Rotas para o Caos
Via Intermitência

Iberê L. Caldas

Sumário

Dois exemplos comuns de rotas para o caos

I - Intermitência do tipo 1

II - Dobramento de período

III - Surgimento de Órbitas Periódicas
no Mapa Logístico

IV – Universalidade (dobramento de períodos)

I – Rota para o Caos: Intermitência do Tipo 1

Ruelle, D. & Takens, F. 1971. On the nature of turbulence. *Communications in Mathematical Physics*, 20: 167–192

Exemplo a seguir:

Mapa unidimensional

$$u' = u + \varepsilon + u^2$$

ε: parâmetro de controle

We consider the instability of a Poincaré map due to the crossing of the unit circle at (+1) by an eigenvalue of the Floquet matrix.

This corresponds to the specific case of *Type I intermittency*.

Let u be the coordinate in the plane of the Poincaré section that points in the direction of the eigenvector whose eigenvalue λ crosses $+1$.

The lowest-order approximation of the 1-D map constructed along this line is

$$u' = \lambda(r)u. \tag{39}$$

Taking $\lambda(r_i) = 1$ at the intermittency threshold, we have

$$u' = \lambda(r_i)u = u. \tag{40}$$

Origem do Mapa

We consider this to be the leading term of a Taylor series expansion of $u'(u,r)$ in the neighborhood of $u = 0$ and $r = r_i$.

Expand to first order in $(r - r_i)$ and second order in u:

$$u'(u,r) \simeq u'(0,r_i) + u \cdot \left.\frac{\partial u'}{\partial u}\right|_{0,r_i} + \frac{1}{2}u^2 \cdot \left.\frac{\partial^2 u'}{\partial u^2}\right|_{0,r_i} + (r - r_i) \left.\frac{\partial u'}{\partial r}\right|_{0,r_i}$$

Evaluating equation (39), we find that the first term vanishes:

$$u'(u = 0, r = r_i) = 0.$$

$$\left.\frac{\partial u'}{\partial u}\right|_{0,r_i} = \lambda(r_i) = 1.$$

Finally, rescale u such that

$$\frac{1}{2}\frac{\partial^2 u'}{\partial u^2}\bigg|_{0,r_i} = 1$$

and set

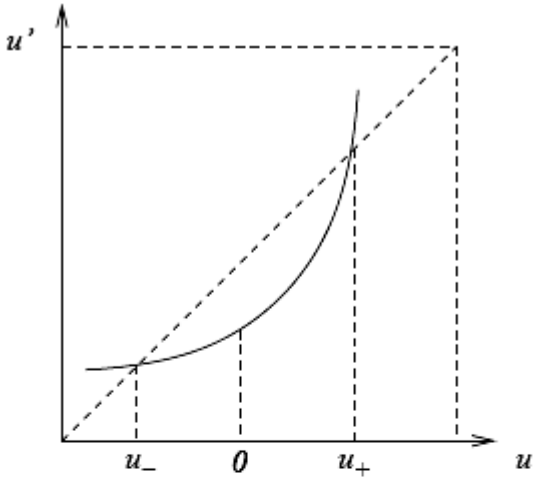
$$\varepsilon \propto (r - r_i).$$

The model now reads

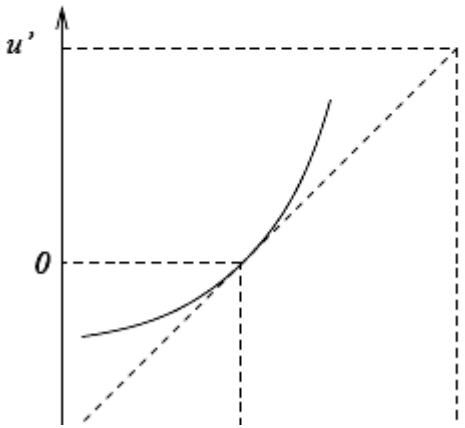
$$u' = u + \varepsilon + u^2,$$

where ε is now the control parameter.

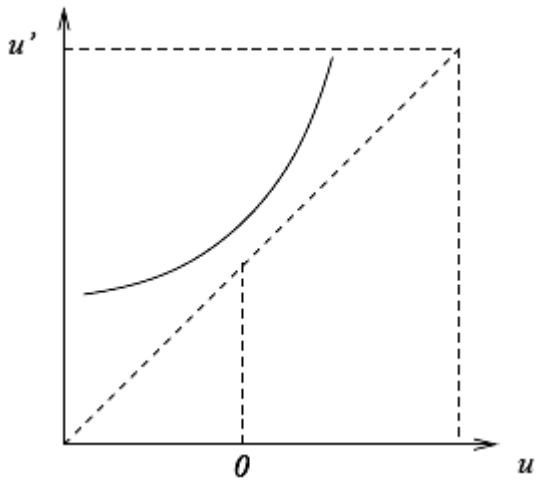
$$u' = u + \varepsilon + u^2$$



- $\varepsilon < 0$, i.e. $r < r_i$.
- u_- is stable fixed point.
- u_+ is unstable.



- $\varepsilon = 0$, i.e. $r = r_i$.
- u' is tangent to indentity map.
- $u_- = u_+ = 0$ is marginally stable.

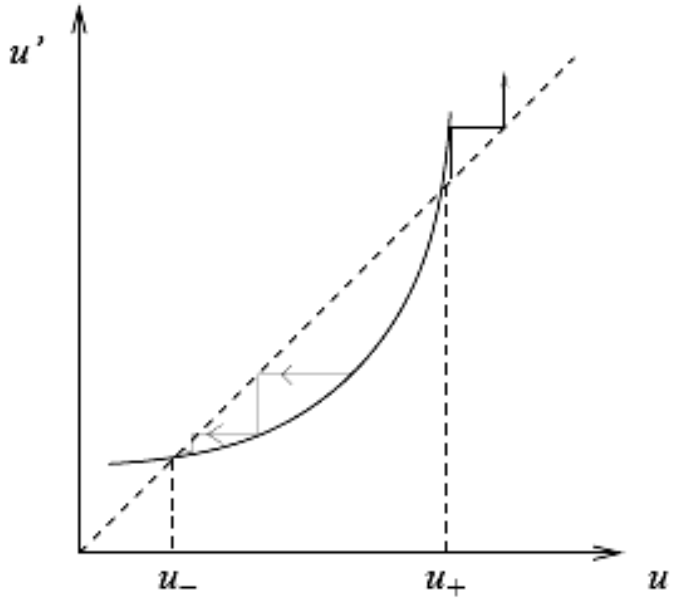


- $\varepsilon > 0$, i.e. $r > r_i$.
- no fixed points.

$$u' = u + \varepsilon + u^2$$

For $\varepsilon < 0$, the iterations look like

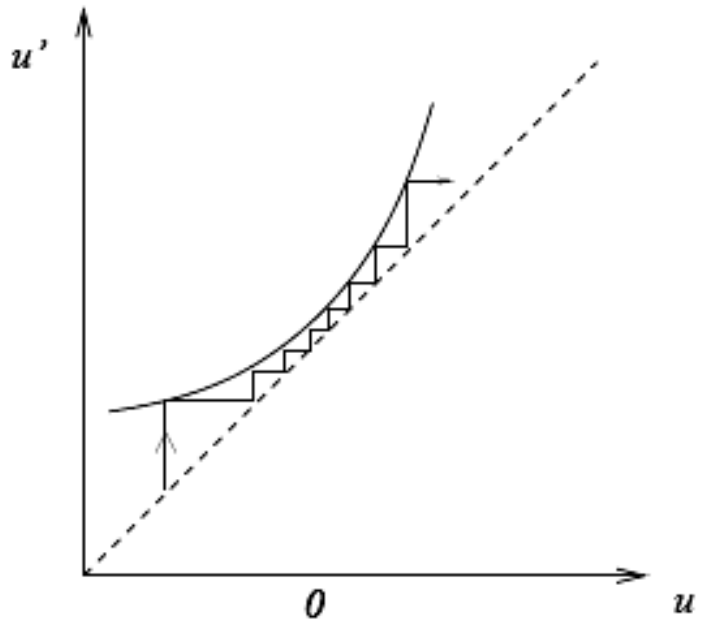
- u_- is an attractor for initial conditions $u < u_+$.
- For initial conditions $u > u_+$, the iterations diverge.



$u' = u + \varepsilon + u^2$

The situation changes for $\varepsilon > 0$, i.e. $r > r_i$:

- No fixed points.
- Iterations beginning at $u < 0$ drift towards $u > 0$.

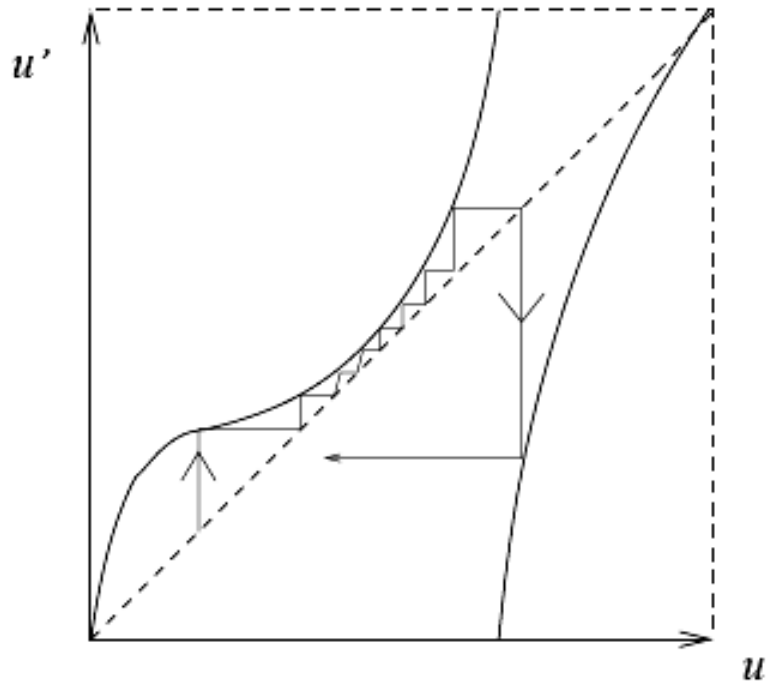


However, when $\varepsilon > 0$, there is no fixed point, and thus no periodic solution.

The iterations eventually run away and become unstable—this is the *intermittent* burst of noise.

How does the laminar phase begin again, or "relaminarize"?

Qualitatively, the picture can look like



behavior is called intermittency by Pomeau & Manneville (1980), who were the first to describe the scaling of the time spent in the laminar phase. They looked at the average time T_A spent by solutions in the laminar phase as a function of the parameter r close to the value r_{sn} at which the saddlenode bifurcation occurs. A simple argument based on the passage time of a trajectory of a map close to a tangency with the diagonal (the condition for the saddlenode bifurcation) establishes that the average time in the laminar phase diverges as a power law:

$$T_A \sim |r - r_{sn}|^{-\frac{1}{2}} . \tag{6}$$

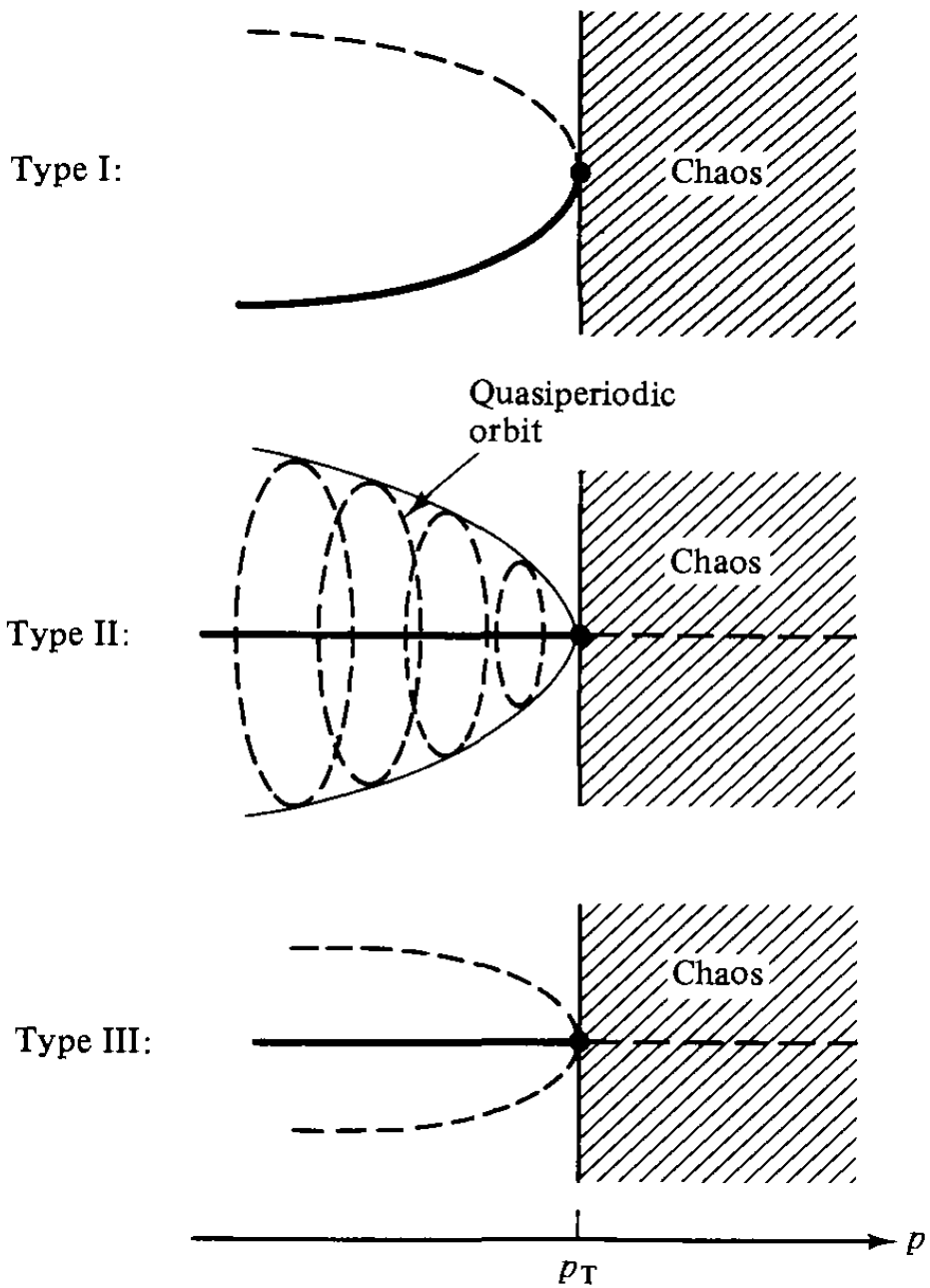
Note that the precise timing of the turbulent burst is unpredictable.

The discontinuity is *not* inconsistent with the presumed continuity of the underlying equations of motion—this is a map, not a flow.

Moreover the Lorenz map itself contains a discontinuity, corresponding to the location of the unstable fixed point.

Rotas Para o Caios Via Intermitência

Figure 8.6 Schematic illustration of the three types of intermittency transitions to chaos.



Permanência Média no Regime Laminar

$$\bar{T}(p) \sim \begin{cases} (p - p_{\mathrm{T}})^{-1/2} & \text{for Type I,} \\ (p - p_{\mathrm{T}})^{-1} & \text{for Type II,} \\ (p - p_{\mathrm{T}})^{-1} & \text{for Type III.} \end{cases}$$

II – Rota para o Caos: Dobramento de Períodos

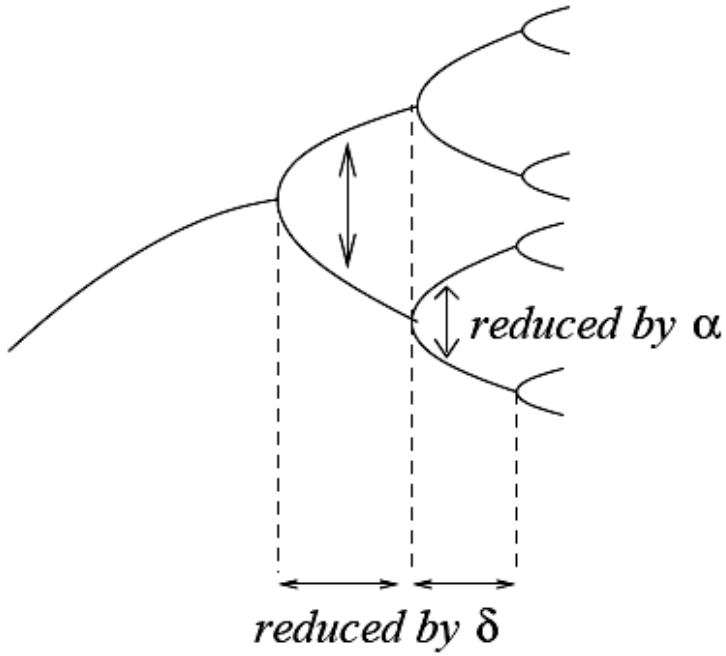
Feigenbaum, M.J. 1978. Quantitative universality for a class of nonlinear transformations. *Journal of Statistical Physics*, 19: 25–52

$$x_{n+1} = r x_n (1 - x_n)$$

As r increases, the periodic orbit of period 2^n is created from the orbit of period 2^{n-1} by a period-doubling bifurcation. If this bifurcation occurs with $r = r_n$ then $r_n \to r_c$ geometrically as $n \to \infty$, with

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \delta \approx 4.66920 \qquad \text{i.e.} \qquad r_n \sim r_c - \kappa\delta^{-n}$$

$$r_c \approx 3.569946,$$



These *quantitative* results hold if a *qualitative* condition—the maximum of f must be locally quadratic—holds.

That is, each increment in μ from one doubling to the next is reduced in size by a factor of $1/\delta$, such that

$$\delta_n = \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} \to \delta \quad \text{for large } n.$$

The truly amazing result, however, is not the scaling law itself, but that

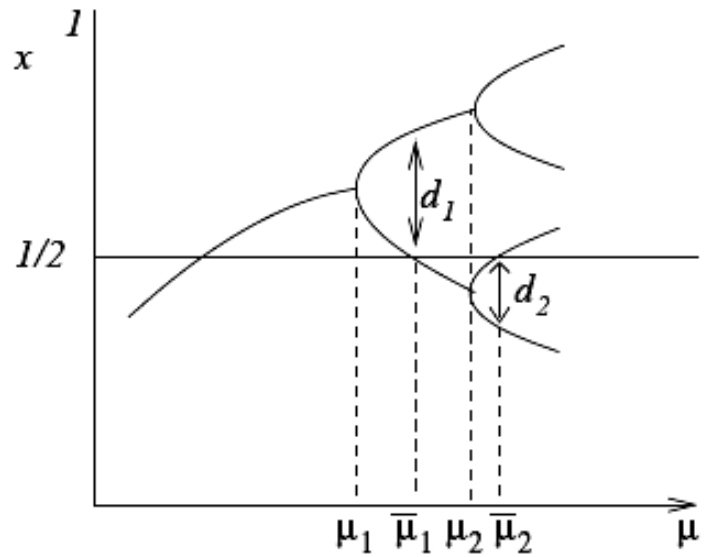
$$\boxed{\delta = 4.669\ldots}$$

is **universal**, valid for *any* unimodal map with quadratic maximum.

The quantitative universality in parameter space described by the scaling δ has a counterpart in phase space. If x_n denotes the point on the periodic orbit of period 2^{n-1} that is closest to the critical point (or turning point) of the map with $r = r_n$, then

$$\lim_{n\to\infty} \frac{x_{n+1} - \frac{1}{2}}{x_n - \frac{1}{2}} = -\alpha \tag{3}$$

where α is another universal constant, which, for maps with a quadratic turning point, takes the value $\alpha \approx 0.3995 = 1/2.50\ldots$.



Define d_n = distance from $x = 1/2$ to nearest value of x that appears in the superstable 2^n cycle (for $\mu = \bar{\mu}_n$).

From one doubling to the next, this separation is reduced by the same scale factor:

$$\frac{d_n}{d_{n+1}} \simeq -\alpha.$$

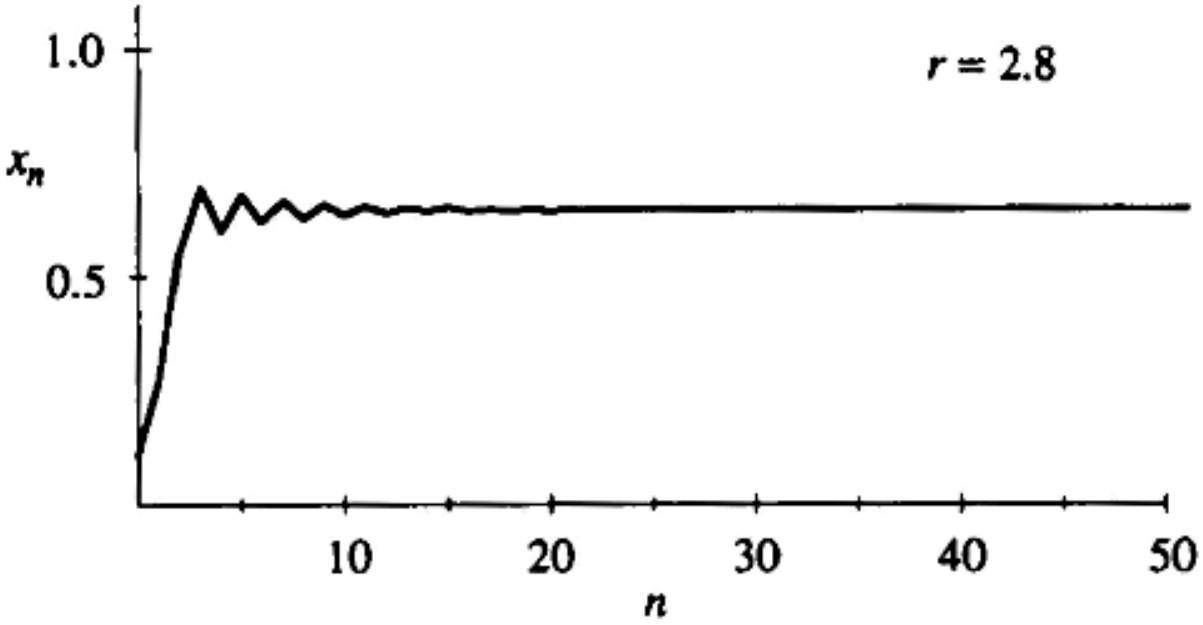
The negative sign arises because the adjacent fixed point is alternately greater than and less than $x = 1/2$.

We shall see that α is also universal:

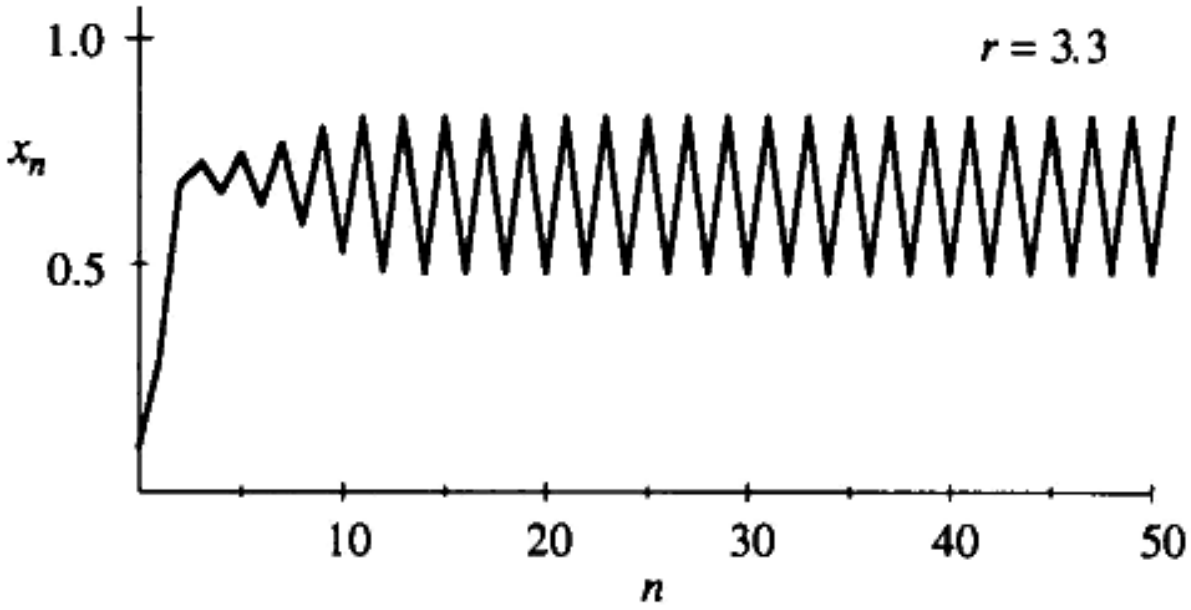
$$\alpha = 2.502\ldots$$

III - Surgimento de Órbitas Periódicas no Mapa Logístico

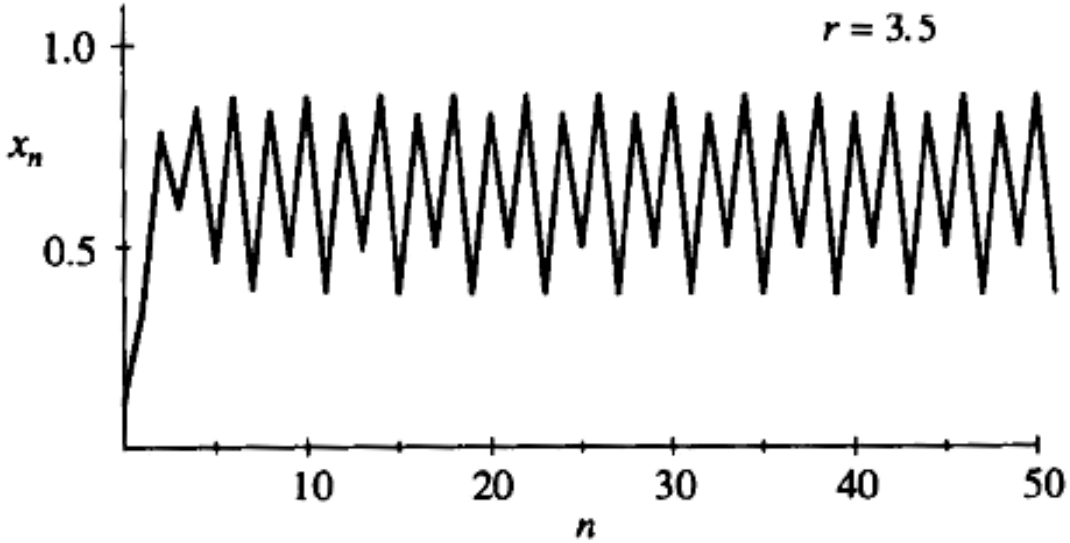
For $1 < r < 3, x_n$ grows as n increases, reaching a non-zero steady state.



For larger r (e.g. $r = 3.3$) x_n eventually oscillates about the former steady state $\Rightarrow$ *period-2 cycle*.



At still larger r (e.g. $r = 3.5$), x_n approaches a cycle which repeats every 4 generations $\Rightarrow$ *period-4 cycle*.



Further *period doublings* to cycles of period 8, 16, 32... occur as r increases. Computer experiments show that

$$
\begin{aligned}
r_1 &= 3 \quad \text{(period 2 is born)} \\
r_2 &= 3.449... \quad \text{(period 4 is born)} \\
r_3 &= 3.54409... \quad \text{(period 8 is born)} \\
r_4 &= 3.5644... \quad \text{(period 16 is born)} \\
&\;\;\vdots \\
r_\infty &= 3.569946... \quad \text{(period } \infty \text{ is born)}
\end{aligned}
$$

The successive bifurcations come faster and faster as r increases.

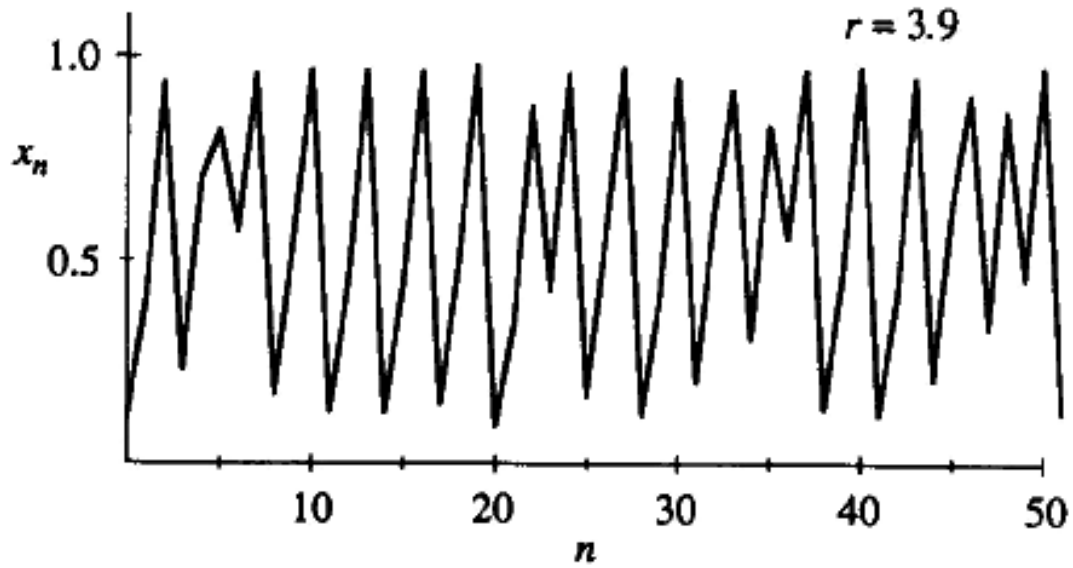
The r_n converge to a limiting value r_∞.

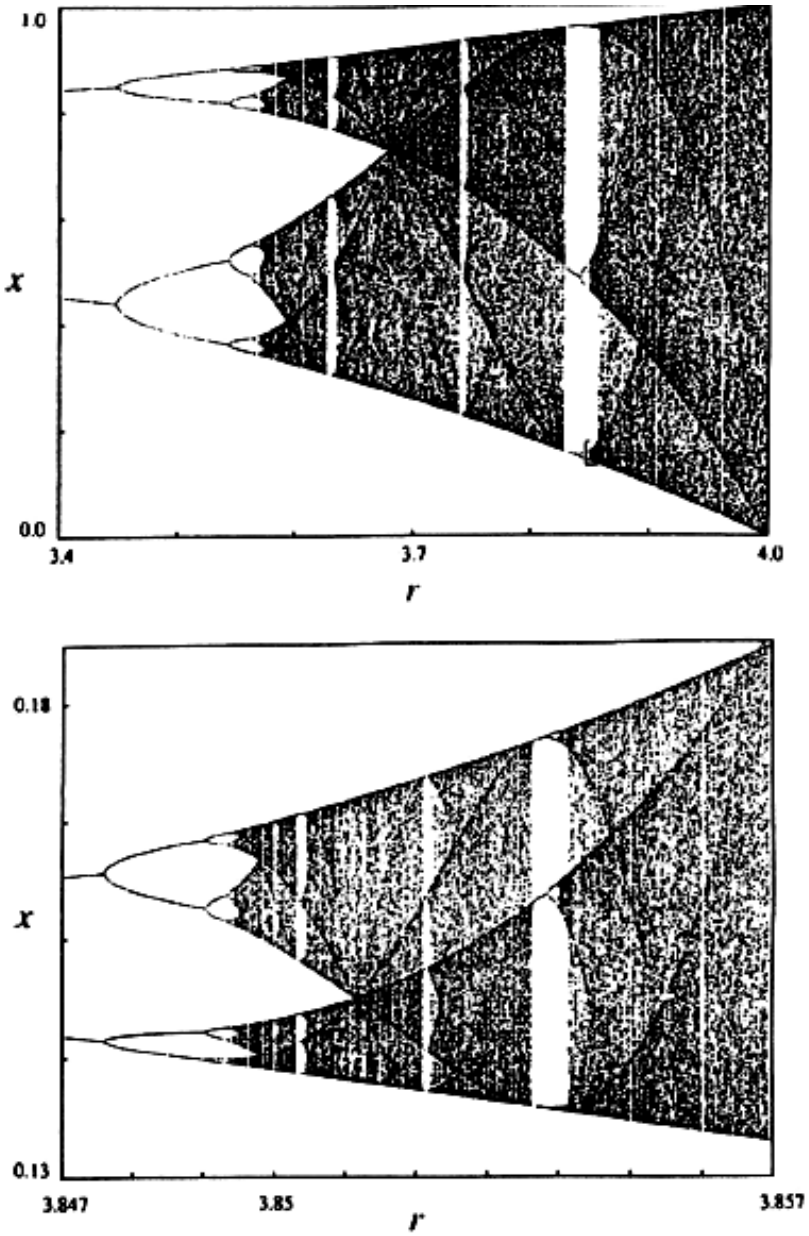
For large n, the distance between successive transitions shrinks by a constant factor

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

Chaos and periodic windows

What happens for $r > r_\infty$? The answer is complicated! For many values of r, the sequence $\{x_n\}$ never settles down to a fixed point or a periodic orbit - the long term behaviour is *aperiodic*.





blown-up
version around
$r = 3.85$

At $r = 3.4$ the attractor is a *period-2 cycle*.

As r increases, both branches split, giving a *period-4 cycle* - i.e. a period-doubling bifurcation has occurred.

A cascade of further period-doublings occurs as r increases, until at $r = r_\infty \simeq 3.57$, the *map becomes chaotic* and the attractor changes from a finite to an infinite set of points.

For $r > r_\infty$, the orbit reveals a mixture of order and chaos, with periodic windows interspersed with chaotic clouds of dots.

The large window near $r \simeq 3.83$ contains a stable period-3 cycle. A blow-up of part of this window shows that a *copy* of the orbit diagram *reappears in miniature!*

Logistic Map: analysis

Consider $x_{n+1} = rx_n(1 - x_n)$; $\quad 0 \leq x_n \leq 1$
and $0 \leq r \leq 4$.

Fixed points $x^* = f(x^*) = rx^*(1 - x^*)$
$\Rightarrow x^* = 0$ or $1 - 1/r$.

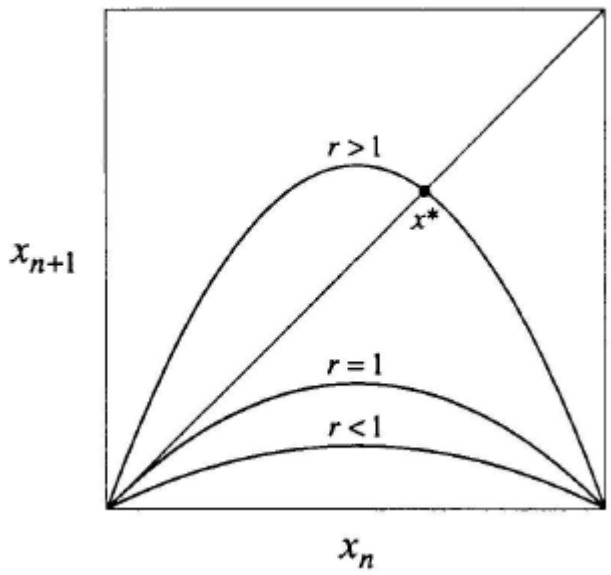
- $x^* = 0$ is a fixed point for all r

- $x^* = 1 - 1/r$ is a fixed point only if $r \geq 1$ (recall $0 \leq x_n \leq 1$).

Stability depends on $f'(x^*) = r - 2rx^*$

$x^* = 0$ is stable for $r < 1$ and unstable for $r > 1$.

$x^* = 1 - 1/r$ is stable for $-1 < (2 - r) < 1$, i.e. for $1 < r < 3$, and unstable for $r > 3$.

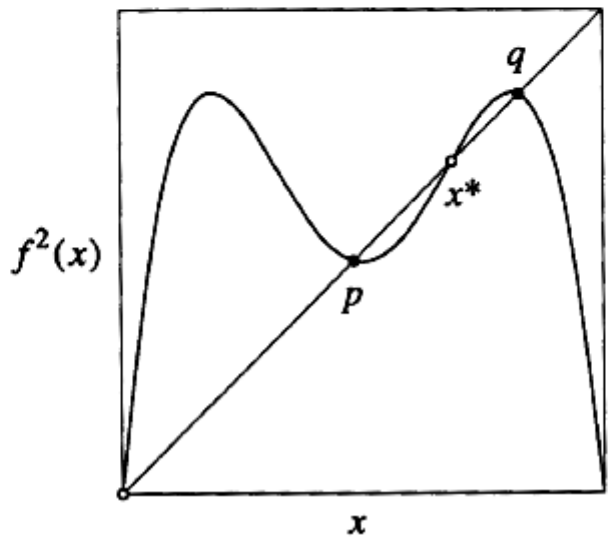


At $r = 1$, x^* bifurcates from the origin in a *transcritical bifurcation*.

As r increases beyond 1, the slope at x^* gets increasingly steep. The critical slope $f'(x^*) = -1$ is attained when $r = 3$ - the resulting bifurcation is called a *flip bifurcation* $\Rightarrow$ 2-cycle!

We now go on to show that the logistic map has a 2-cycle for all $r > 3 \ldots$

A 2-cycle exists if and only if there are two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently, such a p must satisfy $f(f(p)) = p$ where $f(x) = rx(1-x)$. Hence, p is a fixed point of the *second iterate map* $f^2(x) = f(f(x))$. Since $f(x)$ is a quadratic map, $f^2(x)$ is a quartic polynomial. Its graph for $r > 3$ is...



We must now solve $f^2(x) = x$.

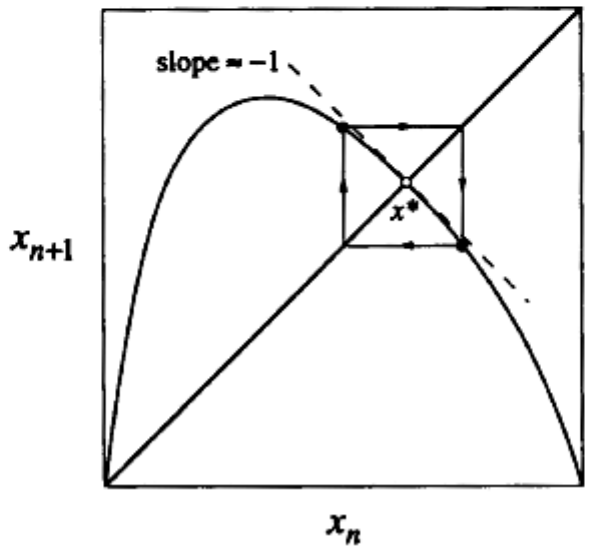
$x^* = 0$ and $x^* = 1 - 1/r$ are trivial solutions.
The other 2 solutions are

$$p, q = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r},$$

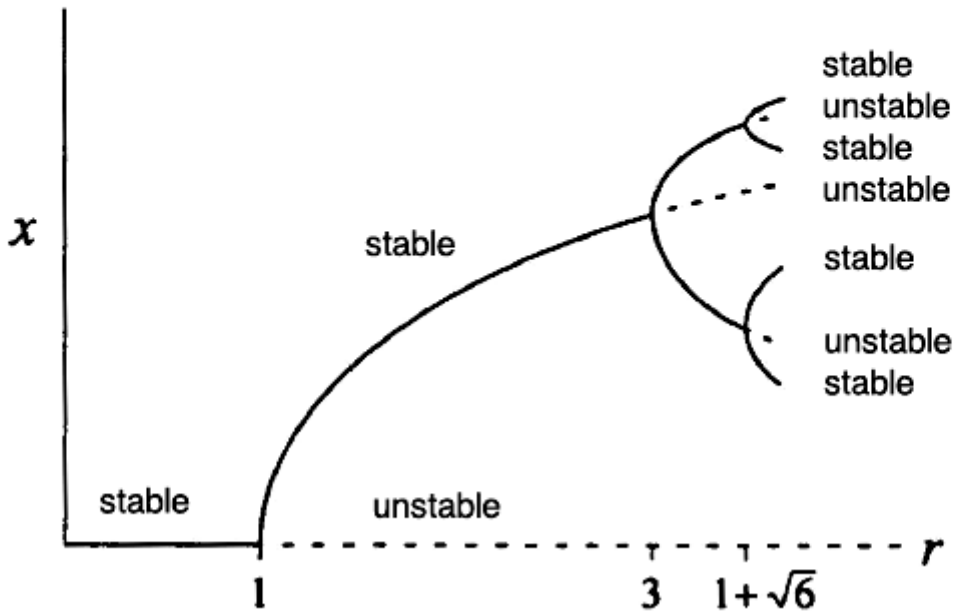
wich are *real* for $r > 3$.

Hence *a 2-cycle exists for all $r > 3$ as claimed!*

A *cobweb diagram* reveals how *flip bifurcations* can give rise to *period-doubling*. Consider any map f, and look at the local picture near a fixed point where $f'(x^*) \simeq -1 \ldots$



A partial bifurcation diagram for the logistic map based mainly on the results so far look like ...

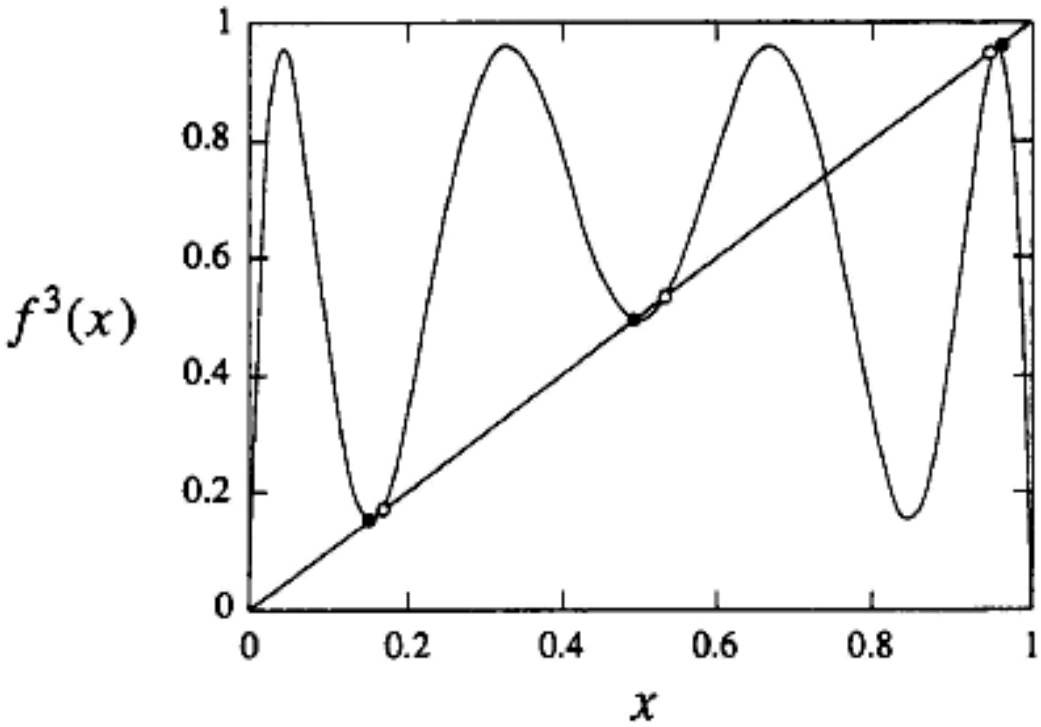


Periodic Windows

We now consider periodic windows for $r > r_\infty$ e.g. the period-3 window that occurs near $3.8284 \leq r \leq 3.8415$ [the same mechanism will account for the creation of all other similar windows!]

Let $f(x) = rx(1-x)$ so that the logistic map is $x_{n+1} = f(x_n)$. Then $x_{n+2} = f(f(x_n))$ or more simply, $x_{n+2} = f^2(x_n)$. Similarly $x_{n+3} = f^3(x_n)$. This *third-iterate map* is the key to understanding the birth of the period-3 cycle.

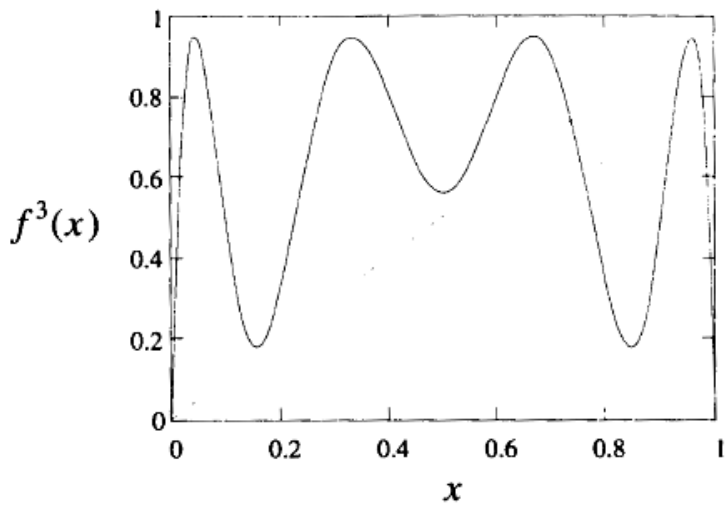
Any point p in a period-3 cycle repeats every three iterates, so such points satisfy $p = f^3(p)$. Consider $f^3(x)$ for $r = 3.835\ldots$



Of the 8 solutions, two are period-1 points for which $f(x^*) = x^*$. The other six are shown on Fig. 7.4.1.

Black dots are *stable* period-3 cycles
Open dots are *unstable* period-3 cycles

Now suppose we decrease r towards the chaotic regime...
Consider $r = 3.8\ldots$



The 6 solutions have vanished! [only the 2 period-1 points are left]

Hence for some r where $3.8 < r < 3.835$ the graph of $f^3(x)$ must have become a tangent to the diagonal $\Rightarrow$ stable and unstable period-3 cycles coalesce and annihilate in a *tangent bifurcation*. This transition defines the beginning of the periodic window.

Intermittency

For r just below the period-3 window one finds...

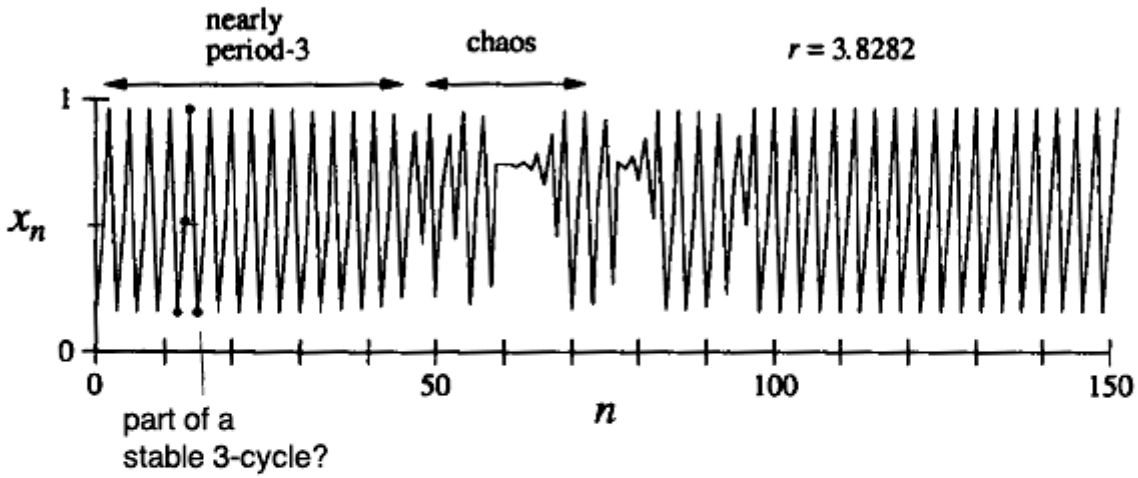
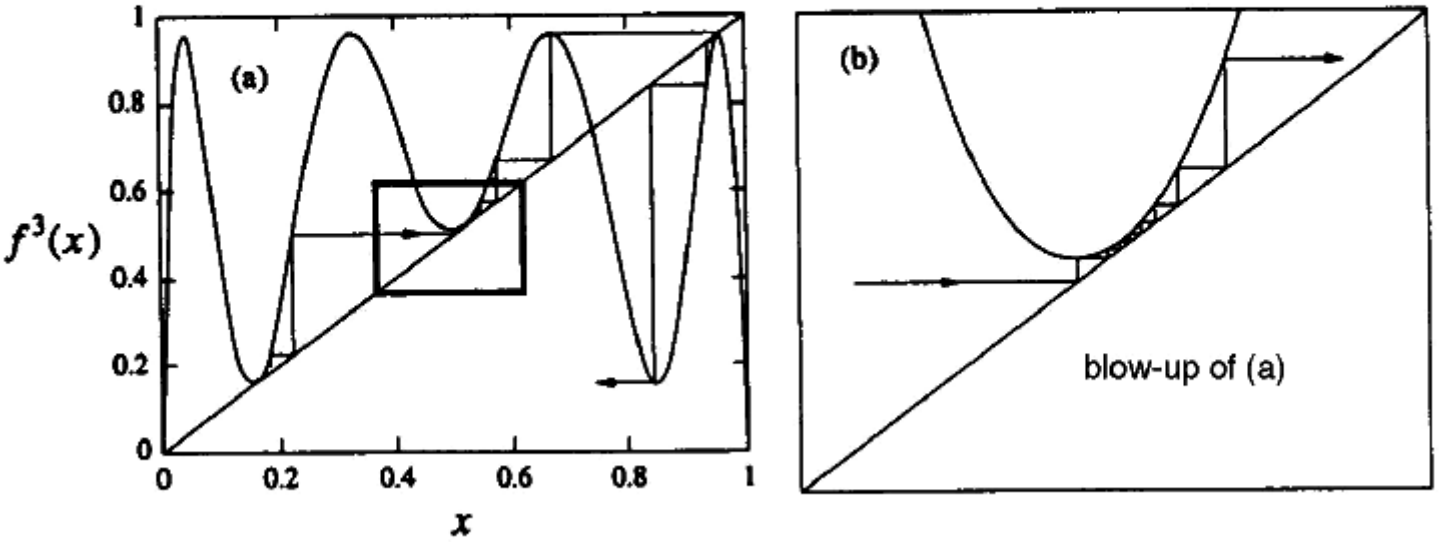


Fig. 7.4.3

where black dots indicate part of the orbit which looks like a stable 3-cycle. This is spooky, since the 3-cycle no longer exists...? We are seeing the "ghost" of the 3-cycle... since the tangent bifurcation is essentially just a *saddle-node bifurcation*.

The new feature is that we have intermittent behaviour of nearly period-3 → chaos → nearly period-3 because...



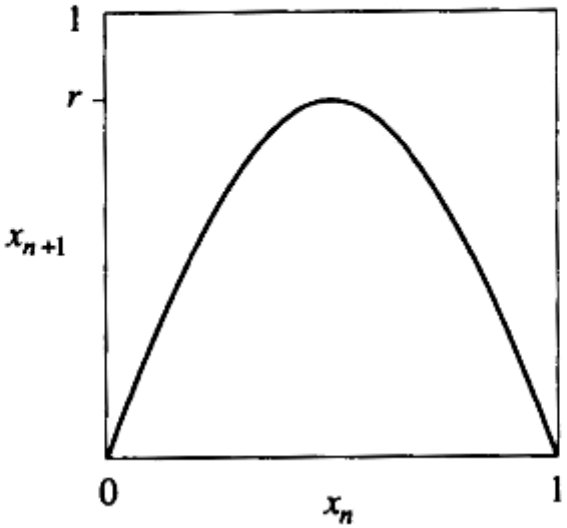
Such intermittency is fairly common. The time between irregular bursts in experimental systems is statistically distributed, much like a random variable, even though the system is completely deterministic! As the control parameter is moved further away from the periodic window, the irregular bursts become more frequent until the system is fully chaotic. This progression is known as *the intermittency route to chaos*.

Period-doubling in the window

Recall Fig. 7.2.7 where a copy of the orbit diagram appears in miniature in the period-3 window. The same mechanism operates here as in the original period-doubling cascade, but now produces orbits of period $3 \cdot 2^n$. A similar period-doubling cascade can be found in *all* of the periodic windows.

IV - Universalidade
(Dobramento de Períodos)

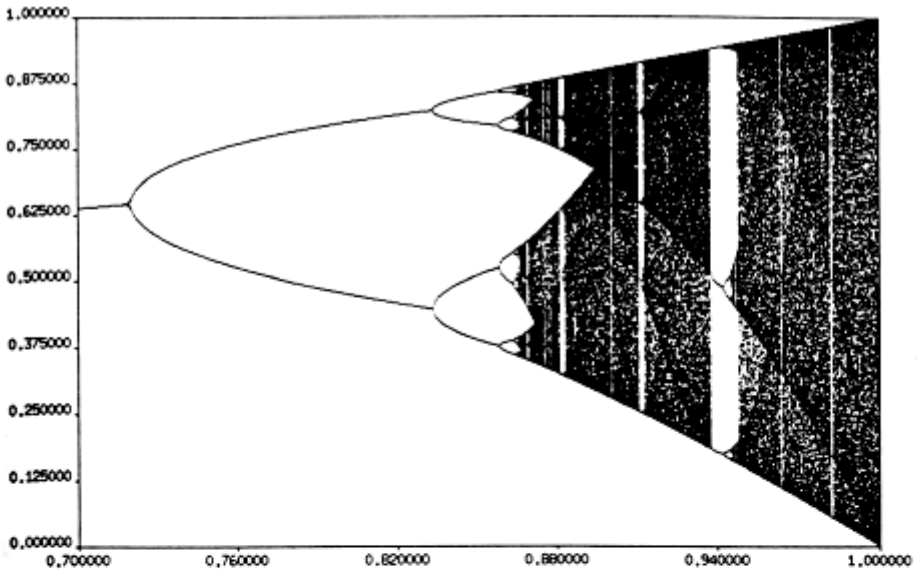
Consider the *sine map* $x_{n+1} = r \sin \pi x_n$ for $0 \leq r \leq 1$ and $0 \leq x \leq 1$.



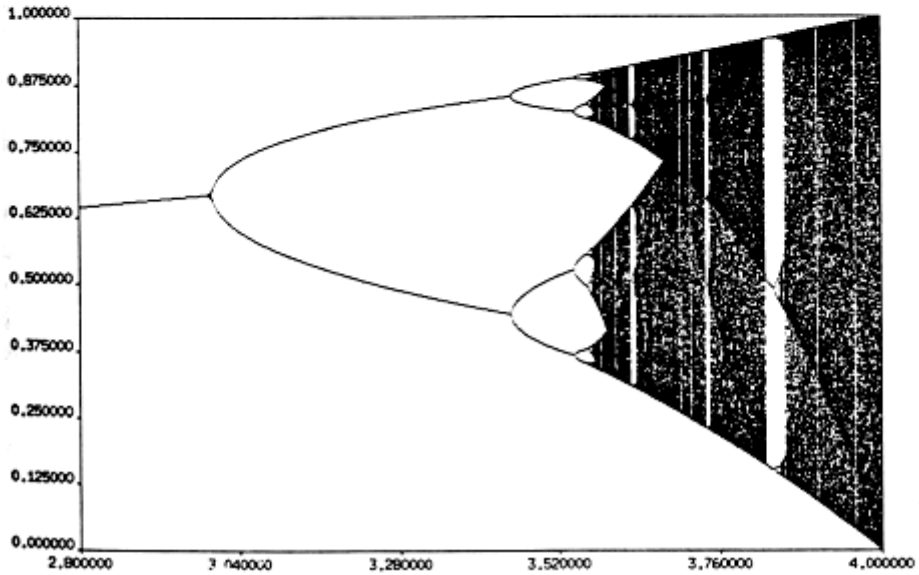
It has qualitatively the same shape as the logistic map - such maps are called *unimodal*.

We now compare the orbit diagrams for the sine map and the logistic map...

the resemblance is quite amazing...



sine ma



logistic map

The *qualitative* dynamics of the two maps are identical! Metropolis (1973) proved that all unimodal maps have periodic attractors (i.e. stable periodic solutions) occurring in the same sequence. This implies that the algebraic form of the map $f(x)$ is irrelevant - only its overall shape matters!

There is an even more amazing *quantitative* universality in 1-dimensional maps...

In 1975, Mitch Feigenbaum was trying to develop a theory to predict r_n, the value of r where a 2^n-cycle first appears. He found that, no matter what unimodal map is iterated, the same convergence rate appears!

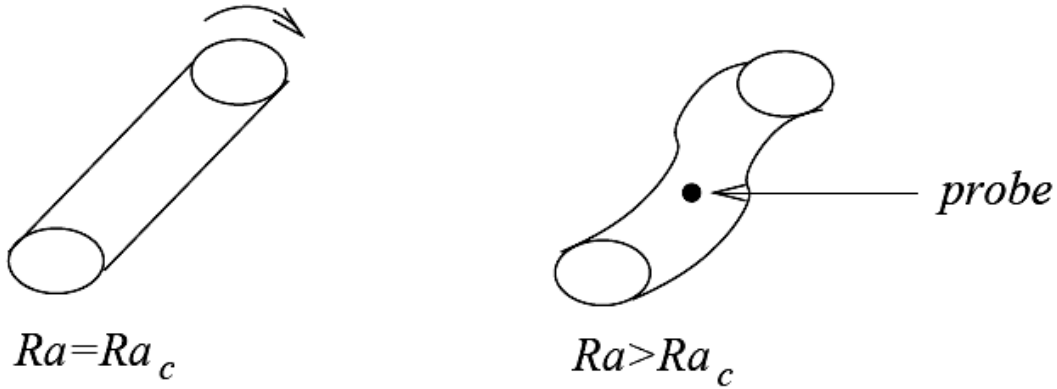
$$\text{i.e. } \delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

is universal! It is a new mathematical constant, as basic to period-doubling as π is to circles.

At first glance this result may appear to pertain only to mathematical maps. However we have seen that more complicated systems can also behave as if they depend on only a few degrees of freedom. Due to dissipation, one may expect that a one-dimensional map is contained, so to speak, within them.

The first experimental verification of this idea was due to Libchaber, in a Rayleigh-Bénard system.

As the Rayleigh number increases beyond its critical value, a single convection roll develops an oscillatory wave:



A probe of temperature $X(t)$ is then oscillatory with frequency f_1 and period $1/f_1$.

Successive increases of Ra then yield a sequence of period doubling bifurcations at Rayleigh numbers

$$\mathrm{Ra}_1 < \mathrm{Ra}_2 < \mathrm{Ra}_3 < \ldots$$

Identifying Ra with the control parameter μ in Feigenbaum's theory, Libchaber found

$$\delta \simeq 4.4$$

which is amazingly close to Feigenbaum's prediction, $\delta = 4.669\ldots$.

He explained why δ is universal, based on the idea of *renormalization* from statistical physics. He thereby found an analogy between δ and the universal exponents observed in experiments on *second-order phase transitions* in magnets, fluids and other physical systems. This has been confirmed in experiments...

What do 1-D maps have to do with science?

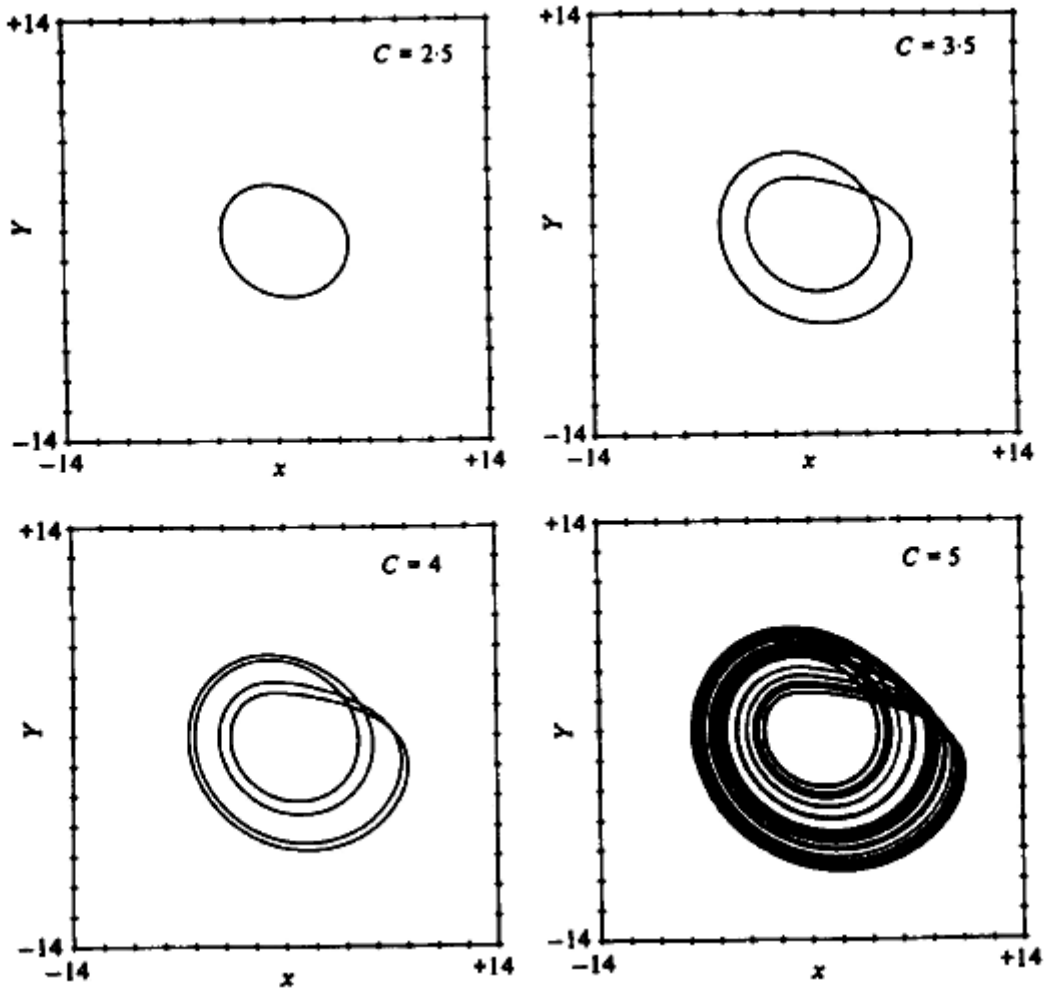
Real systems often have tremendously many degrees of freedom. How can all that complexity be captured by a 1-dimensional map? To try and answer this, we start by considering the so-called *Rössler model*...

The *Rössler model* is a set of 3 differential equations designed to exhibit the *simplest possible strange attractor*

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned}$$

where a, b, c are parameters. The term zx is the only nonlinear term (recall that Lorenz has two!)

We consider the Rössler system with $a = b =$ 0.2 held fixed, and vary c


C = 2·5
C = 3·5
C = 4
C = 5
+14
−14
x
y

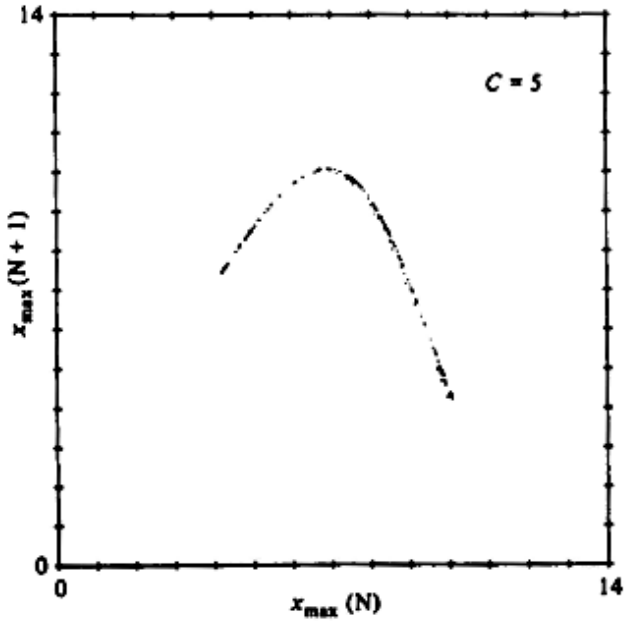
$c = 2.5$ - attractor is a simple limit cycle

$c = 3.5$ - period-doubling in a continuous-time system! Hence, a *period-doubling bifurcation of cycles* must have occurred somewhere between 2.5 and 3.5

$c = 4$ - another period-doubling bifurcation creates the 4-loop shown at $c = 4$

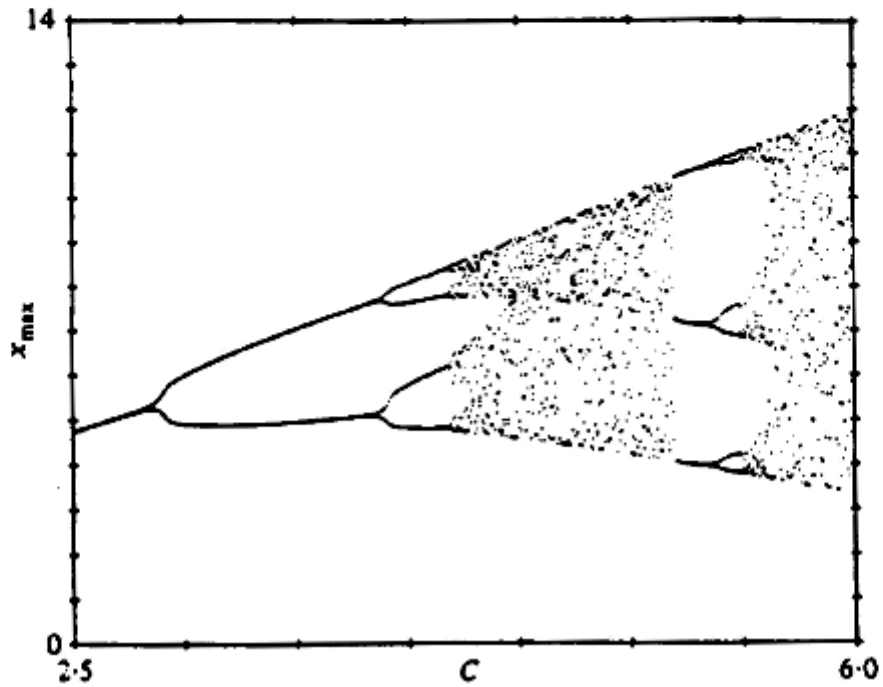
$c = 5$ - after an infinite cascade of further period-doublings, one obtains the strange attractor shown at $c = 5$.

To compare these results to those for 1-dimensional maps, we use Lorenz's trick for obtaining a map from a flow (see Lecture 6). For a given c, record successive local maxima of $x(t)$ for a trajectory on the strange attractor. Then plot x_{n+1} vs x_n where x_n denotes the nth local maximum.



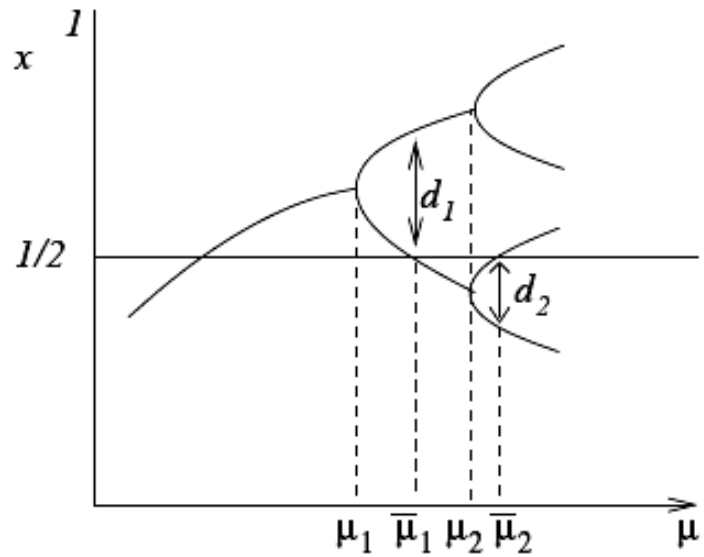
Data points fall very nearly onto a 1-D curve - note uncanny resemblance to the (unimodal) logistic map!

To compute an orbit diagram for the Rössler model, we allow c to vary. Above each c we plot *all* the local maxima x_n on the attractor for that value of c. The number of different maxima tells us the "period" of the attractor...



Now we can see why *certain* physical systems are governed by Feigenbaum's universality theory - *if the system's Lorenz map is nearly one-dimensional and unimodal, then the theory applies!*

For the Lorenz map to be almost 1-dimensional, the strange attractor has to be *very flat* i.e. only slightly more than 2-dimensional. This requires the system to be *highly dissipative*; only 2 or 3 degrees of freedom are truly active - the rest follow on slavishly.



Define d_n = distance from $x = 1/2$ to nearest value of x that appears in the superstable 2^n cycle (for $\mu = \bar{\mu}_n$).

From one doubling to the next, this separation is reduced by the same scale factor:

$$\frac{d_n}{d_{n+1}} \simeq -\alpha.$$

The negative sign arises because the adjacent fixed point is alternately greater than and less than $x = 1/2$.

We shall see that α is also universal:

$$\alpha = 2.502\ldots$$