

Estabilidade de Pontos Fixos

Sistema Não Autônomo

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F}{m} \cos(\omega t)$$

Introduzindo as variáveis y, z

$$z = t \rightarrow \dot{z} = 1$$

$$y = \dot{x}$$

obtemos

$$\dot{x} = y$$

$$\dot{y} = \frac{F}{m} \cos(\omega z) - \gamma y - \omega_0^2 x$$

$$\dot{z} = 1$$

Sistema autônomo

Ponto Fixo

$$\frac{dx}{dt} = \dot{x} = f(x^*, y^*) = 0$$

$$\frac{dy}{dt} = \dot{y} = g(x^*, y^*) = 0$$

O ponto $P(x^, y^*)$ é um ponto fixo*

Estabilidade do Ponto Fixo

Ponto P é assintoticamente estável se

$$\lim_{t \rightarrow \infty} (x, y) \rightarrow (x^*, y^*)$$

P é um atrator

Ponto P é estável se

$$\lim_{t \rightarrow \infty} (x, y) \approx (x^*, y^*)$$

Caso contrário é instável

Estabilidade Estrutural

As soluções da equação do pêndulo ideal

$$\ddot{x} + \omega_0^2 x = 0$$

são alteradas se considerarmos atrito ($\gamma \neq 0$)

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

Ponto de Sela

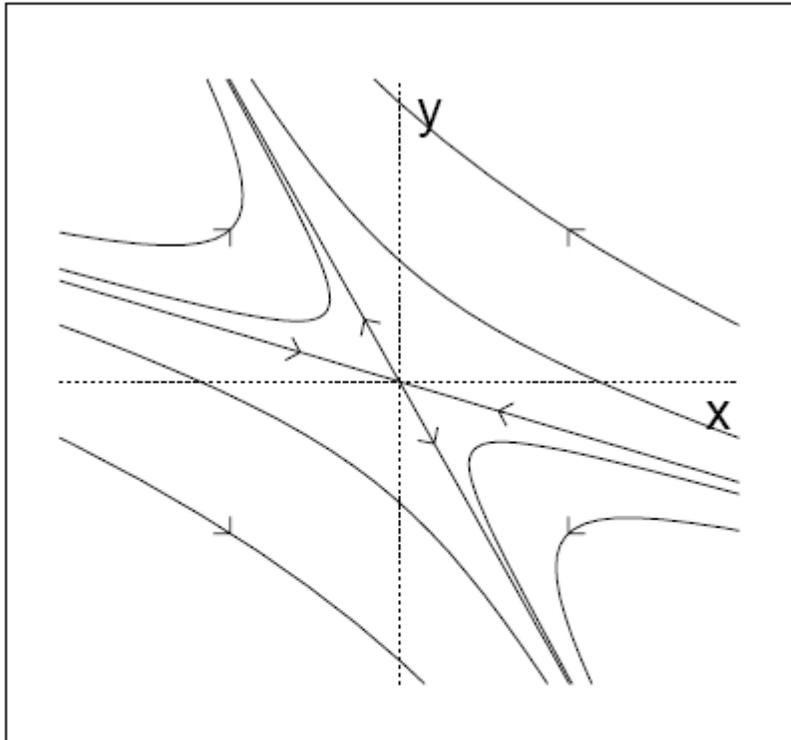


Figure 7.7 Phase plane for a saddle equilibrium.

For (7.16), the origin is an equilibrium. Except for two solutions that approach the origin along the direction of the vector $(3, -1)$, solutions diverge toward infinity, although not in finite time.

Ponto Fixo Repulsor

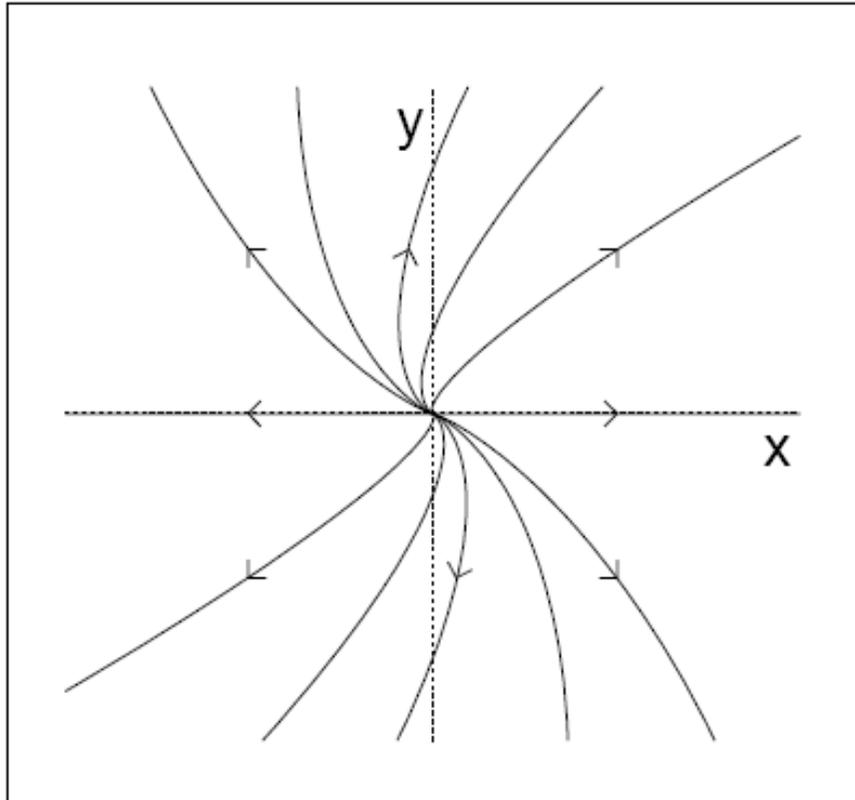


Figure 7.8 Phase plane for Equation (7.17).

The coefficient matrix \mathbf{A} for this system has only one eigenvector, which lies along the x -axis. All solutions except for the equilibrium diverge to infinity.

Auto Valores Imaginários

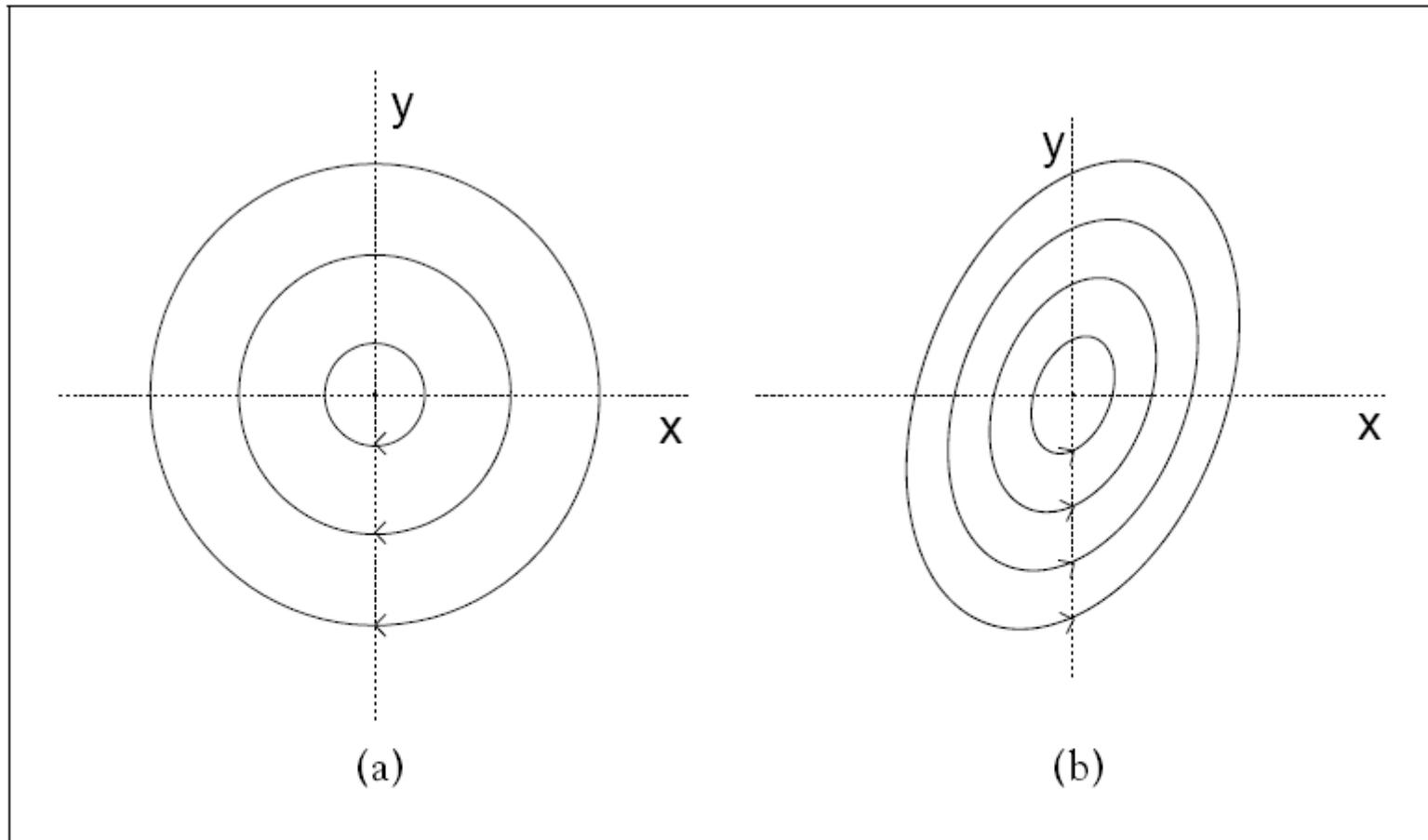


Figure 7.9 Phase planes for pure imaginary eigenvalues.

(a) In (7.18), the eigenvalues are $\pm i$. All solutions are circles around the origin, which is an equilibrium. (b) In (7.23), the eigenvalues are again pure imaginary. Solutions are elliptical. Note that for this equilibrium, some points initially move farther away, but not too far away. The origin is (Lyapunov) stable but not attracting.

Ponto Atrator

Ponto Repulsor

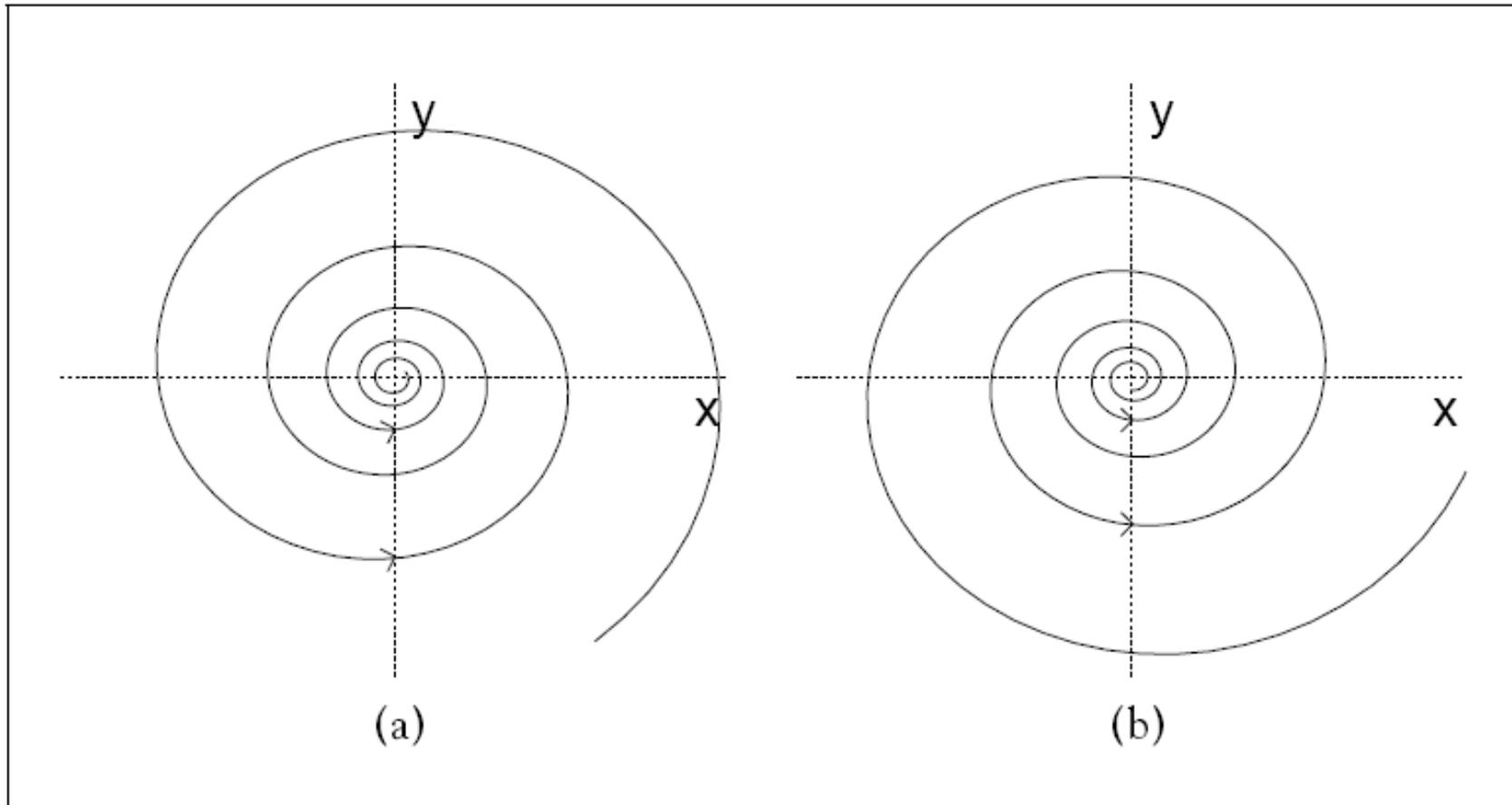


Figure 7.10 Phase planes for complex eigenvalues with nonzero real part.

(a) Under (7.21), trajectories spiral in to a sink at the origin. The eigenvalues of the coefficient matrix A have negative real part. (b) For (7.22), the trajectories spiral out from a source at the origin.

Mapa Logístico

Órbita / Ponto Fixo

$$\text{Órbita : } \{x_0, x_1, x_2, \dots, x_n, \dots\} = \{x_0, f(x_0), f^2(x_0), \dots, f^n(x_0), \dots\}$$

x_0 : ponto inicial

Ponto fixo p : $f^k(p) = p$

Para encontrar o ponto fixo usamos : $x_{n+1} = x_n$

Para o Mapa Logístico, $x_{n+1} = 2x_n(1-x_n) \Rightarrow x_n = 0; 0,5$

Estabilidade de Pontos Fixos

$N_\varepsilon(p)$: vizinhança ε em torno de $x=p$ ($\varepsilon \cong 0$ e $\varepsilon > 0$)

$$N_\varepsilon(p) \equiv \{x \in \mathbb{R} : |x - p| < \varepsilon\}$$

Ponto fixo de atração:

$$\text{Para } \varepsilon > 0 \text{ e } x \in N_\varepsilon(p) \Rightarrow \lim_{k \rightarrow \infty} f^k(x) = p$$

Ponto fixo de repulsão:

$$\text{Para } \varepsilon > 0 \text{ e } x \in N_\varepsilon(p) \Rightarrow \lim_{k \rightarrow \infty} f^k(x) \notin N_\varepsilon(p)$$

Exemplo

Mapa Logístico $x_{n+1} = 2x_n(1 - x_n)$

Pontos fixos: $0; 0,5$ $f(0)=0$ e $f(0,5)=0$

$f'(0) = 2 > 1 \Rightarrow$ *ponto de repulsão*

$f'(0,5) = 0 < 1 \Rightarrow$ *ponto de atração*

Abacia de atração de $x=0,5$ é o intervalo $0 < x < 1$

Pontos periódicos

Mapa Logístico $x_{n+1} = f(x_n) = a x_n (1 - x_n)$ $a = 3,3$

Pontos fixos instáveis: $x = 0$; $0,696969\dots$

Órbitas periódicas estáveis com período 2:

$$f^2(0,4794) = 0,4794 \quad f^2(0,8236) = 0,8236$$

$$f^{2'}(0,4794) = |-0,2904| < 1 \quad f^{2'}(0,8236) < 1$$

Mapa Logístico

Mapa Logístico $x_{n+1} = g_a(x_n) = a x_n (1 - x_n)$

Para $0 < a < 1$, $\forall x_0 \in 0 < x_0 < 1 \Rightarrow x \rightarrow 0$

Para $1 < a < 3$, $\forall x_0 \in 0 < x_0 < 1 \Rightarrow x \rightarrow 0; \frac{a-1}{a}$

$g'(0) = a > 1 \Rightarrow$ ponto de repulsão

$\left| g'\left(\frac{a-1}{a}\right) \right| = |-a + 2| < 1 \Rightarrow$ ponto de atração

Mapa de Hénon

Hénon (Comm. Math. 50, 69, 1976) introduziu o mapa

$$(x_{n+1}, y_{n+1}) = f(x_n, y_n) = (a - x_n^2 + b y_n, x_n)$$

a, b: parâmetros de controle

Atratores, Repulsores e Pontos de Sela

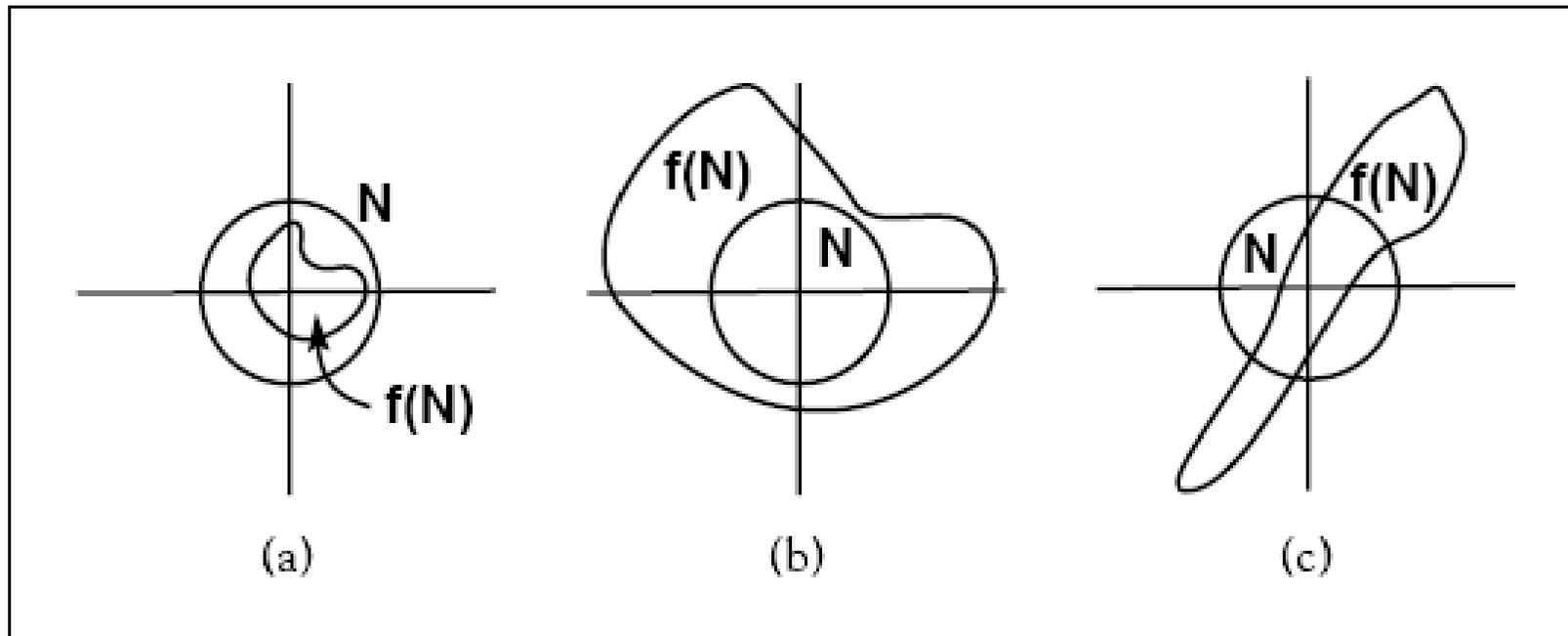


Figure 2.8 Local dynamics near a fixed point.

The origin is (a) a sink, (b) a source, and (c) a saddle. Shown is a disk N and its iterate under the map f .

(Alligood et al.
Chaos...)

5 - Mapas Lineares

$$\mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix} \quad \vec{V} \equiv \begin{pmatrix} x \\ y \end{pmatrix}$$

Definição: A é linear se $A(a\vec{v} + b\vec{w}) = aA(\vec{v}) + bA(\vec{w})$

λ é um auto-valor da matriz A se (para $\vec{v} \neq 0$)

$$A\vec{v} = \lambda\vec{v}$$

$$\vec{v}_{n+1} = A\vec{v}_n \Rightarrow \vec{v}_{n+1} = \lambda^{n+1}\vec{v}_0$$

Exemplo

Para a matriz $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a \text{ é auto - valor e } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ auto - vetor}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow b \text{ é auto - valor e } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ auto - vetor}$$

Matriz Jacobiana

Em uma dimensão, com um ponto fixo $p = f(p)$,

$$f(p + h) \cong f(p) + h f'(p)$$

$$|f'(p)| < 1 \Rightarrow p \text{ é um atrator}$$

$$|f'(p)| > 1 \Rightarrow p \text{ é um repulsor}$$

Em duas dimensões, com um ponto fixo $\vec{p} = \vec{f}(\vec{p})$

$$\vec{f}(\vec{p} + \vec{h}) \cong \vec{f}(\vec{p}) + \vec{D}\vec{f}(\vec{p}) \cdot \vec{h} = \vec{p} + \vec{D}\vec{f}(\vec{p}) \cdot \vec{h}$$

Teorema :

1 - Se os módulos dos auto-valores da matriz $\vec{D}\vec{f}(\vec{p})$ forem menores que 1, \vec{p} é um atrator.

2 - Se os módulos dos auto-valores da matriz $\vec{D}\vec{f}(\vec{p})$ forem maiores que 1, \vec{p} é um repulsor.

Definição : Se um auto-valor for maior que 1 e o outro menor que 1, \vec{p} é um ponto de sela.

Em uma dimensão, $f^2(p_0)' = f(p_0)' f(p_1)'$

Em duas dimensões, $D\vec{f}^2(\vec{p}_0) = D\vec{f}(\vec{p}_0) D\vec{f}(\vec{p}_1)$

Exemplo: Mapa de Hénon $f(x, y) = (a - x^2 + b y, x)$ $a = 0$ e $b = -0,4$

Ponto fixo $(0, 0)$

$$J = \begin{pmatrix} \partial_x (a - x^2 + b y) & \partial_y (a - x^2 + b y) \\ \partial_x x & \partial_y x \end{pmatrix} = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0,4 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 - \lambda & b \\ 1 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow |\lambda| = |\pm \sqrt{0,4}| < 1 \Rightarrow \text{atrator}$$

Ponto fixo $(-0,6, -0,6)$

$$J = \begin{pmatrix} \partial_x (a - x^2 + b y) & \partial_y (a - x^2 + b y) \\ \partial_x x & \partial_y x \end{pmatrix} = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1,2 & 0,4 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1,2 - \lambda & b \\ 1 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow |\lambda_1| = |1,472| > 1 ; |\lambda_2| = |-0,272| < 1 \Rightarrow \text{ponto de sela}$$

Exemplo: Mapa de Hénon $f(x, y) = (a - x^2 + by, x)$ $a=0,43$ $b=0,4$

Órbita de período 2: $(0.7, -0,1) \Leftrightarrow (-0,1, 0.7)$

$$J^2 = JJ = \begin{pmatrix} -2(-0.1) & 0.4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2(0.7) & 0.4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0.12 & 0.008 \\ -1.4 & 0.4 \end{pmatrix}$$

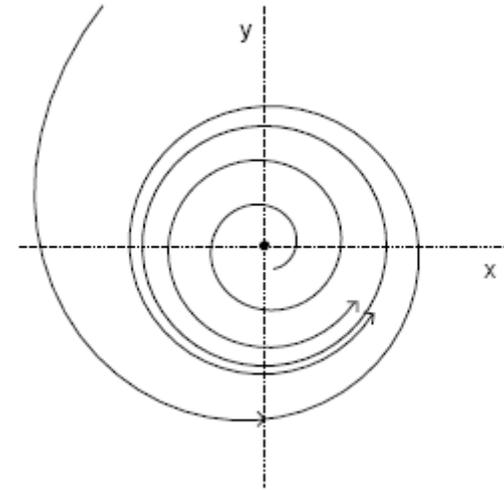
$$\begin{vmatrix} 0.12 - \lambda & 0.008 \\ -1.4 & 0.4 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0.26 \pm i 0.30 \Rightarrow |\lambda| = 0.40 < 1 \text{ atrator}$$

Ciclos Limite

Exemplo:

- $\dot{r} = r(1 - r)$

- $\dot{\theta} = 8$



Ponto de equilíbrio instável: $(0, 0)$

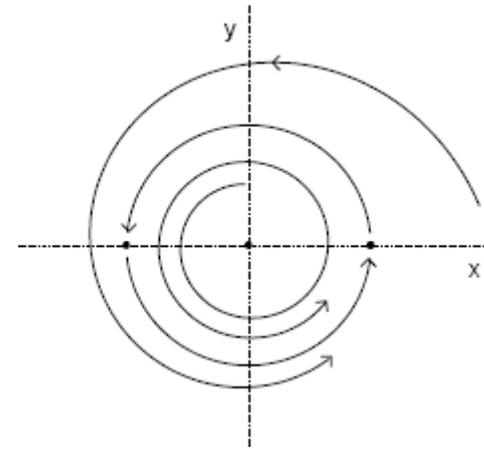
$r = 1$ órbita periódica \rightarrow ciclo limite estável

$$r = \frac{c e^t}{c e^t - 1} \quad \theta = 8 t + d$$

Exemplo :

- $\dot{r} = -r(1 - r)$

- $\dot{\theta} = 8$



Ponto de equilíbrio estável: $(0,0)$

$r=1$ órbita periódica instável \rightarrow ciclo limite instável

Exemplo

•
 $\dot{x} = x(a - x)$, $a > 0$

Pontos de equilíbrio: $x = 0$ e $x = a$

Para $x_0 > 0$, $\omega(x_0) = \{a\}$

Para $x_0 = 0$, $\omega(x_0) = \{0\}$

Para $x_0 < 0$, $\omega(x_0) = \{ \}$: conjunto vazio

Exemplo

$$\dot{r} = r(a - r)$$

$$\dot{\theta} = b$$

Origem é ponto de equilíbrio estável

$$\omega(0) = \{0\}$$

$$\omega(r_0, \theta_0) = \{r = a\} ; r_0 \neq 0$$

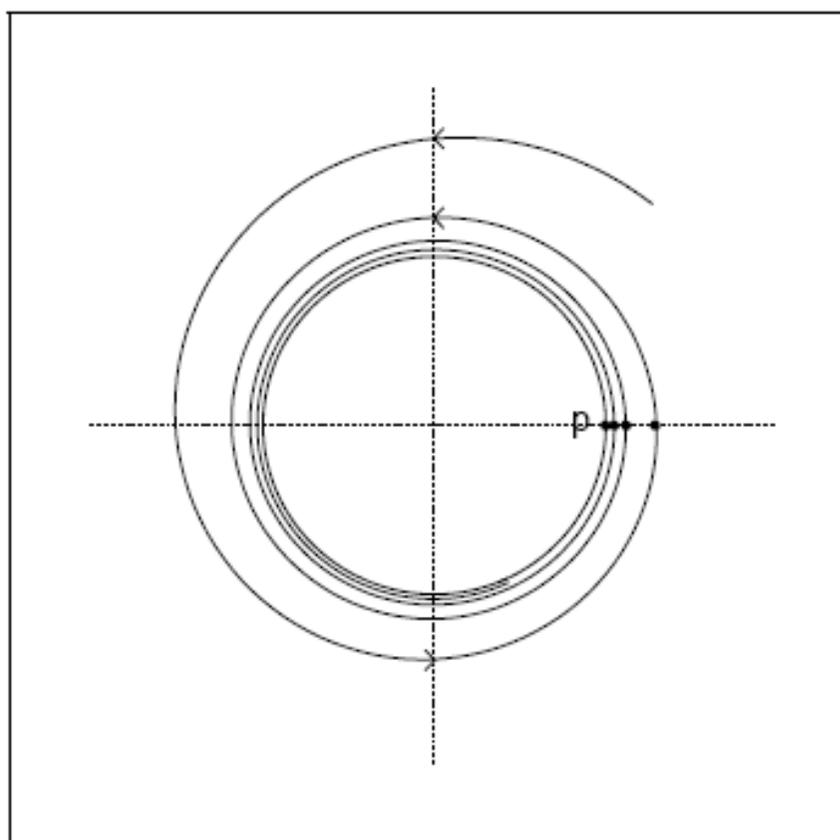


Figure 8.1 The definition of an ω -limit set.

The point p is in the ω -limit set of the spiraling trajectory because there are points $F(t_1, v_0)$, $F(t_2, v_0)$, $F(t_3, v_0) \dots$ of the trajectory, indicated by dots, that converge to p . The same argument can be made for any point in the entire limiting circle of the spiral solution, so the circle is the ω -limit set.

Exemplo

$$\dot{r} = r(a - r)$$

$$\dot{\theta} = \text{sen}^2\theta + (r - a)^2$$

$$\text{Pontos de equilíbrio: } \begin{cases} (a, 0) \\ (a, \pi) \\ (0, 0) \end{cases}$$

$$\omega(0) = \{0\}$$

$$\omega(r_0, \theta_0) = \{r = a\} ; \quad \forall r_0 \neq 0$$

$$\omega(a, \theta_0) = (a, \theta = 0, \pi)$$

Bifurcações de Ciclos Limites

Sistemas bidimensionais

Estabilidade Estrutural

- Estabilidade da solução depende dos parâmetros de controle.
- Mudança qualitativa na solução, com a alteração de um parâmetro, caracteriza uma bifurcação.

Bifurcação Sela-Nó

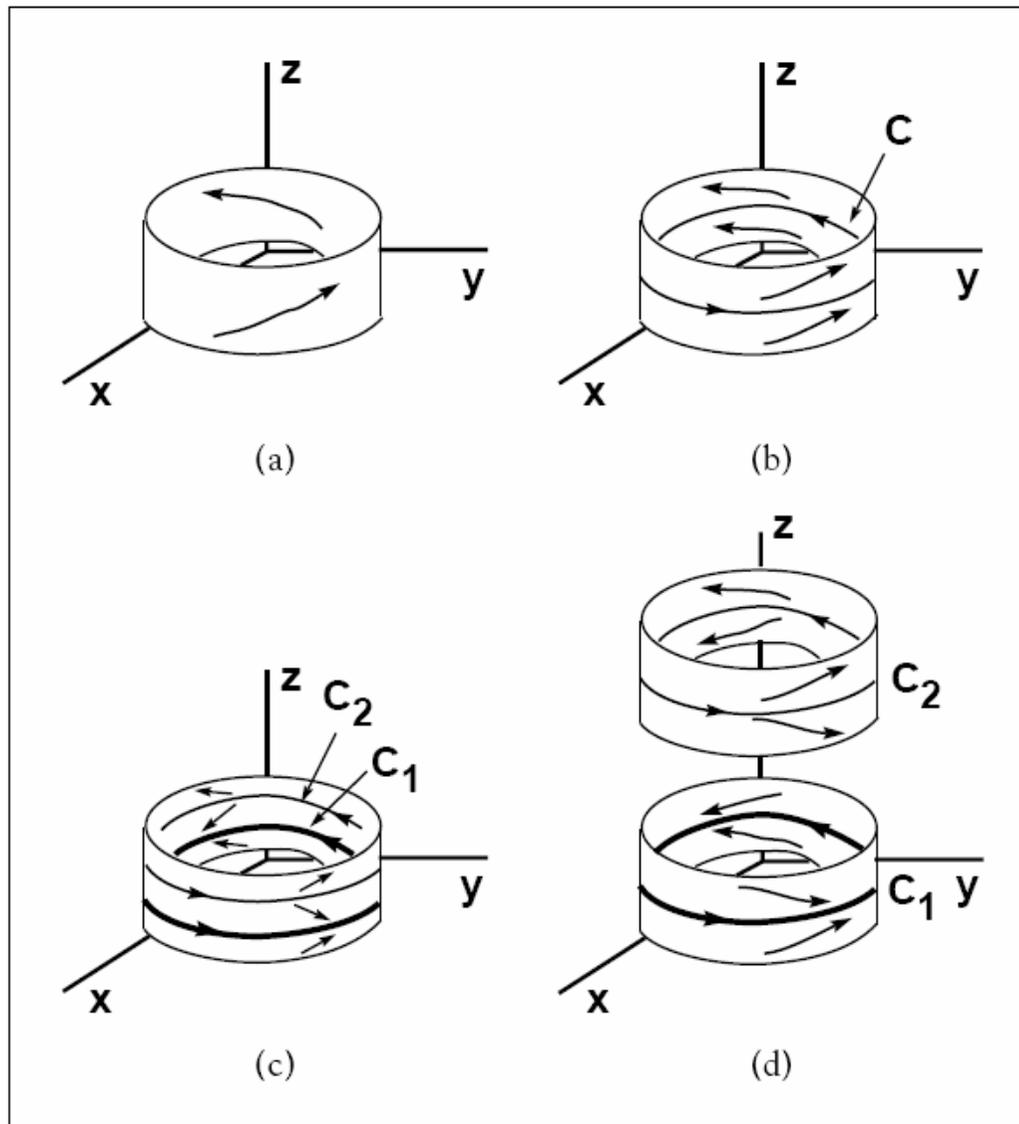


Figure 11.21 A saddle-node bifurcation for a three-dimensional flow.

(a) The system begins with no periodic orbits. (b) The parameter is increased, and a saddle-node periodic orbit C appears. (c) The saddle-node orbit splits into two periodic orbits C_1 and C_2 , which then move apart, as shown in (d).

Dobramento de Período

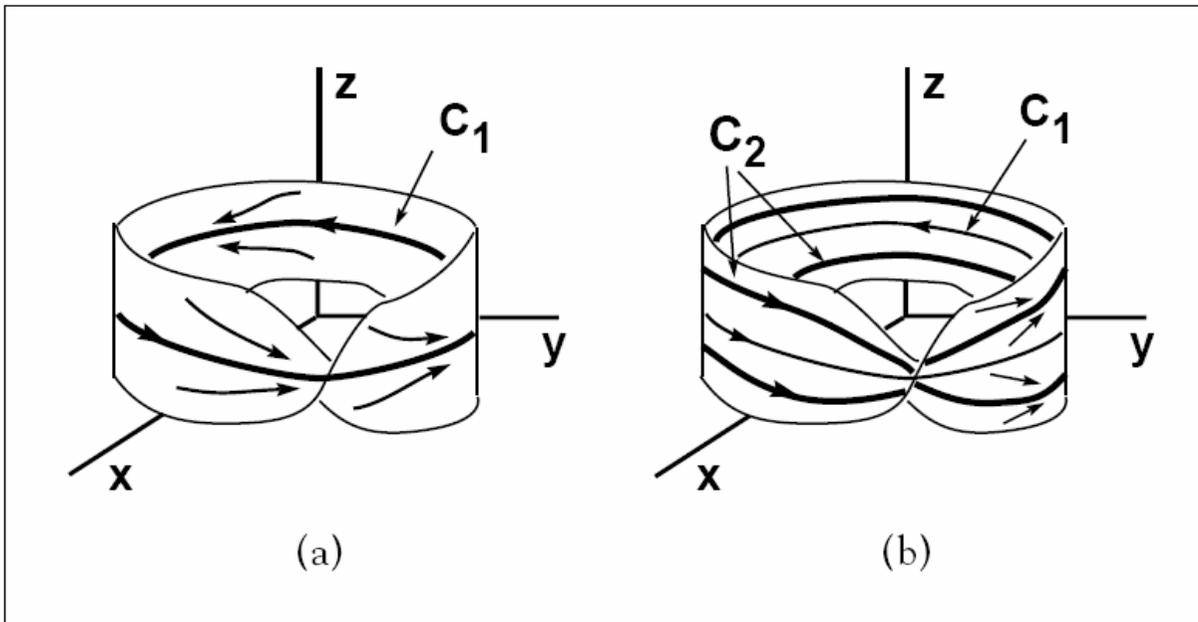


Figure 11.22 A period-doubling bifurcation for a three-dimensional flow.

The system begins with a periodic orbit C_1 , which has one multiplier between 0 and -1 , as shown in (a). As the parameter is increased, this multiplier crosses -1 , and a second orbit C_2 of roughly twice the period of C_1 bifurcates. This orbit wraps twice around the Möbius strip shown in (b).

Bifurcação de Hopf

Exemplo

$$\dot{x} = -y + x(a - x^2 - y^2)$$

$$\dot{y} = x + y(a - x^2 - y^2)$$

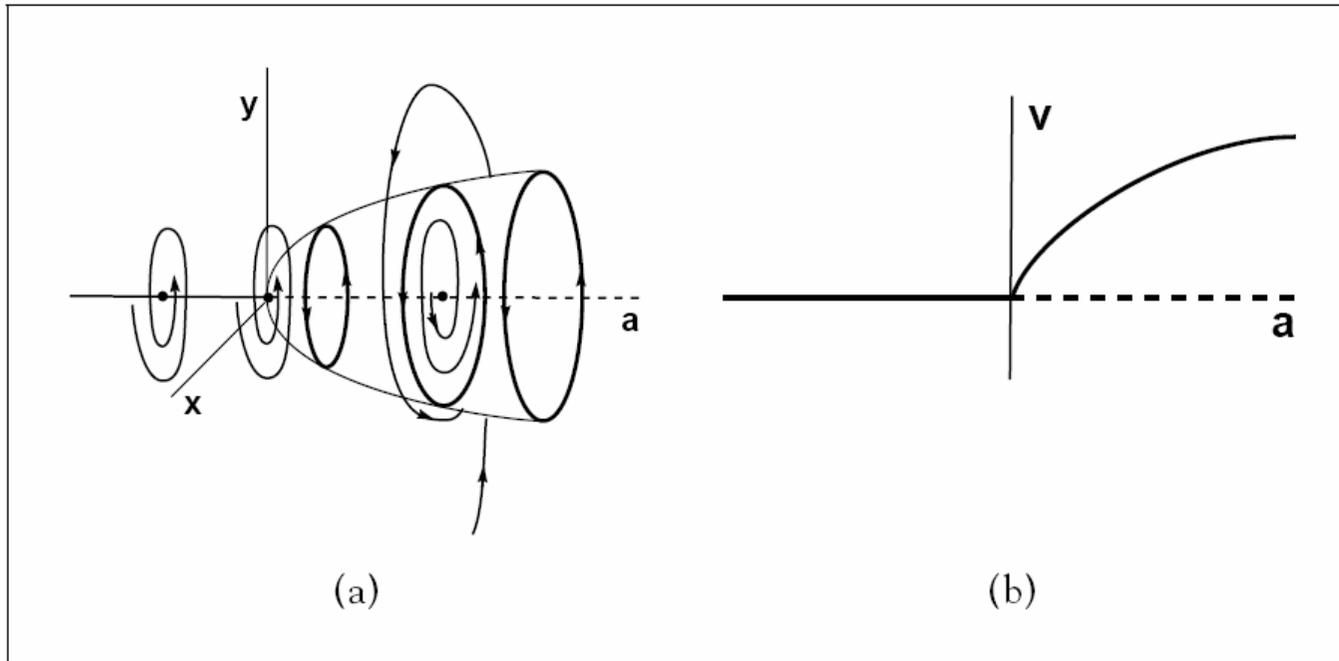


Figure 11.23 Supercritical Hopf bifurcation.

(a) The path $\{(a, 0, 0)\}$ of equilibria changes stability at $a = 0$. A stable equilibrium for $a < 0$ is replaced by a stable periodic orbit for $a > 0$. (b) Schematic path diagram of the bifurcation. Solid curves are stable orbits, dashed curves are unstable.

Ilustração da Bifurcação de Hopf

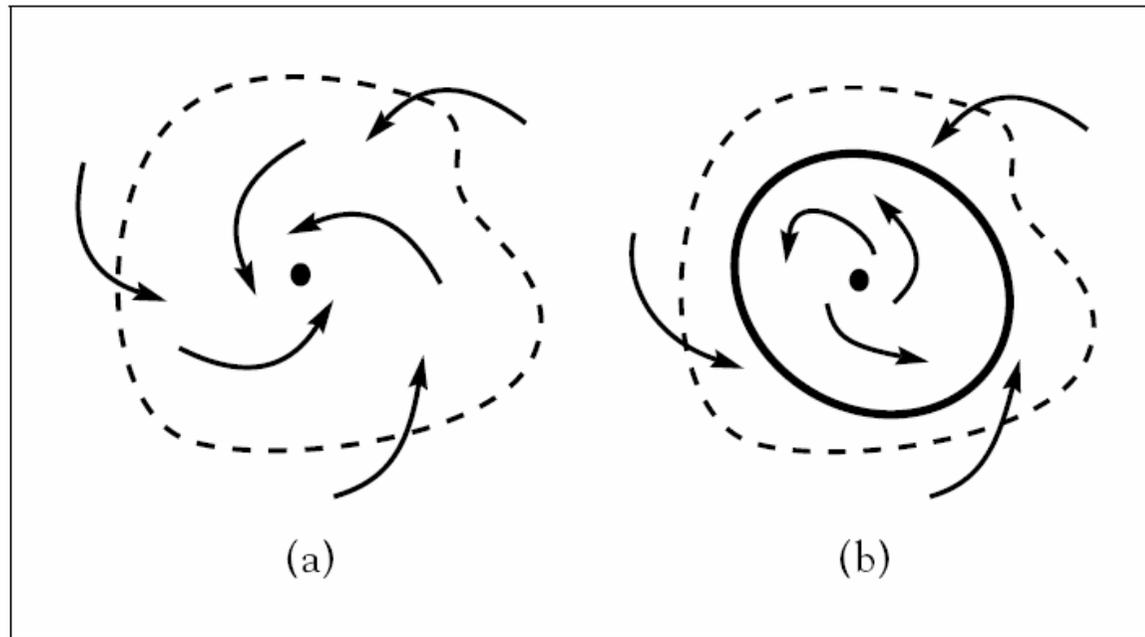


Figure 11.24 A periodic orbit bifurcates from an equilibrium in the plane. As the equilibrium goes from attracting (a) to repelling (b), a periodic orbit appears.

Bifurcação de Hopf Sub-Crítica

$$\dot{r} = ar + 2r^3 - r^5$$

$$\dot{\theta} = 1$$

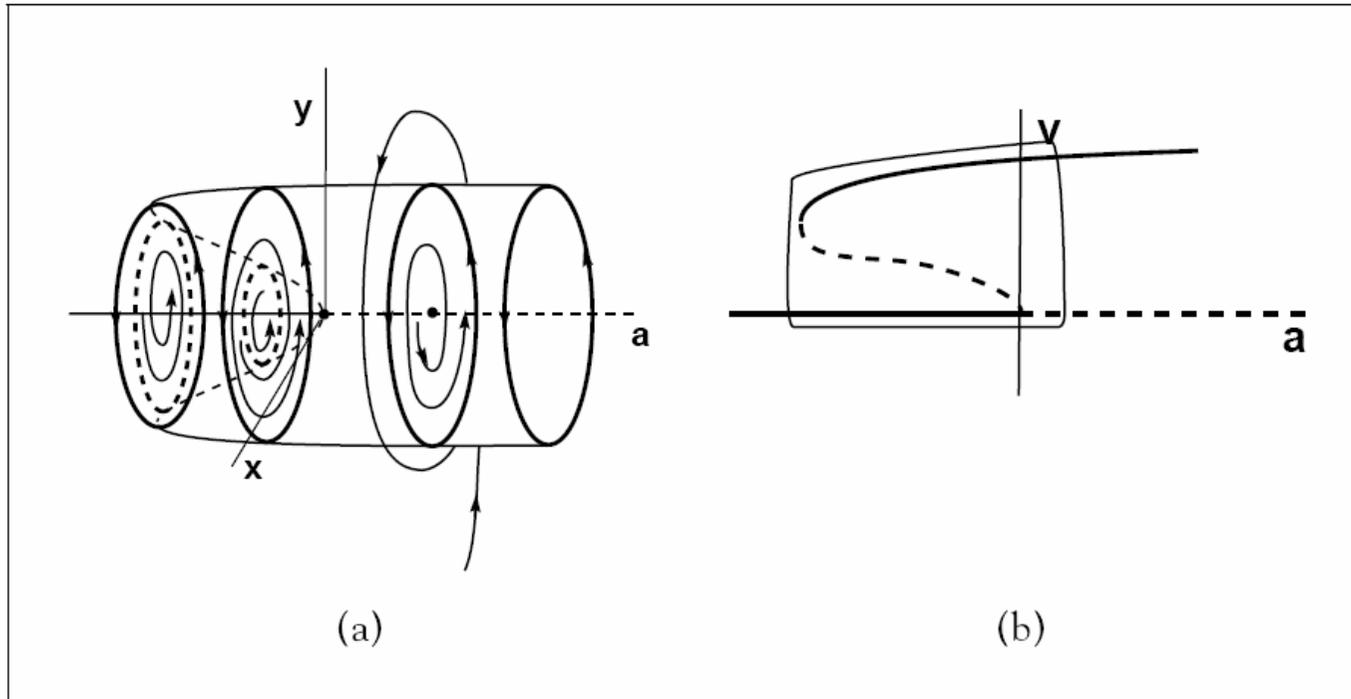


Figure 11.25 A subcritical Hopf bifurcation with hysteresis.

(a) There is a bifurcation at $a = 0$ from the path $r = 0$ of equilibria. At this point the equilibria go from stable to unstable, and a path of unstable periodic orbits bifurcates. The periodic orbits are unstable and extend back through negative parameter values, ending in saddle node at $a = -1$. An additional path of attracting periodic orbits emanates from the saddle node. (b) Schematic diagram of bifurcation

Bifurcação de Hopf Sub-Crítica nas Equações de Lorenz

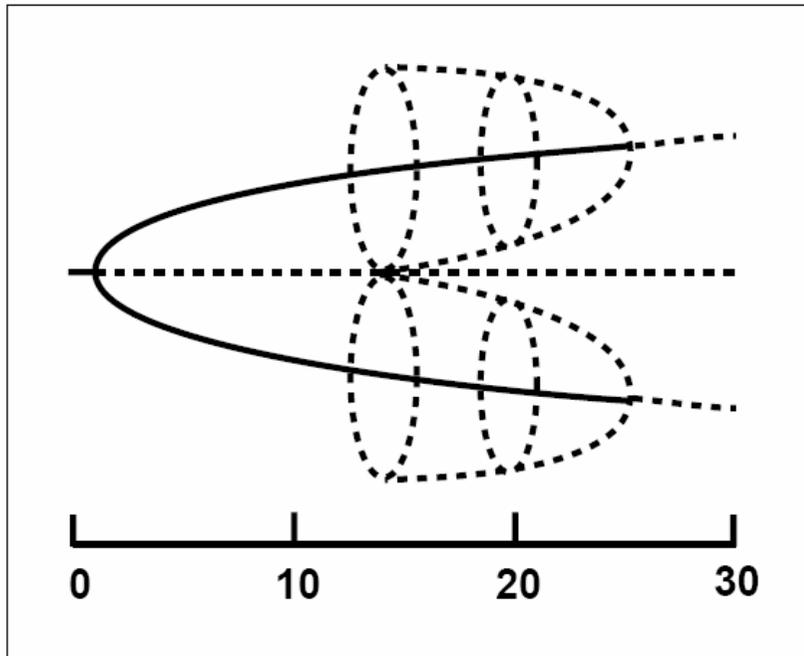


Figure 11.26 A Hopf bifurcation in the Lorenz equations.

A bifurcation diagram of the Lorenz equations for $\sigma = 10$, $b = 8/3$, and $0 \leq r \leq 30$, is shown. At $r = 1$, the origin goes from stable to unstable, as two attracting equilibria bifurcate. Paths of stable orbits are indicated by solid curves, while paths of unstable orbits are represented by dashed curves and circles. A subcritical Hopf bifurcation occurs at $r = 24.74$, at which point two families of unstable orbits bifurcate simultaneously as the two attracting equilibria lose their stability. Typical trajectories in computer simulations then move toward the chaotic attractor, first observed to occur at the crisis value $r = 24.06$.