Applications of Finite Time Lyapunov Exponents in Hamiltonian Systems

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Presentation Outline

- Hamiltonian Systems and Dynamical Traps
- Ergodic hypothesis ?
- Finite Time Lyapunov Exponents (FTLE)
- Non Twist Systems
- Lagragian Coherent Structures and FTLE

Introduction and Motivation

General characteristics of non-integrable Hamiltonian systems of 2 degrees of freedom

- The dynamics is neither entirely regular, nor entirely chaotic
- The boundaries between regular and irregular motion has a complicated surface
- Presence of sticks domains in phase space

Dynamics of standard map

The standard map represents the discrete form of the equations for the kicked rotor characterized by the Hamiltonian

$$H(p,x,t) = \frac{1}{2}p^2 - K\cos x \sum_{n=-\infty}^{\infty} \delta(t-n), \quad (1)$$

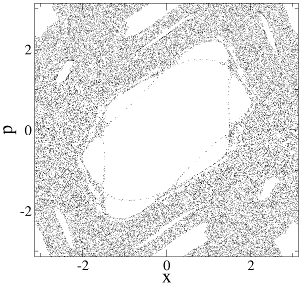
where p and x are the rotor angular momentum and positions, and K is the so-called non-linearity parameter.

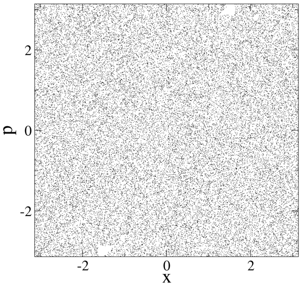
$$\begin{aligned} p_{n+1} &= p_n - K\sin x_n, \\ x_{n+1} &= x_n + p_{n+1}, \end{aligned} \quad (2)$$

Dynamics of standard map

Phase space for the standard map for

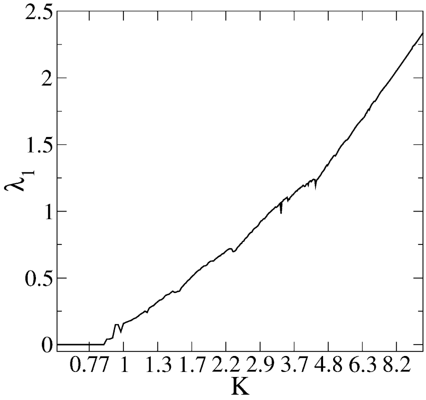
$K = 1.5$ $K = 6.0$





Infinite Time Lyapunov Exponent

Maximum Lyapunov exponent as function of parameter K

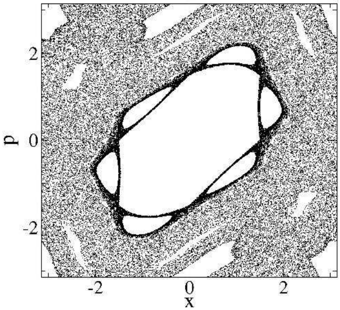


$$\lambda_{1,2} = \lim_{n\to\infty} \frac{1}{n} \ln(||D\mathcal{M}^n(p_0, x_0)\mathcal{U}_k||)$$

However...

The same previous case of $K = 1.5$ but for longer times of iteration.

Sticky regions



Ergodic hypothesis

The idea behind the ergodic hyphotesis is that trajectories in phase space Γ pass near every point in phase space if you wait long enough.

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How much is long enough?

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How much is long enough?

Could we compute the Lyapunov Exponents?

Finite-time Lyapunov Exponents (FTLE)

We define the k-th time-n Lyapunov exponent associated with the point (p_0, x_0) as

$$\lambda_k(p_0, x_0, n) = \frac{1}{n} \ln(||D\mathcal{M}^n(p_0, x_0)\mathcal{U}_k||) \ (k = 1, 2) \tag{3}$$

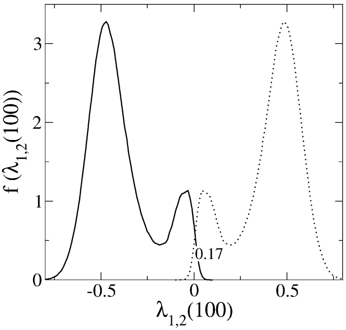
The FTLEs depend on the initial condition (p_0, x_0), whereas their infinite-time counterparts

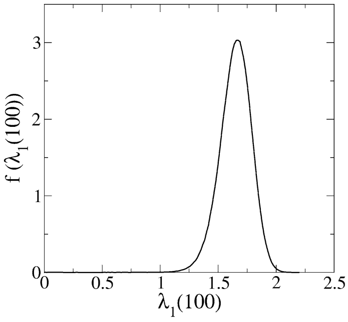
$$\lambda_k = \lim_{n \to \infty} \lambda_k(p_0, x_0, n) \tag{4}$$

Finite-time Lyapunov Exponents (FTLE)

Finite-time Lyapunov distribution for the standard map for:

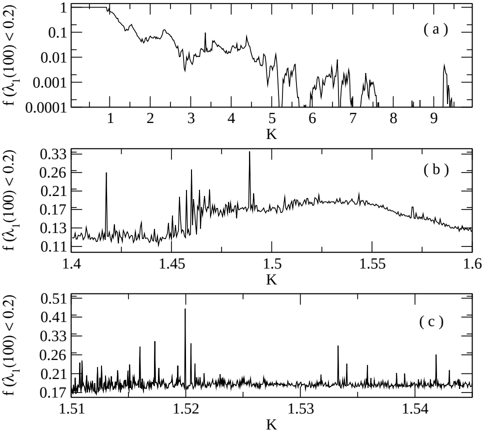
$K = 1.5$ $K = 6.0$





Total trapping time

Behavior of the finite-time Lyapunov exponent distribution as a function of the parameter K.



Standard Nontwist Map

Standard nontwist map

The equations of the so-called nontwist map

$$
\begin{aligned}
y_{n+1} &= y_n - b\sin(2\pi x_n), \\
x_{n+1} &= x_n + a(1 - y_{n+1}^2),
\end{aligned}
$$

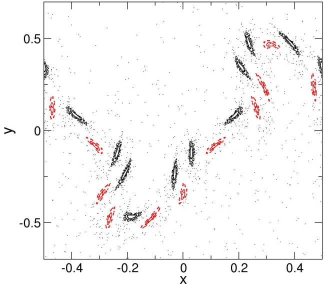
where $x \in [-1/2, +1/2)$ and $y \in \mathbb{R}$ the Hamiltonian with kicks corresponding to the map is

$$
H(x, y, n) = ay\left(1 - \frac{y^2}{3}\right) - \frac{b}{2\pi}\cos(2\pi x)\sum_{m=-\infty}^{\infty}\delta(n - m).
$$

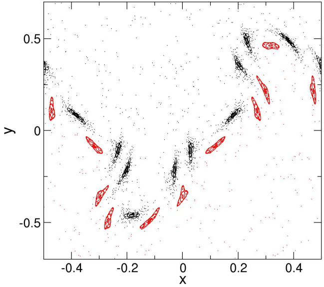
For all next results we will keep the parameter $b = 0.6$ and take a as our tunable parameter.

Phase Space for the two cases

$a = 0.80552$



$a = 0.80630$

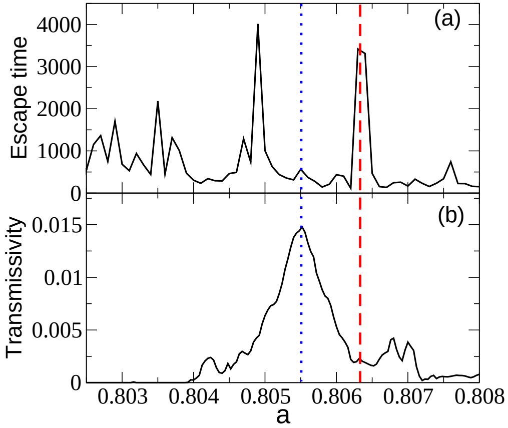


Mean escape time and transmissivity

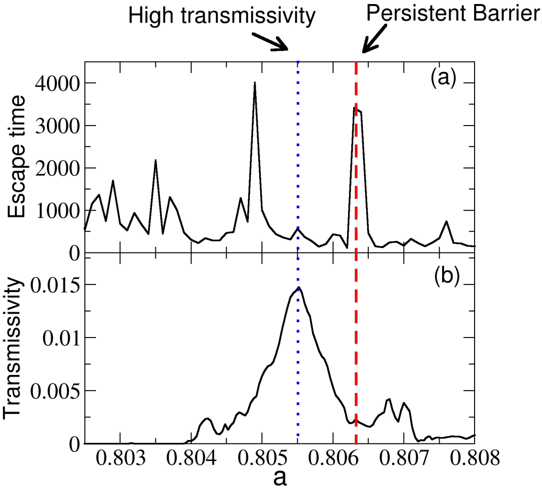
Escape time $\longrightarrow$ *average time of a square of initial conditions*

$$[-0.5, 0.5] \times [-0.9, 0.9] \textit{reach}\; y_b = \pm 2.0$$

Transmissivity $\longrightarrow$ *number of the trajectories that cross the barrier*

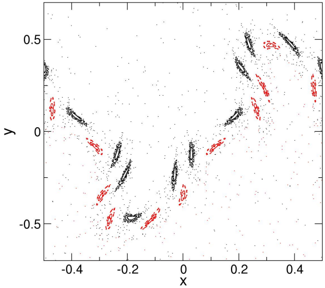


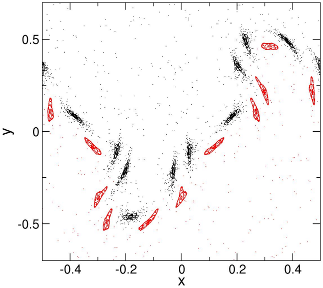
Mean escape time and transmissivity



Phase Space for the two cases

High transmissivity	Persistent Barrier
$a = 0.80552$	$a = 0.80630$

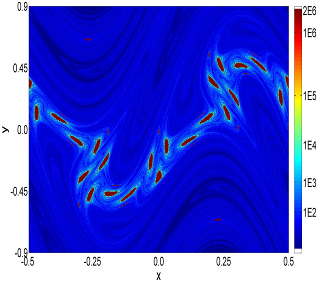




Time escape for the two cases

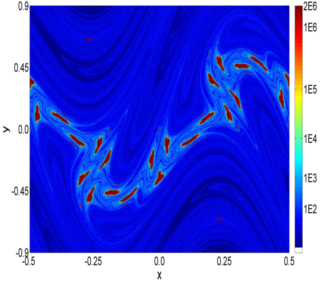
High transmissivity

$a = 0.80552$



Persistent Barrier

$a = 0.80630$



Time escape for the two cases

High transmissivity

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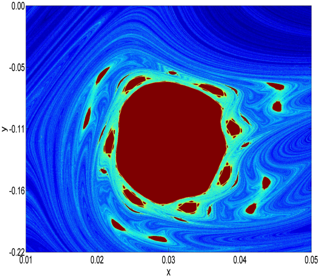

Persistent Barrier

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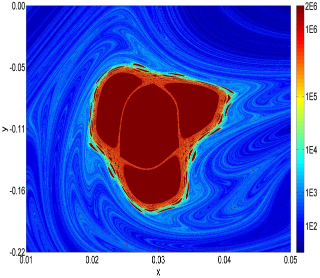
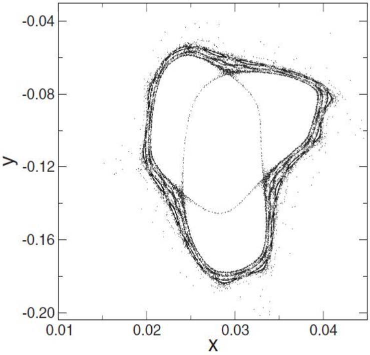
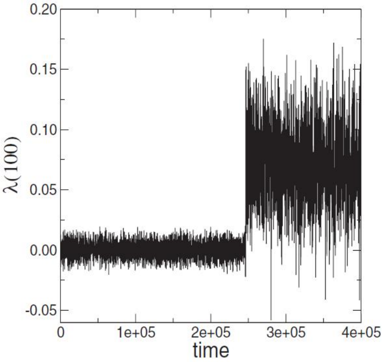


Comparison between a Sticky trajectory and FTLE

Persistent Barrier Case ($a = 0.8063$)

Space Phase and FTLE time series





Lagrangian Coherent Structures

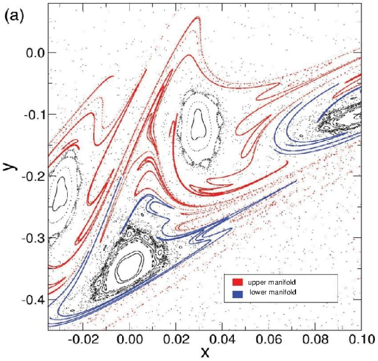
Lagrangian Coherent Structures (LCS)

- LCS are structures which separate dynamically distinct regions in systems.
- Finite-time Lyapunov Exponents can be used to find LCS, which are often analogous to stable and unstable manifolds of time-independent systems.
- These structures divide dynamically distinct regions in the phase space.

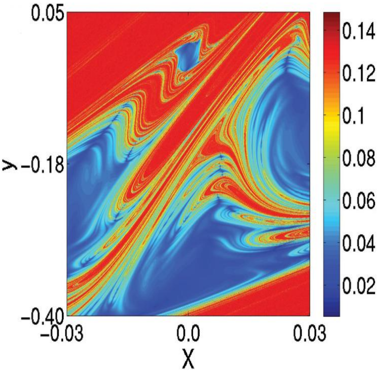
Comparison between manifolds and FTLE

High transmissivity case ($a = 0.80552$)

Manifolds



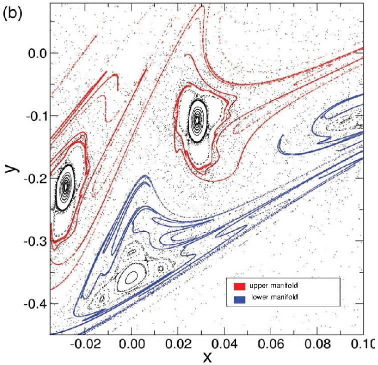
FTLE



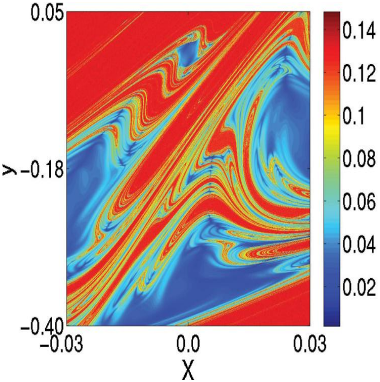
Comparison between manifolds and FTLE

Persistent Barrier Case ($a = 0.8063$)

Manifolds



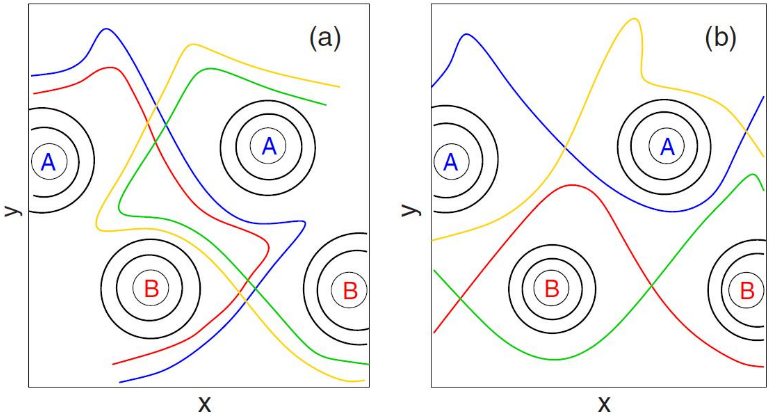
FTLE



Schematic depiction

Dominant Intercrossing

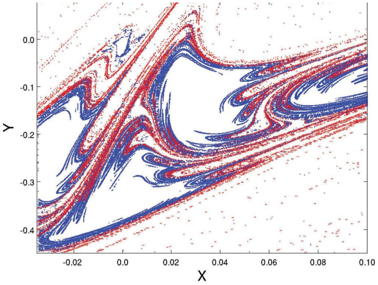
Dominant Intracrossing



Ridges of FTLE

- Marked in Blue - Ridges of FTLE.
- Marked in Red - The trajectories transmitted.

High Transmissivity
$a = 0.80552$



Persistent Barrier Case
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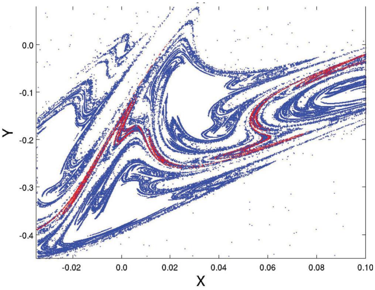

Conclusions

- FTLE distributions can be useful to analyze the influence of trapping domains.
- When we look to distribution of the orbits that suffer trapping effects, they develop a mode near zero in FTLE.
- Even after the barrier breakdown, the region of the shearless curve continues to reduce transport.
- Manifold intercrossings increase the transmissivity. Otherwise, the manifold intracrossings slow down the transmissivity and increase the average escape time.

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Thank You!