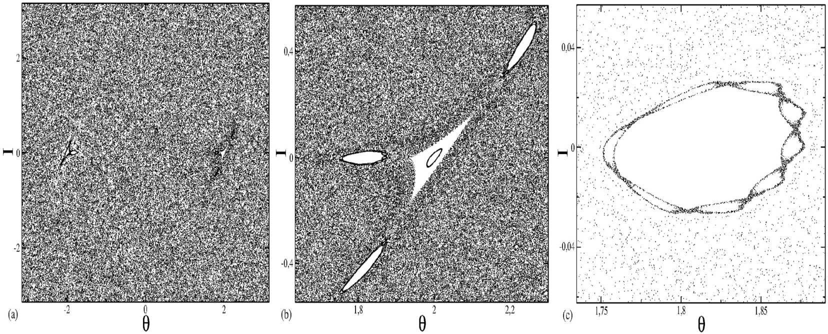
- Typical Hamiltonian systems have non-ergodic and non-integrable dynamics. Their phase space are divided into regions with regular and chaotic dynamics, where we can observe KAM islands and invariant tori surrounded by chaotic seas[1].



[1] *Lichtenberg and Lieberman, "Regular and Chaotic Dynamics",(1992)*

Mixed phase space and Stickiness

- The division in the mixed phase space leads to the stickiness phenomenon, which is manifested through the fact that a chaotic orbit passing near enough a KAM island or a cantori, may get trapped around the stability region for a finite time [2].



- The stickiness phenomenon has several applications in many areas of research, such as Fluid Mechanics, Astronomy, Biology, Plasma Physics, among others.

[2] *Zaslavsky, "Hamiltonian Chaos and Fractional Dynamics", (2008)*

Transport and Diffusion

- The diffusive behaviour is set as the way the transport of orbits occurs in the phase space.
- The introduction of a leakage or hole, as a pre-defined subset of the phase space, allow us to set $\rho(\vec{r},t)$, which is the probability of an ensemble of initial conditions to survive the leaking [3].
- The dynamical region is separated in particles that have escaped and particles that still have not escaped.
- Considering the difference of concentration of particles among both regions, and the continuity equation we end up with the diffusion equation.

$$\frac{\partial \rho(\vec{r},t)}{\partial t} = D\nabla^2 \rho(\vec{r},t) \ . \qquad (2)$$

- The transport is then investigated considering the decay rate of the survival probability [4].
- *(i) Chaotic Sea:* The decay rate of $\rho(r,\vec{t})$ is exponential, as $\rho(\vec{r},t) \propto e^{-\zeta t}$.
- *(ii) Near the stability islands:* There is a trapping by stickiness influence, and the decay rate of $\rho(\vec{r},t)$ is slower like a power law: $\rho(\vec{r},t) \propto t^{-\gamma}$, or like a stretched exponential: $\rho(\vec{r},t) \propto e^{-\xi t^{\alpha}}$, where $\alpha \in [0,1]$.

---

[3] P. Gaspard, *"Chaos, Scattering and Statistical Mechanics"*, (1998)

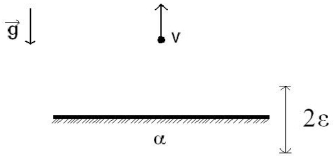
[4] E. G. Altmann, et. al., Rev. Mod. Phys., 85, 869, (2013)

Fermi Acceleration and the Bouncing ball model

- In 1949, Fermi claimed that charged cosmic particles could acquire energy (in average) from the moving magnetic fields present in the cosmos [5]. Such mechanism that was an attempt to explain the orign of the high energy of the cormic rays, and was called Fermi Acceleration.

- The model consists of a free particle (making allusions to the cosmic particles) which is falling under influence of a constant gravitational field g (a mechanism to inject the particle back to the collision zone) and suffering collisions with a heavy and time-periodic moving wall (denoting the magnetic fields).

- Dissipation: Inelastic collisions with coefficient $\alpha \in [0, 1]$ (as here), and/or damping or kinetic friction.



- The moving wall is located at $y = 0$, and vibrates according $y_w(t) = \epsilon' \cos(wt)$, where ε and w are respectively the amplitude and the frequency of oscillation.

5 *E. Fermi, Phys. Rev.,75, 1169 (1949)*

Mapping and Collisions

- The dynamics are described by the velocity v_n and time t_n after a collision with the moving wall.
- Between the collisions with the moving wall, the particle travels in a uniform accelerate movement. The exact collision time is hence obtained from $y_p(t) = y_w(t)$.
- Defining some dimensionless variables as: $V_n = v_n w/g$, $\epsilon = \varepsilon w^2/g$ e measuring the time in terms of the phase of the moving wall $\phi_n = wt_n$, we end up with

$$T : \begin{cases} V_{n+1} = V_n^* - 2\epsilon \sin(\phi_{n+1}) \\ \phi_{n+1} = [\phi_n + \Delta T_n] \mod(2\pi) \end{cases} . \tag{3}$$

- For multiple collisions inside the collision zone, $y \in [-\epsilon, +\epsilon]$ we have: $V_n^* = V_n$ and $\Delta T = \phi_c$, where ϕ_c is the solution of the equation $G(\phi_c) = 0$, where

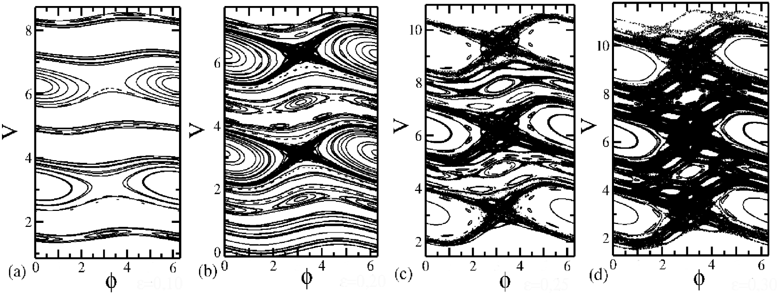
$$G(\phi_c) = \epsilon \cos(\phi_n + \phi_c) - \epsilon \cos(\phi_n) - V_n \phi_c + \frac{1}{2}\phi_c^2 . \tag{4}$$

- For single collisions , we have: $V_n^* = -\sqrt{V_n^2 + 2\epsilon(\cos(\phi_n) - 1)}$ and $\Delta T_n = \phi_u + \phi_d + \phi_c$, where $\phi_u = V_n$ and $\phi_d = \sqrt{V_n^2 + 2\epsilon(\cos(\phi_n) - 1)}$ are up and down flight times. Again, ϕ_c is the solution of the equation $F(\phi_c) = 0$, where

$$F(\phi_c) = \epsilon \cos(\phi_n + \phi_u + \phi_d + \phi_c) - \epsilon - V_n^* \phi_c + \frac{1}{2}\phi_c^2 . \tag{5}$$

Phase space

- For $\epsilon = 0$, the system is integrable, and when we increase ϵ, there is a transition from local chaos, to global chaos.
- Such transition is crucial for the FA phenomenon to occur. Here, we have the destruction of the invariant spanning curves, allowing the union of the local chaotic seas. So a chaotic orbit has a “free path” to diffuse along the velocity axis [6].

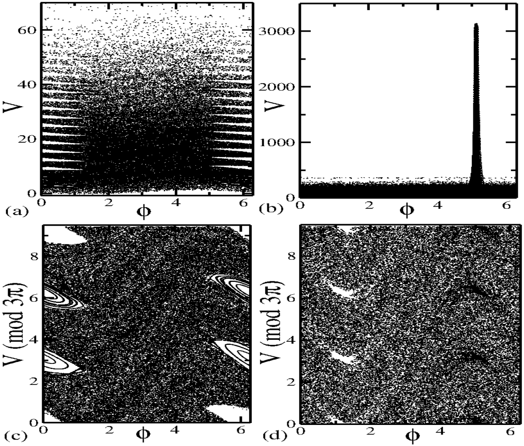


Phase Space parameters. *In (a)* $\epsilon = 0.1$*, (b)* $\epsilon = 0.2$*, (c)* $\epsilon = 0.2425$*, and (d)* $\epsilon = 0.3$*.*

[6] A. L. P. Livorati, et. al., Phys. Rev. E., 86, 036203, (2012)

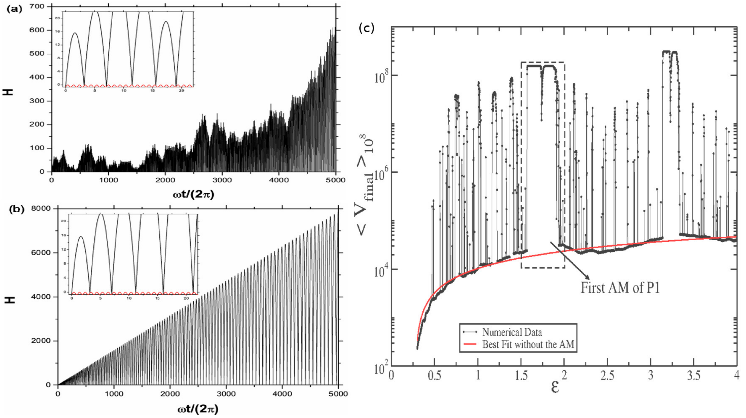
Fermi Acceleration and Accelerator modes

- As the number of collisions evolve, the velocity grows in two different manners.
- It can grow following a normal diffusion, also known as Regular Fermi Acceleration (RFA). Or, the velocity can obey a super diffusive regime of growth, because of the influence of the Accelerator Modes (AM), roughly described as featured resonances, where we observe Ballistic Fermi Acceleration (BFA).



Acceleration as function of ϵ

- The difference between the two growths of velocity lies in the vibraring plataform.
- For the RFA, the impacts may occur with the platform in a descendent movement and this promotes an instantaneous loss of energy, but in the average a growth can be observed after several impacts.
- In contrast, for the BFA, the gain of energy is always ascendent, where the impacts always occur for an ascendent movement of the platform.



Root mean square velocity and diffusion

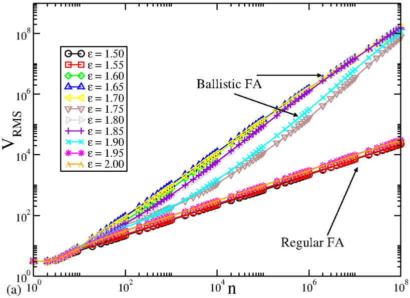
- Let us set numerically the behaviour of the Root mean square velocity V_{RMS}, as .

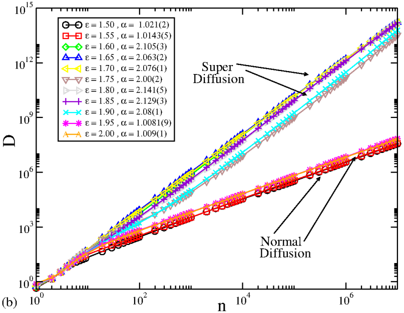
$$V_{RMS} = \sqrt{\langle V^2 \rangle} = \frac{1}{M}\sum_{i=1}^{M}\frac{1}{n}\sum_{j=1}^{n} V_{i,j}{}^2 \ , \tag{6}$$

- The dispersion of the mean square velocity and the diffusion coefficient can be given by

$$\langle \Delta V^2 \rangle = \lim_{NP\to\infty}\frac{1}{NP}\sum_{i=1}^{NP}(V_n^i - V_0{}^i)^2 \qquad ; \qquad D = \lim_{n\to\infty}\frac{1}{2n}\langle \Delta V^2 \rangle^{\alpha} \ . \tag{7}$$

For $\alpha < 1$, we have a sub diffusive regime, if $\alpha = 1$ the normal diffusion (random walk) takes place, and finally if $\alpha > 1$ we have the super diffusive regime.





Fixed Points and Bifurcation of the AM

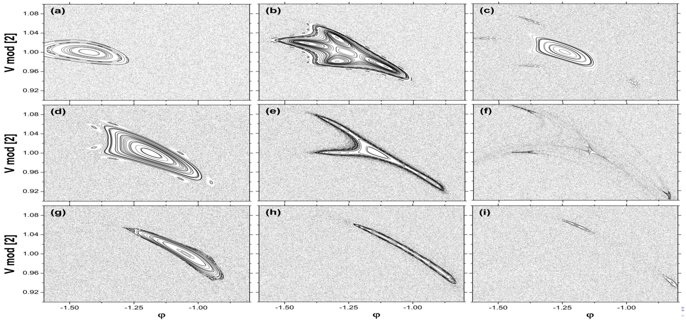
- The period-1 fixed points can be obtained considering the repetition structure for the velocity in the phase space. So, we have

$$V^{ac} = \pi m \text{ where; } \quad m = 1, 2, 3, \ldots \qquad \phi^{ac} = \arcsin\left(-V^{ac}/2\epsilon\right). \tag{8}$$

- We linearize the system around (V^{ac}, ϕ^{ac}) via Jacobian matrix, where the stability condition, $|Tr\ J| < 2$ will be satisfied since the eigenvalues are complex (elliptical fixed points).

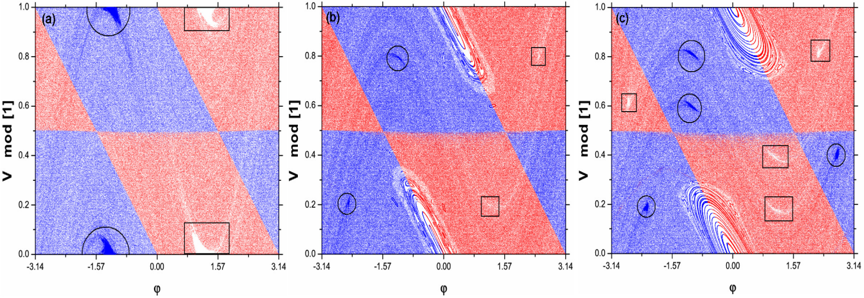
- So, the stability condition as a function of parameter ϵ is

$$\frac{\pi m}{2} < \epsilon < \sqrt{1 + \frac{\pi^2 m^2}{4}}. \tag{9}$$



Phase space according $det(J)$

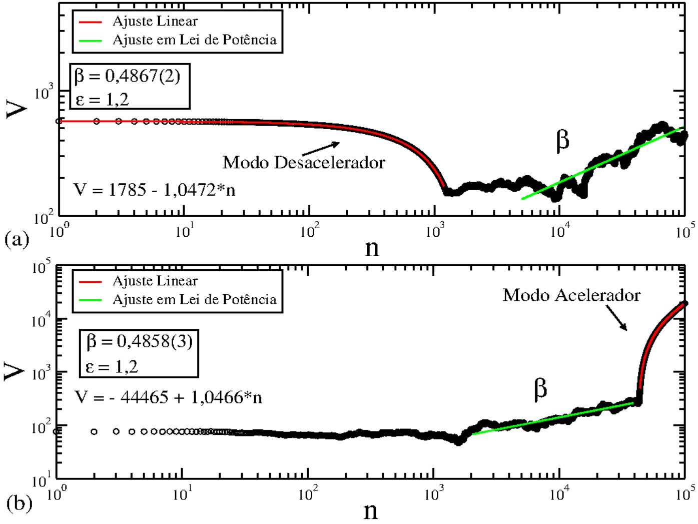
- Since the complete mapping is not symplectic, the $det(J) = \frac{V_n + \epsilon \sin(\phi_n)}{V_{n+1} + \epsilon \sin(\phi_{n+1})}$, present values larger or smaller than the unity. So, the phase space presents contract (red) regions and expansion (blue) regions [7].
- In these regions, we found deaccelerator modes (DM) and accelerator modes (AM) in the anti-symmetric position of each other.



Phase space for: (a) $\epsilon = 1.69$, (b) $\epsilon = 1.01$ and (c) $\epsilon = 0.744$.

[7] T. Kroetz, A. L. P. Livorati, et. al., Phys. Rev. E., 92, 012905 (2015)

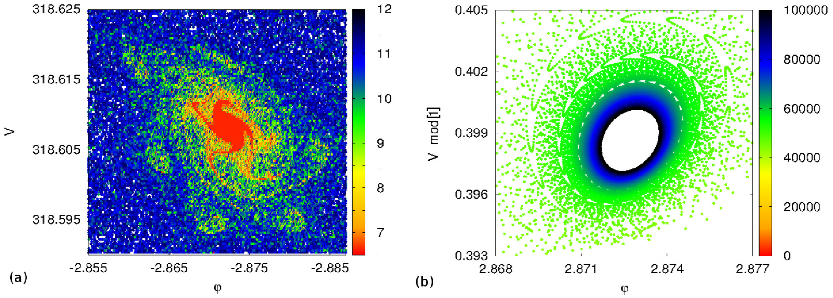
Evolution for DM and AM



Evolution for a single initial condition for period-3: (a) DM and (b) AM.

Repelling and attracting nature

- Considering the modulated phase space V, we can see the repelling nature for the DM, and the attracting nature of the AM.[7].

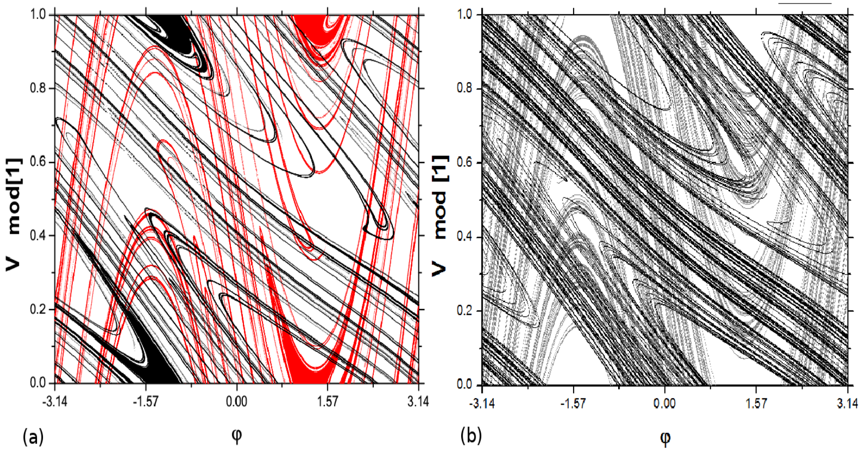


In (a) basin of repelling for the DM and in (b) the "attracting" nature for the AM.

[7] T. Kroetz, A. L. P. Livorati, et. al., Phys. Rev. E., 92, 012905 (2015)

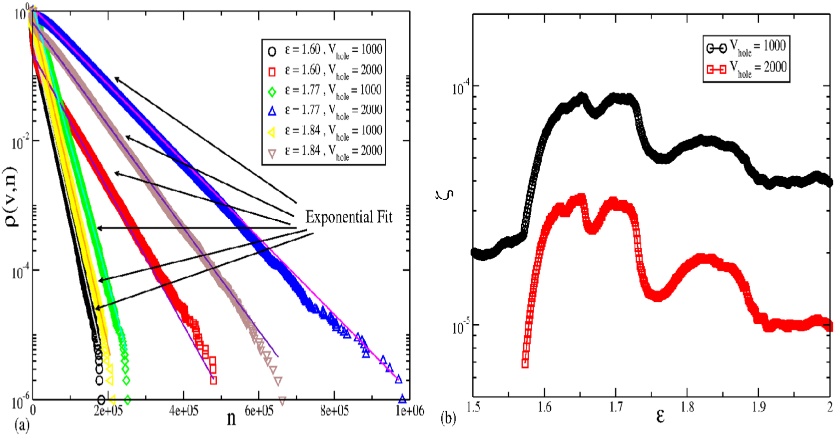
Manifolds and escape basins

- Since we have the nature of attracting and repelling for respective AM and DM in the modulated phase space, we may consider escape (attracting) basins for the AM, and repelling basins for the DM.
- We drawn the stable and unstable manifold for the central saddle point, and confirm that these manifolds are respectively drawing the boundary of the escape and repelling basins, for AM and DM.



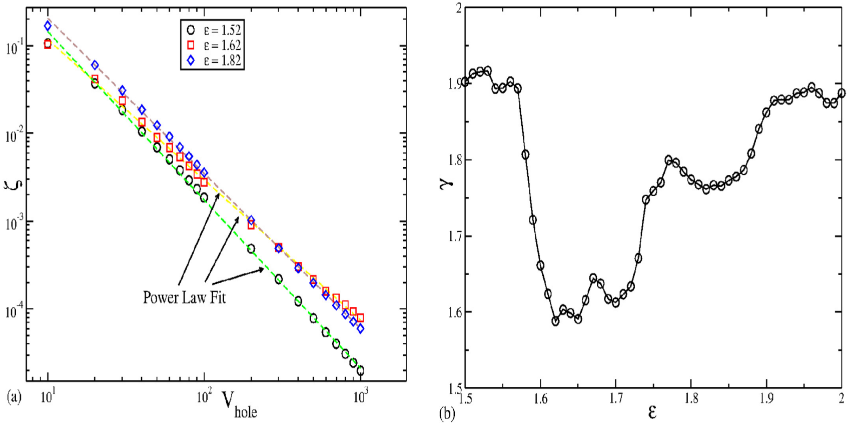
Transport Analysis

- The survival probability $\rho(v, n)$, has an exponential decay rate for some values of ϵ where the AM of period-1 is active.



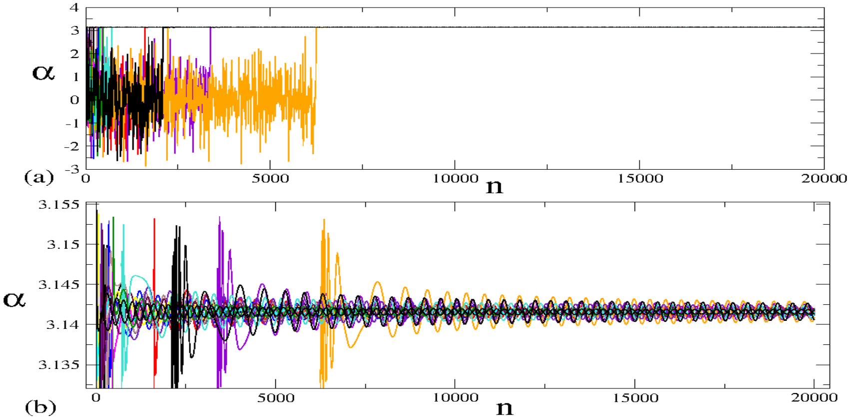
Escape Scenario

- Considering distinct velocity holes, for each ϵ where the AM is active, we can obtain a transport scenario.
- The decay rate of the survival probability ζ follows a power law dependence with the velocity hole V_{hole}.



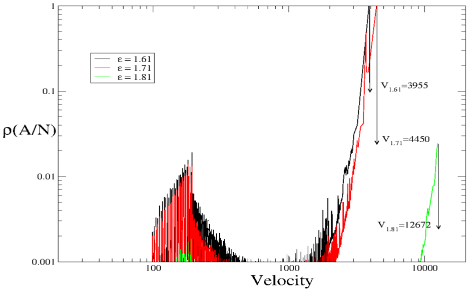
Convergence to the AM via linear regression

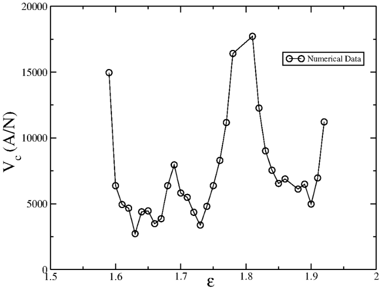
- How do we know if an orbit reached the AM?
- For the AM of period-1, there is a step-size of $V = \pi$ in the velocity axis, and there is a linear growth of the V_{RMS}.
- So, a linear regression of the type $V_n = \alpha n + \beta$ should provide us $\alpha \approx \pi$.



Probability of achieve the AM as function of the velocity

- We created histograms of frequencies as function of the velocity of the orbit for two dynamical cases:
- *(i)* before reach the AM, which we will label as *N* (for normal diffusion), and *(ii)* when the orbit is at the AM, which we will label as *A* (for accelerator mode).
- At each collision, we keep adding an unity to the relevant *N* box, until the linear coefficient reaches the value of $\alpha \approx \pi$. After that, we know (according our linear regression criteria) that the orbit reached the AM, then we add an unit to the *A* box, stop the simulation and start a new initial condition.





Summary and final remarks

- The dynamics of the bouncing model was investigated using a two dimensional mapping. Chaotic properties where characterized, and transition from local to global chaos allows the phenomenon of Fermi Acceleration (FA) to occur.
- Two growing regimes were characterized for FA. The Regular Fermi Acceleration (RFA), set by a normal diffusion; and the Ballistic Fermi Acceleration (BFA), related with featured resonances known as accelerator modes (AM), causing super diffusion.
- Through the analysis of the dispersion of the root mean square velocity, we were able to characterize a diffusive transition in a range were a period-1 AM is active.
- Considering transport properties, a description of the super diffusive scenario for the AM was achieved, where the probability of an ensemble to reach the AM, as function of the velocity and as function of the number of collision, present the same layout, with distinguished peaks in the ϵ range.
- As a next step, we intent to investigate how different and higher periods of the AM influence the transport properties and the transition from normal to ballistic diffusion from a local and global points of view.

Acknowledgements

FAPESP

Fundunesp

Fundação para o Desenvolvimento da UNESP

CNPq

Conselho Nacional de Desenvolvimento Científico e Tecnológico

CAPES

University of BRISTOL

unesp