Applications of Finite Time Lyapunov Exponents in Hamiltonian Systems

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April, 2014
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Applications of FTLE in Hamiltonian Systems

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Presentation Outline

• Hamiltonian Systems and Dynamical Traps
• Ergodic hypothesis ?
• Finite Time Lyapunov Exponents (FTLE)
• Non Twist Systems
• Lagragian Coherent Structures and FTLE
Introduction and Motivation

General characteristics of non-integrable Hamiltonian systems of 2 degrees of freedom

- The dynamics is neither entirely regular, nor entirely chaotic
- The boundaries between regular and irregular motion has a complicated surface
- Presence of sticks domains in phase space
Dynamics of standard map

The standard map represents the discrete form of the equations for the kicked rotor characterized by the Hamiltonian

\[ H(p, x, t) = \frac{1}{2}p^2 - K \cos x \sum_{n=-\infty}^{\infty} \delta(t - n), \quad (1) \]

where \( p \) and \( x \) are the rotor angular momentum and positions, and \( K \) is the so-called non-linearity parameter.

\[ p_{n+1} = p_n - K \sin x_n, \]

\[ x_{n+1} = x_n + p_{n+1}, \quad (2) \]
Dynamics of standard map

Phase space for the standard map for

\[ K = 1.5 \quad \text{and} \quad K = 6.0 \]
Infinite Time Lyapunov Exponent

Maximum Lyapunov exponent as function of parameter $K$

$$\lambda_{1,2} = \lim_{n \to \infty} \frac{1}{n} \ln(||D M^n(p_0, x_0) U_k||)$$
The same previous case of $K = 1.5$ but for longer times of iteration.

Sticky regions
Ergodic hypothesis

The idea behind the ergodic hypothesis is that trajectories in phase space $\Gamma$ pass near every point in phase space if you wait long enough.
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How much is long enough?
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How much is long enough?

Could we compute the Lyapunov Exponents?
Finite-time Lyapunov Exponents (FTLE)

We define the $k$-th time-$n$ Lyapunov exponent associated with the point $(p_0, x_0)$ as

$$\lambda_k(p_0, x_0, n) = \frac{1}{n} \ln(||D\mathcal{M}^n(p_0, x_0)\mathcal{U}_k||) \quad (k = 1, 2) \quad (3)$$

The FTLEs depend on the initial condition $(p_0, x_0)$, whereas their infinite-time counterparts

$$\lambda_k = \lim_{n \to \infty} \lambda_k(p_0, x_0, n) \quad (4)$$
Finite-time Lyapunov Exponents (FTLE)

Finite-time Lyapunov distribution for the standard map for:

$K = 1.5$

$K = 6.0$
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Total trapping time

Behavior of the finite-time Lyapunov exponent distribution as a function of the parameter $K$. 

![Graphs showing the behavior of the finite-time Lyapunov exponent distribution as a function of $K$.](image)
Standard Nontwist Map
Standard nontwist map

The equations of the so-called nontwist map

\[ \begin{align*}
  y_{n+1} &= y_n - b \sin(2\pi x_n), \\
  x_{n+1} &= x_n + a(1 - y_{n+1}^2),
\end{align*} \]

where \( x \in [-1/2, +1/2) \) and \( y \in \mathbb{R} \) the Hamiltonian with kicks corresponding to the map is

\[ H(x, y, n) = ay \left(1 - \frac{y^2}{3}\right) - \frac{b}{2\pi} \cos(2\pi x) \sum_{m=-\infty}^{\infty} \delta(n - m). \]

For all next results we will keep the parameter \( b = 0.6 \) and take \( a \) as our tunable parameter.
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Phase Space for the two cases

\[ a = 0.80552 \]

\[ a = 0.80630 \]
Mean escape time and transmissivity

Escape time $\rightarrow$ average time of a square of initial conditions

$[-0.5, 0.5] \times [-0.9, 0.9]$ reach $y_b = \pm 2.0$

Transmissivity $\rightarrow$ number of the trajectories that cross the barrier

![Graph showing mean escape time and transmissivity](image)
Mean escape time and transmissivity

High transmissivity

Persistent Barrier

![Graph showing mean escape time and transmissivity](attachment:image.png)
Phase Space for the two cases

**High transmissivity**

\[ a = 0.80552 \]

**Persistent Barrier**

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Time escape for the two cases

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Time escape for the two cases

High transmissivity

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Persistent Barrier

\( a = 0.80630 \)
Comparison between a Sticky trajectory and FTLE

Persistent Barrier Case \((a = 0.8063)\)

Space Phase and FTLE time series
Lagrangian Coherent Structures
Lagrangian Coherent Structures (LCS)

- LCS are structures which separate dynamically distinct regions in systems.
- Finite-time Lyapunov Exponents can be used to find LCS, which are often analogous to stable and unstable manifolds of time-independent systems.
- These structures divide dynamically distinct regions in the phase space.
Comparison between manifolds and FTLE

High transmissivity case ($a = 0.80552$)

Manifolds

FTLE
Comparison between manifolds and FTLE

Persistent Barrier Case \( (a = 0.8063) \)
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Schematic depiction

Dominant Intercrossing

Dominant Intracrossing
Ridges of FTLE

- Marked in Blue - Ridges of FTLE.
- Marked in Red - The trajectories transmitted.

High Transmissivity

$$a = 0.80552$$

Persistent Barrier Case

$$a = 0.8063$$
Conclusions

- FTLE distributions can be useful to analyze the influence of trapping domains.
- When we look to distribution of the orbits that suffer trapping effects, they develop a mode near zero in FTLE.
- Even after the barrier breakdown, the region of the shearless curve continues to reduce transport.
- Manifold intercrossings increase the transmissivity. Otherwise, the manifold intracrossings slow down the transmissivity and increase the average escape time.
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References:


Thank You!