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Direction coherence in scale-free lattices of chaotic maps

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Abstract

We considered a coupled chaotic logistic map lattice exhibiting the scale-free property: the outreach connectivity of each node obeys a power-law distribution. We analyzed a weak form of coherent spatio-temporal behavior (direction coherence) which presents features common to completely synchronized states, like a transitional behavior with intermittent bursting. We studied such phenomena and their dependence on the parameters of the coupled scale-free lattice. Prospective applications in neuronal networks are emphasized.

Keywords: Scale-free lattices; Direction coherence; Phase direction

1. Introduction

Networks with complex topology are ubiquitous both in scientific and technological applications [1]. Such complex networks have nodes representing individuals or organizations, and the links stand for the interactions among them [2]. A class of complex networks which has received a lot of attention is the *scale-free* network, for which the connectivity, or the number of connections for each node, presents a statistical power-law dependence [3]. Hence in scale-free networks a few nodes are connected with a large number of other ones, whereas most of the nodes are connected with a few others only. More specifically, if $P(k)\,\mathrm{d}k$ denotes the probability of finding a node with connectivity between k and $k+\mathrm{d}k$, for scale-free lattices one has $P(k){\sim}k^{-\gamma}$ where $\gamma>1$, what turns to be an example of Lévy distribution [4]. The power-law distribution of connectivities is regarded as a consequence of two generic mechanisms [3]: (i) networks expand continuously by the addition of new nodes; and (ii) new nodes attach preferentially to already well-connected nodes.

A plethora of networks of physical, biological, and social interest have been found to exhibit a scale-free connectivity: the World Wide Web [5], earthquakes [6], large computer programs [7], epidemic spreading [8],

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human sexual contacts [9], protein domain distributions [10], cellular metabolic chains [11], human brain functional networks [12], just to mention a few representative examples.

While many problems involving scale-free networks have been treated from the graph–theoretical point of view, such that the Euclidean distance between nodes does not play a significant role, practical applications of scale-free networks often involve the use of a lattice embedded in a Euclidean space [13]. For example, neuronal networks are embedded in a three-dimensional lattice in the brain, where the nodes are the neurons, connected by axons and dendrites [14]. A recent work used a scale-free neural network to implement Hopfield pattern recognition [15]. Hence, in order to investigate dynamical models for the brain function, for example, it is necessary to develop methods to generate and analyze scale-free lattices of coupled dynamical systems.

In this paper we consider a lattice of coupled maps, which are discrete-time dynamical systems playing the role of information-processing neurons, and a coupling prescription with the power-law dependence characteristic of scale-free networks. Since there are experimental evidences that the neurons exhibit chaotic behavior we use coupled logistic maps as simple models [16]. However, other discrete-time maps [17] or continuous-time flows [18,19] could be used as well. In particular, the choice of logistic maps limits us to the case of a single timescale, thus emulating the spiking dynamics only, whereas the models of Refs. [17–19] are able to describe also the bursting timescale.

The spatio-temporal dynamics of coupled map lattices with the related *small-world* property (a small average distance between sites, while retaining a reasonable degree of clustering [20]) has been studied in previous papers [21,22]. We found that, for small-world lattices, synchronization of chaotic motion is enhanced due to the non-local couplings added to an otherwise purely regular coupled map lattice. Lind et al. have studied completely synchronized states in a scale-free lattice for which the coupling is heterogeneous (proportional to some power of the neighbor connectivity) [23]. For the scale-free lattices considered in this work the coupling is homogeneous (i.e. the coupling strength takes on the same value for all coupled sites) and we cannot find complete synchronization. In spite of this, we observed a weaker form of synchronization, which we called coherence direction, and which is an analogue of phase synchronization for discrete-time systems.

The main purpose of this work is to consider the transition to phase synchronization and its dependence with the coupling strength and the nonlinearity of the sites. We found this transition to share many features with those observed for completely synchronized states, specially an intermittent behavior near the transition threshold. This paper is organized as follows: in Section 2 we introduce the coupled logistic map lattice and explain how we obtained a scale-free lattice. Section 3 deals with the direction coherence and the transition to it through intermittency. Our conclusions are left to the last section.

2. Scale-free coupled chaotic map lattice

Coupled map lattices are widely reckoned as simple but paradigmatic models for complex systems like neural networks, excitable media, oscillator chains, etc. [24]. They present both space and time as discrete variables, while retaining a continuous state variable that is capable to undergo a smooth nonlinear dynamics. We examine, in particular, a chain of N coupled logistic maps $x \mapsto f(x) = rx(1-x)$, where $x_n^{(i)} \in [0,1]$ represents the state variable for the site i ($i = 1, 2, \ldots, N$) at time n. The chaotic region in the bifurcation diagram of an isolated chaotic map starts at the Feigenbaum point $r_\infty = 3.569\,945\,672\ldots$ and ends at the crisis point $r_{CR} = 4.0$. In spite of this region having an infinite number of periodic windows, the probability of getting a chaotic orbit by randomly choosing r in the interval $(r_\infty, r_{CR}]$ is non-zero [25].

In this paper we use the following coupling prescription:

$$x_{n+1}^{(i)} = (1-\varepsilon) f(x_n^{(i)}) + \frac{\varepsilon}{k^{(i)}} \sum_{j \in I} f(x_n^{(j)}), \tag{1}$$

where $\varepsilon > 0$ is the coupling strength and we assumed that each site i is coupled with itself and with a set I comprising $k^{(i)}$ other sites randomly chosen along the lattice according to an assigned scale-free probability distribution $P(k) \sim k^{-\gamma}$, where k is the connectivity, or the number of connections *per* site. We use free boundary conditions for the lattice and random initial conditions $x_0^{(i)}$.

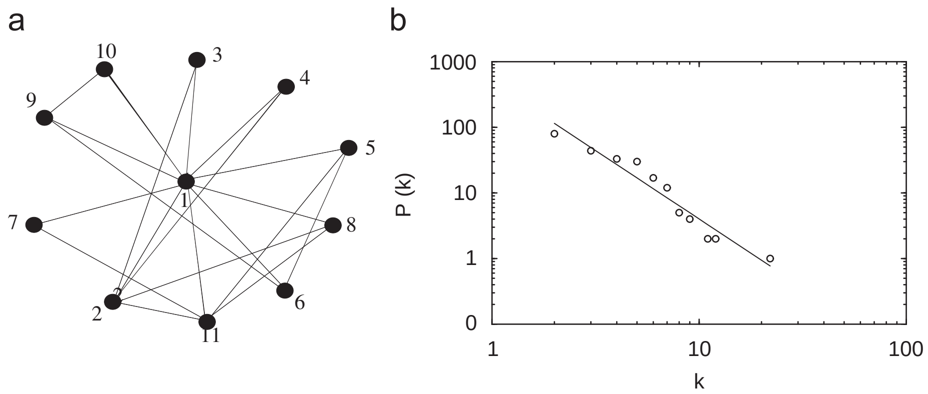

Fig. 1. (a) Seed lattice with $N_0 = 11$ sites used to start a sequence of steps towards a scale-free lattice. (b) Probability distribution for connectivity of a coupled map lattice of the form (1) with $N = 230$ sites. The solid line is a least-squares fit with slope -2.08.

We built the scale-free lattice by means of a sequence of steps $s = 0, 1, 2, \ldots, s_{max}$; starting from a seed lattice with $N_0 = 11$ sites (Fig. 1(a)). At each step s a new site is inserted in the lattice of size N_s, such that it is connected to $\ell = 2$ randomly chosen sites. According to the scale-free distribution, these connections occur preferentially with the more connected sites. This is done using a different probability for each site $P_s^{(i)} = k_s^{(i)}/N_s$, where $k_s^{(i)}$ is the number of connections *per* site at the step s. The process is repeated until we achieve a desired lattice size N, which we choose as $N = 230$ in the numerical simulations we perform in this work. After a number s_{max} of steps we have $k^{(i)}$ connections *per* site, corresponding to a probability $P^{(i)} = k^{(i)}/N$. Fig. 1(b) shows a non-normalized histogram for the number of sites with a connectivity k, obtained through this procedure for $N = 230$ sites. The numerical approximation to the probability actually displays the scale-free signature of a power-law scaling $k^{-\gamma}$ with slope $\gamma = 2.08$.

We can also rewrite (1) in the form

$$x_{n+1}^{(i)} = (1 - \varepsilon) f(x_n^{(i)}) + \frac{\varepsilon}{k^{(i)}} \sum_{j=1}^{N} g_{ij} f(x_n^{(j)}), \tag{2}$$

where g_{ij} are the elements of a $N \times N$ connectivity matrix, where $g_{ij} = 1$ if the sites i and j are connected, and zero otherwise. Since the connectivity *per* site is different, each line of the matrix g_{ij} has a different number of ones distributed over the columns, the remaining elements being padded with zeroes. However, the connectivity matrix is symmetric ($g_{ij} = g_{ji}$) due to the process of construction of the scale-free lattice, i.e. the connectivity matrix evolves through a finite number of steps conserving its symmetry.

3. Phase coherence in coupled maps

Even though the sites may undergone chaotic dynamics by themselves, when they are coupled the resulting behavior is strongly affected by the connectivity of the lattice sites. One of the most studied phenomena is complete synchronization, by which a given number $M \leqslant N$ of lattice sites have the same amplitudes for m map iterations: $x_n^{(i)} = x_n^{(i+1)} = \cdots = x_n^{(i+M)}$, $n = 0, 1, \ldots, m$. If this synchronized state does exist it must be a valid solution of Eq. (1), what is generally *not* the case for the scale-free lattice given by Eq. (1) with homogeneous coupling, due to the randomness in the connectivity matrix elements. In spite of not having completely synchronized states, the sites may still present some kind of spatial coherence as the coupling parameters vary over a given range, opening the possibility of weaker but equally important collective spatio-temporal phenomena. We stress, however, that for heterogeneous couplings, scale-free lattices can present completely synchronized states [23].

There are many situations of physical interest in which two or more continuous-time oscillators may have different amplitudes, even in a chaotic regime, but with a well-expressed *phase coherence*. The oscillator phase can be defined in various ways for continuous-time systems, the simplest one being a geometrical phase for a bounded attractor [26,27]. For coupled map lattices, however, this procedure cannot be carried over, since there is no vanishing Lyapunov exponent which would enable an interaction of the coupled phases, in order to yield phase synchronization. Instead of phase synchronization, coupled maps can display a coherence with respect to the direction of their temporal evolution.

Direction-coherent maps are defined as those showing local maxima or minima for their amplitudes at the same time [28], such that the direction is provided by two sequential iterations of the coupled maps [29,30]. A lattice site $x_n^{(j)}$ thus has a direction at a fixed time n given by

$$d_n^{(j)} = \begin{cases} 1 & \text{if } x_n^{(j)} > x_{n-1}^{(j)}, \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

in such a way that a direction-coherent cluster is a union of adjacent maps with the same value of $d_n^{(j)}$.

In Fig. 2 we superposed two spatial profiles for two successive times and after a large number of transients have decayed, for a lattice of coupled chaotic logistic maps, where the directions are indicated by arrows. On the basis of the previous definition we can say that between the times at which both profiles were generated, all sites in Fig. 2 are direction-coherent. For further times, a certain number of the sites (or the entire lattice) may remain coherent or become non-coherent. If the only relevant information one needs is whether or not the site amplitudes are increasing or decreasing, this definition of direction is sufficient.

We denote by $\mathcal{N}_n^{(0)} = \sum_{j=1}^{N}(d_n^{(j)} = 0)$ and $\mathcal{N}_n^{(1)} = \sum_{j=1}^{N}(d_n^{(j)} = 1)$ the number of lattice sites at a time n with directions d_n equal to 0 and 1, respectively, and define a coherence ratio ρ_n as [29,30]

$$\rho_n \equiv \frac{1}{N}\max(\mathcal{N}_n^{(0)}, \mathcal{N}_n^{(1)}), \tag{4}$$

in such a way that, if the directions of all lattice sites flip randomly between 0 and 1, the ratio approaches a constant value; whereas, if $\rho = 1$, all lattice sites are direction-coherent. The minimum value for this ratio is $\rho = \frac{1}{2}$, a situation in which half of the sites have $d_n^{(j)} = 0$.

As the lattice pattern evolves with time, this ratio may change in distinct ways. For some parameter values (Fig. 3(a)) the ratio increases monotonically and saturates at unity after a number of iterations. On the other hand, other parameter sets make the coherence ratio to vary in an intermittent fashion, as illustrated by Fig. 3(b), where ρ_n has laminar phases at 1.0 with irregular bursts of lower values. In order to analyze both situations into a same framework we define the quantity $F = N_\rho/\Delta_n$, where N_ρ is the number of occurrences of direction-coherent sites (i.e., for which $\rho_n = 1$) we find within a time interval Δ_n. We can thus interpret F as the fraction of completely direction-coherent maps in a given time interval. For example, in Fig. 3(a), if we

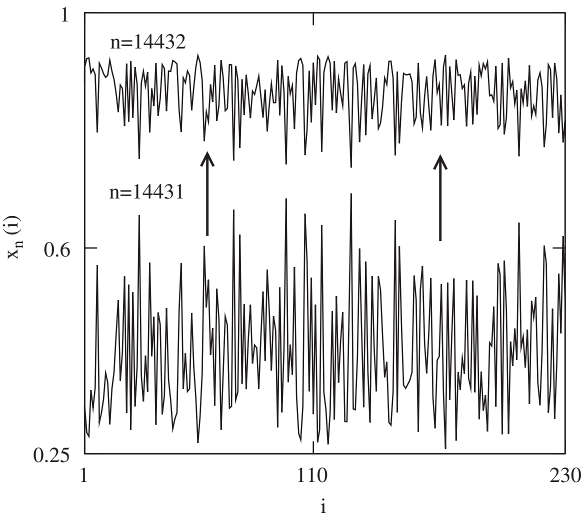

Fig. 2. Overlap of two spatial patterns at times $n = 14431$ and 14432 for a scale-free lattice of $N = 230$ maps with $r = 3.72$ and $\varepsilon = 0.9$. The arrows indicate the phase direction.

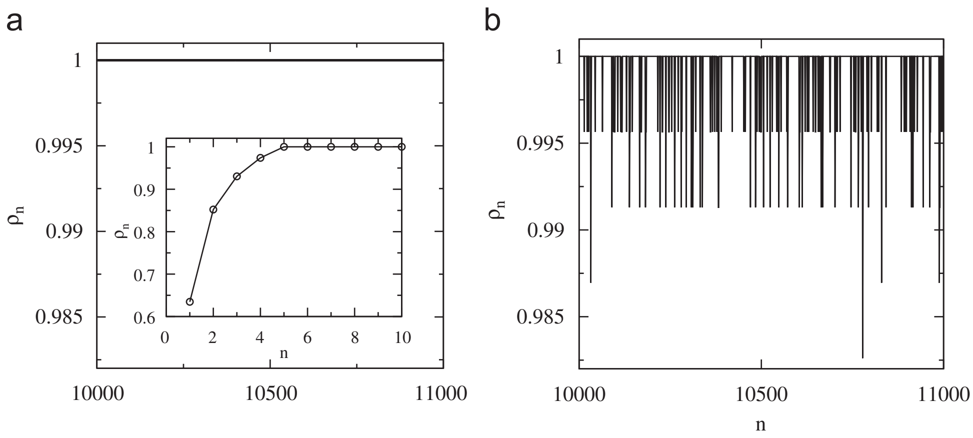

Fig. 3. Time series of the direction-coherence ratio for $N = 230$: (a) $r = 3.69$, $\varepsilon = 0.9$; and (b) $r = 3.72$, $\varepsilon = 0.33831$.

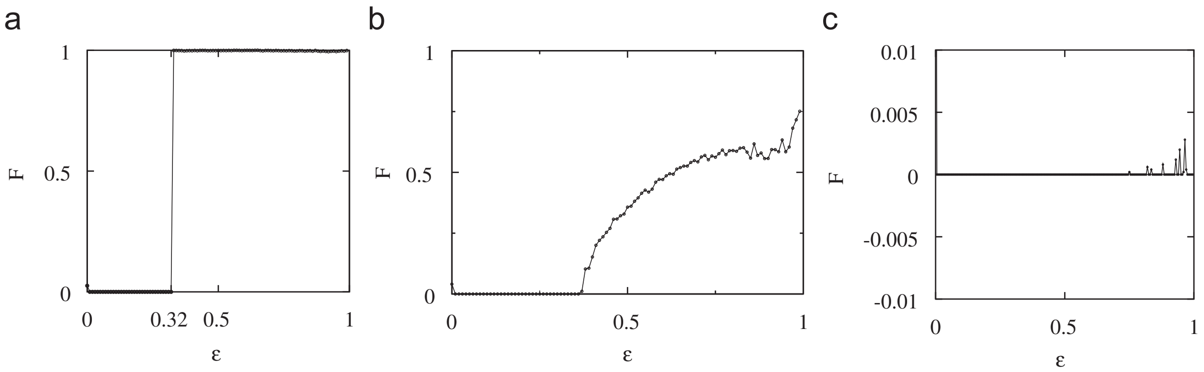

Fig. 4. Fraction of time where all lattice sites are coherent *versus* coupling strength for a scale-free lattice with $N = 230$: (a) $r = 3.69$; (b) $r = 3.77$; and (c) $r = 4.00$.

consider the entire time interval $\Delta_n = 10$, we have a fraction of $\approx 60\%$ of times for which the lattice exhibits direction-coherence. In Fig. 3(b) this fraction is considerably lower due to the many bursts for which $\rho_n < 1$, in the time interval $\Delta_n = 1000$.

The fraction of time where all lattice sites are coherent depends on the coupling strength ε in a way showing a critical transition, as depicted by Fig. 4(a), where, for coupled chaotic logistic maps with $r = 3.69$, there is an abrupt transition of F from zero to unity at $\varepsilon^* \approx 0.32$. This means that, as we increase the coupling strength, a completely non-coherent lattice can suddenly become completely coherent. There are actually two critical points, since this transition is generally not so abrupt, occurring for other nonlinearity parameters, with $\varepsilon_c \approx 0.37$ (Fig. 4(b)). In general we shall distinguish between ε_c, the value for which F ceases to be zero; and ε^*, the value for which F becomes equal to unity (notice that, in Fig. 4(a), $\varepsilon_c \approx \varepsilon^*$). For other parameter values, however, these transitions do not occur at all (Fig. 4(c)).

One of the key features we observed in our scale-free lattice is that, for $\varepsilon > \varepsilon_c$, there is an intermittent switching between laminar regions of $\rho = 1$ and bursts with $\rho < 1$, like those observed in Fig. 3(b). The laminar phases have different lengths and their normalized histograms show, for two widely different post-critical values of the coupling strength, a probability distribution well-fitted by an exponential $\mathscr{P}(\tau) \sim e^{-\alpha\tau}$ (Fig. 5), with values of α different according to the coupling strength.

The critical coupling strength ε_c does depend on the nonlinearity parameter r, but seems not to depend on the lattice size, provided it is large enough. Fig. 6 plots the value of ε_c *versus* the lattice size N for $r = 3.72$, and we observe that ε_c exhibits fluctuations around 0.35 of less amplitude as N increases.

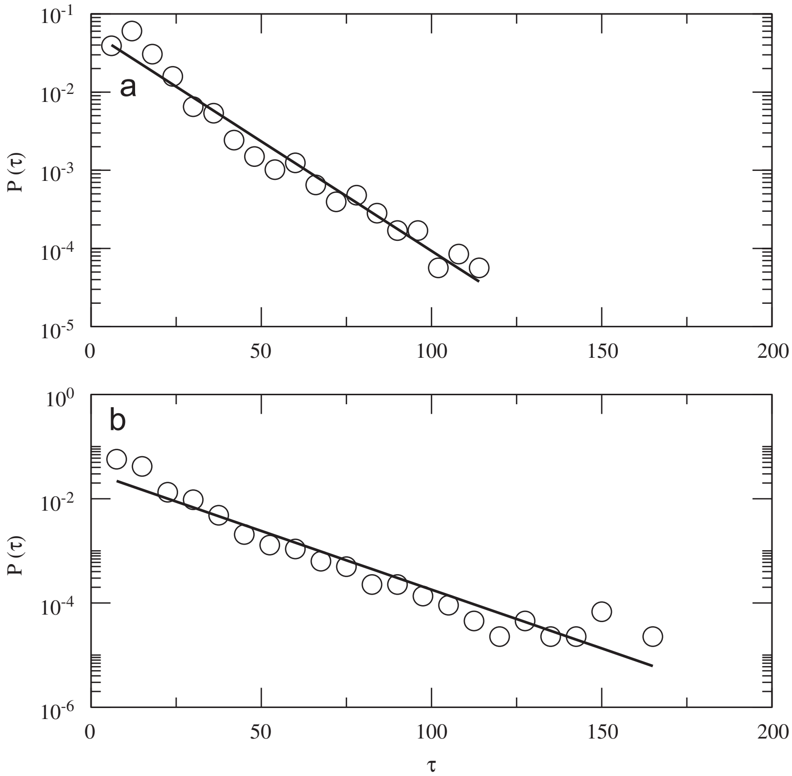

Fig. 5. Probability distributions of the relative number of direction-coherent plateaus for a scale-free lattice with $N = 230$, $r = 3.72$: (a) $\varepsilon = 0.33831$; and (b) $\varepsilon = 0.7$. The solid lines are least-squares fits with slopes -0.13 and -0.04, respectively.

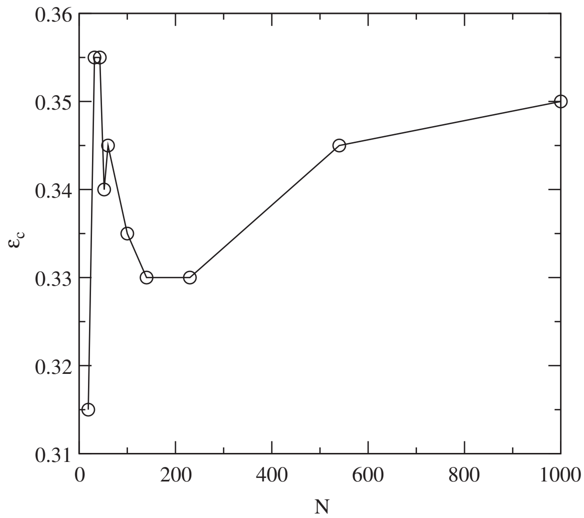

Fig. 6. Critical coupling strength for the onset of intermittent behavior of the direction-coherence ratio *versus* lattice size N for $r = 3.72$.

The second transition we observed, namely from an intermittent switching between coherence and non-coherence, to a completely coherent behavior, occurs for a value ε^* of the coupling strength which increases monotonically with the nonlinearity parameter r, as depicted in Fig. 7. Those values of r in Fig. 7 for which

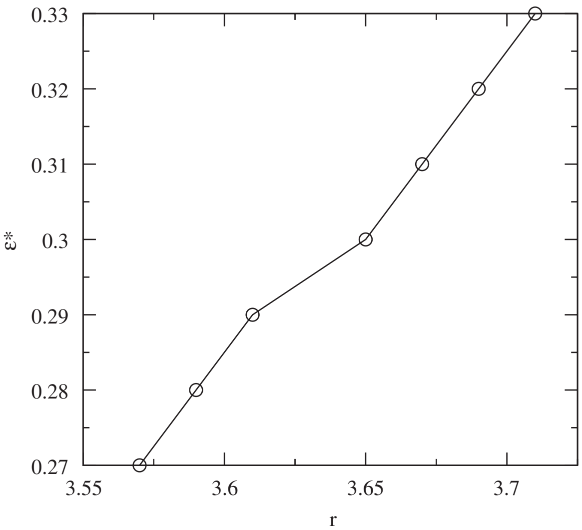

Fig. 7. Critical coupling strength for the end of intermittent behavior of the direction-coherence ratio as a function of the nonlinearity parameter r for $N = 230$.

there are not ε^*-values, correspond to situations for which the system does not present such transition (cf. Fig. 4(c) for an example).

4. Conclusions

We constructed a scale-free lattice of coupled chaotic logistic maps by adding new sites according to a probability distribution dependent on the connectivity as a power law. The coupling prescription we used is homogeneous albeit bidirectional, what rules out the possibility of having completely synchronized states. We nevertheless observed a weaker form of collective behavior we called direction coherence, and which would correspond to phase synchronization in continuous-time systems.

We found three possible situations, according to the values of the coupling strength and the nonlinearity parameter of the maps: (i) the lattice can be completely direction-coherent, what means that the time evolution of each site amplitude, although different, does share the same trend (i.e. they are either increasing or decreasing); (ii) the lattice alternates between direction-coherent and incoherent phases in an intermittent fashion; and (iii) the lattice amplitudes are totally incoherent.

The probability distribution of inter-burst intervals in the intermittent behavior case was found to obey an exponential decay. Moreover, there are transitions between the three above-mentioned situations, which we identify by computing the corresponding critical values. We still do not have a theory explaining such transitional behavior, although some clues may be found in the previous studies we have made of regular lattices with non-local coupling (the coupling strength decaying with the lattice distance as some negative power) and small-world coupling . In the former case, we have found that such non-locally coupled lattices may present completely synchronized states, and the transition to a non-synchronized behavior also occurs *via* an intermittent scenario [31]. As for the latter case, we were able to identify synchronization transitions for small-world lattices, although without noticeable intermittent behavior [22].

Scale-free lattices are not regular, since a given number of sites is randomly added. However, if the connectivity is large enough, a scale-free lattice becomes increasingly similar to a globally coupled lattice, where each site is coupled to the mean-field produced by all other sites [32]. We conjecture that this is also may be the case in the system treated in this paper, i.e. a completely synchronized state being replaced to totally direction-coherent sites. The main conclusion is that a scale-free network can enhance spatial coherence of sites, just like strongly coupled regular lattices enhance synchronization. This has potentially far-reaching consequences in the modeling of neuronal networks with scale-free lattices of the form we proposed in this work, since neurons—albeit chaotically oscillating—should produce coherent output through a kind of synchronization or at least enhanced coherent behavior in the brain. Our work showed that this is possible for the kind of models here considered.

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References

[1] S. Borhnoldt, H.G. Schuster (Eds.), Handbook of Graphs and Networks, Wiley-VCH Verlag, Berlin, 2003.
[2] R. Albert, A.-L. Barabási, Rev. Mod. Phys. 74 (2002) 47;
S.N. Dorogovtsev, J.F.F. Mendes, Adv. Phys. 51 (2002) 1079.
[3] A.-L. Barabási, R. Albert, Science 286 (1999) 509.
[4] M.F. Shlesinger, J. Klafter, Phys. Rev. Lett. 54 (1985) 2551.
[5] A.-L. Barabási, R. Albert, H. Jeong, Physica A 281 (2000) 69;
A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins, J. Wiener, Comput. Network 33 (2000) 309;
R. Pastor-Satorras, A. Vázquez, A. Vespignani, Phys. Rev. Lett. 87 (2001) 258701.
[6] M. Baiese, M. Paczuski, Phys. Rev. E 69 (2004) 066106.
[7] A.P.S. de Moura, Y.-C. Lai, A.E. Motter, Phys. Rev. E 68 (2003) 017102.
[8] R. Pastor-Satorras, A. Vespignani, Phys. Rev. Lett. 86 (2001) 3200.
[9] F. Lijeros, C.R. Edling, L.A.N. Amaral, H.E. Stanley, Y. Aberg, Nature 411 (2001) 907.
[10] S. Wuchty, Mol. Biol. Evol. 18 (2001) 1694.
[11] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, A.-L. Barabási, Nature 407 (2000) 651;
H. Jeong, S. Mason, A.-L. Barabási, Z.N. Oltvai, Nature 411 (2001) 41;
A.-L. Barabási, Z.N. Oltvai, Nat. Rev. Genet. 5 (2004) 101.
[12] V.M. Equiluz, D.R. Chialvo, G.A. Cecchi, M. Buliki, A.V. Apkarian, Phys. Rev. Lett. 94 (2005) 018102.
[13] A.F. Rozenfeld, R. Cohen, D. ben-Avraham, S. Havlin, Phys. Rev. Lett. 89 (2002) 218701.
[14] H. Nozawa, Chaos 2 (1992) 377;
S. Raghavachari, J.A. Glazier, Phys. Rev. Lett. 74 (1995) 3297;
S. Ishii, M. Sato, Physica D 121 (1998) 344.
[15] D. Stauffer, A. Aharony, L. da Fontoura Costa, J. Adler, Eur. Phys. J. B 32 (2003) 395.
[16] E. Basar, T.H. Bullock, Chaos in Brain Function, Springer, Berlin–New York, 1990.
[17] N.F. Rulkov, Phys. Rev. E 65 (2002) 041922;
N.F. Rulkov, I. Timofeev, M. Bazhenov, J. Comput. Neurosci. 17 (2004) 203.
[18] R. FitzHugh, Biophys. J. 1 (1961) 445;
J.S. Nagumo, S. Arimoto, S. Yoshizawa, Proc. IRE 50 (1962) 2061.
[19] J.L. Hindmarsch, R.M. Rose, Philos. Trans. R. Soc. London Ser. B 221 (1984) 87.
[20] D. Watts, Small Worlds, Princeton University Press, Princeton, 2000.
[21] A.M. Batista, S.E. de S. Pinto, R.L. Viana, S.R. Lopes, Physica A 322 (2003) 118.
[22] A.M. dos Santos, R.L. Viana, S.R. Lopes, S.E. de S. Pinto, A.M. Batista, Physica A 367 (2006) 145.
[23] P.G. Lind, J.A.C. Gallas, H.J. Hermann, Phys. Rev. E 70 (2004) 056207.
[24] K. Kaneko, in: K. Kaneko (Ed.), Theory and Applications of Coupled Map Lattices, Wiley, Chichester, 1993.
[25] J.D. Farmer, Phys. Rev. Lett. 55 (1985) 351.
[26] A. Pikowsky, G. Osipov, M. Rosenblum, M. Zaks, J. Kurths, Phys. Rev. Lett. 79 (1997) 47.
[27] G.V. Osipov, A.S. Pikowsky, M.G. Rosenblum, J. Kurths, Phys. Rev. E 55 (1997) 2353.
[28] B. Hu, Z. Liu, Phys. Rev. E 62 (2000) 2114.
[29] W. Wang, Z. Liu, B. Hu, Phys. Rev. Lett. 84 (2000) 261.
[30] W. Wang, Z. Liu, B. Hu, Phys. Rev. E 55 (1997) 2353.
[31] R.L. Viana, C. Grebogi, S.E. de S. Pinto, S.R. Lopes, A.M. Batista, J. Kurths, Phys. Rev. E 68 (2003) 067204;
R.L. Viana, C. Grebogi, S.E. de S. Pinto, S.R. Lopes, A.M. Batista, J. Kurths, Physica D 206 (2005) 94.
[32] A.M. dos Santos, C.F. Woellner, S.R. Lopes, A.M. Batista, R.L. Viana, Chaos, Solit. Fract. 32 (2007) 702.