Rayleigh-Taylor instability in viscous-resistive plasmas with finite skin depth

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Classical Rayleigh-Taylor instability

- Occurs at the interface of two fluids with different densities

- More dense fluid is supported by less dense fluid against gravity

Rayleigh-Taylor instability in plasmas

- More dense fluid: plasma itself

- Less dense "fluid": magnetic field $\vec{B}$

- "Gravitational" field: centrifugal acceleration

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Gravitational field as centrifugal acceleration

Centrifugal force on guiding-center following a curved $\vec{B}$ line

$$\vec{F} = -m_S v_G^2 (\vec{b} \cdot \nabla) \vec{b}$$

- m_S: species mass
- v_G: guiding-center speed
- $\vec{b}$: unit vector along $\vec{B}$

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Simplest example: circular $\vec{B}$ line of radius r_0

Adopt plane polar coordinates (r, θ)

- Hence $\vec{b} = \hat{\theta}$ and $\vec{v}_G = \hat{\theta} v_G$

- r_0 is constant $\Rightarrow \nabla = \hat{\theta} \dfrac{1}{r_0} \dfrac{\partial}{\partial \theta}$

- $\hat{\theta} \cdot \hat{\theta} = 1$ and $\dfrac{\partial \hat{\theta}}{\partial \theta} = -\hat{r} \Rightarrow \vec{F} = -m_S v_G^2 (\vec{b} \cdot \nabla) \vec{b} = \hat{r} m_S \dfrac{v_G^2}{r_0}$

(usual centrifugal force)

Gravitational field

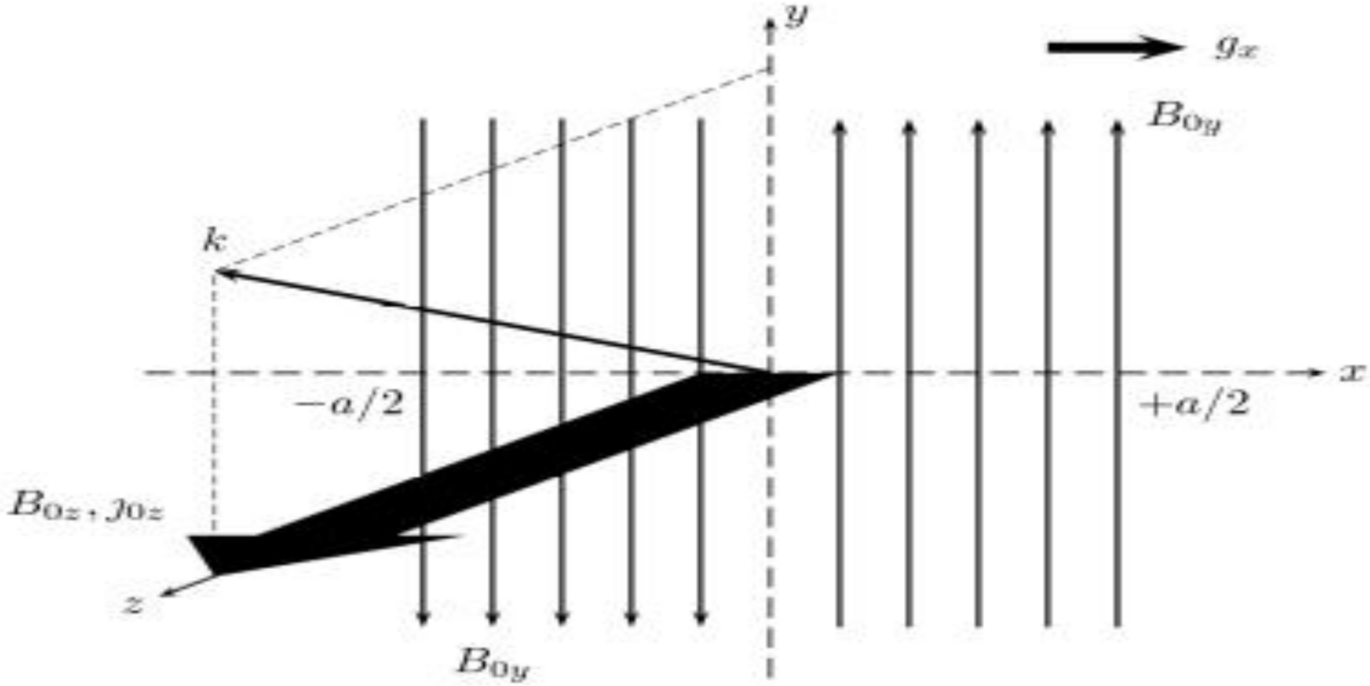
- Average centrifugal force on many gyro-periods

$$\langle \vec{F} \rangle = -m_S \langle v_G^2 \rangle (\vec{b} \cdot \nabla) \vec{b} \equiv m_S \vec{a}$$

- Thermal speed: $v_T = \sqrt{\langle v_G^2 \rangle}$

- Gravitational field: $\vec{g} \equiv \vec{a} = -v_T^2 (\vec{b} \cdot \nabla) \vec{b}$

Current slab



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Infinitesimal perturbation

$$\varphi(\vec{r},t) = \varphi_0(x) + \varphi_1(x)e^{\gamma t + i\vec{k}\cdot\vec{r}}$$

- φ_0: equilibrium field
- φ_1: perturbative field
- $|\varphi_1| \ll |\varphi_0| \Rightarrow$ terms $\sim O(\varphi_1^2) \cong 0$: linear approximation
- $\vec{k} = \hat{y}k_y + \hat{z}k_z$: perturbative vector is perpendicular to x-axis
- $\vec{v}_0 = 0$: static state of equilibrium
- $\nabla \cdot \vec{v}_1 = 0$: incompressible perturbation (replaces adiabaticity)

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Inviscid, perfectly conducting plasma

- ❑ Magnetic flux freezing condition: $\vec{E} + \vec{v} \times \vec{B} = 0$

- ❑ Linearize Faraday's law: $B_{1x} = -\frac{1}{\omega}(\vec{k} \cdot \vec{B}_0) v_{1x}$

- ❑ $\gamma = -i\omega$: stable equilibrium (no dissipative effects)

- ❑ If the direction of $\vec{B}_0$ were constant,
then the direction of $\vec{k}$ could always be chosen
such that $\vec{k} \cdot \vec{B}_0 = 0$ and
no bending of the $\vec{B}_0$ lines would be produced

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Rational surface

- Since the $\vec{B}_0$ lines exhibits a shear, the condition $\vec{k} \cdot \vec{B}_0 = 0$ can be satisfied only at a singular point in the plasma

- In the neighborhood of the singular point, $B_{1x} \neq 0$ and the distortion of the $\vec{B}_0$ lines provokes the appearance of a restoring force which exactly opposes the perturbative force

- This is the well-known stabilizing effect due to shear of the Rayleigh-Taylor instability

- The condition $\vec{k} \cdot \vec{B}_0 = 0$ defines the so-called rational surface of a magnetically confined plasma

Viscous-resistive plasma

- Ohm's law: $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} + \mu_0 \delta_e^2 \dfrac{\partial \vec{j}}{\partial t}$
- Electron skin depth: $\delta_e = \sqrt{\dfrac{m_e}{\mu_0 n_e e^2}}$
- Linearize Ohm's law: $\vec{E}_1 + \vec{v}_1 \times \vec{B}_{0k} = (\eta + \gamma \mu_0 \delta_e^2) \vec{j}_1$
- Take the vector product of linearized equation with $\vec{B}_{0k}$

$$\vec{E}_1 \times \vec{B}_{0k} - \vec{v}_1 B_{0k}^2 + \vec{B}_{0k}\left(\vec{v}_1 \cdot \vec{B}_{0k}\right) = (\eta + \gamma \mu_0 \delta_e^2) \vec{j}_1 \times \vec{B}_{0k}$$

Determination of drift term

- ❑ Linearize Faraday's law: $\hat{x}\times\vec{E}_1' + i\vec{k}\times\vec{E}_1 = -\gamma\vec{B}_{1x}$

- ❑ Linearize Ampère's law: $i\vec{k}\times\vec{B}_{1x} = \mu_0\vec{J}_1$

- ❑ Eliminate $\vec{B}_{1x}$ from linearized equations

$$i\hat{x}\left(\vec{k}\cdot\vec{E}_1'\right) - \vec{k}\left(\vec{k}\cdot\vec{E}_1\right) + k^2\vec{E}_1 = -\gamma\mu_0\vec{J}_1$$

- ❑ Take the vector product of last equation with $\vec{B}_{0k}$

$$i\hat{x}\times\vec{B}_{0k}\left(\vec{k}\cdot\vec{E}_1'\right) + k^2\vec{E}_1\times\vec{B}_{0k} = -\gamma\mu_0\vec{J}_1\times\vec{B}_{0k}$$

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Lorentz force

We have

$$\vec{E}_1 \times \vec{B}_{0k} - \vec{v}_1 B_{0k}^2 + \vec{B}_{0k}\left(\vec{v}_1 \cdot \vec{B}_{0k}\right) = (\eta + \gamma\mu_0\delta_e^2)\vec{j}_1 \times \vec{B}_{0k}$$

$$i\hat{x} \times \vec{B}_{0k}\left(\vec{k} \cdot \vec{E}_1'\right) + k^2 \vec{E}_1 \times \vec{B}_{0k} = -\gamma\mu_0 \vec{j}_1 \times \vec{B}_{0k}$$

Eliminate the drift term from equations above
and take the x-component of the resulting equation

$$\hat{x} \cdot \left(\vec{j}_1 \times \vec{B}_{0k}\right) = -\left[1 + (1 + k^2\delta_e^2)\frac{\gamma\tau_D}{k^2 a^2}\right]^{-1} \frac{B_{0k}^2}{\eta} v_{1x}$$

Magnetic diffusion time scale: $\tau_D = \frac{\mu_0 a^2}{\eta}$

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Viscous force

Its x-component is determined by: $\hat{x} \cdot \nabla^2 \vec{v}_1 = v''_{1x} - k^2 v_{1x}$

- In the neighborhood of the rational surface, the magnetic force on the charged species does not depend on the x-coordinate:

$$(v_{1x} B_{0k})' = 0$$

(otherwise, the gravitational field g_x cannot remain constant)

- Close to the singular point, Taylor-expand B_{0k}: $B_{0k} \cong waB'_{0k}$

Boundary-layer width: $w \ll 1$ (positive dimensionless number)

Therefore: $\nu\rho_0 \hat{x} \cdot \nabla^2 \vec{v}_1 = \nu\rho_0 \left(\frac{3}{w^2 a^2} - k^2\right) v_{1x}$

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Restoring force

- The x-component of Lorentz + viscous force:

$$\hat{x} \cdot \left(\vec{j}_1 \times \vec{B}_{0k} + \nu \rho_0 \nabla^2 \vec{v}_1\right)$$

- In the neighborhood of the rational surface, it approaches the gravitational force: $\rho_1 g_x$

- Linearize the continuity equation: $\gamma \rho_1 + \rho_0' v_{1x} = 0$ (recall $\nabla \cdot \vec{v}_1 = 0$)

Therefore: $\rho_1 g_x = -\frac{\rho_0' g_x}{\gamma} v_{1x}$

Put it all together and you will find...

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... how to evaluate the boundary-layer width

$$\frac{w^4}{w_\eta^4} - \left(\frac{3 - k^2 a^2 w_\nu^2}{3}\right)\frac{w^2}{w_\eta^2} - \frac{w_v^2}{w_\eta^2} = 0$$

❑ Resistive width: $w_\eta = \sqrt{\left[1 + (1 + k^2\delta_e^2)\,\frac{\gamma\tau_D}{k^2 a^2}\right]\frac{\kappa\alpha}{\gamma\tau_D}}$

❑ Viscous width: $w_\nu = \sqrt{\frac{\gamma\tau_D}{\kappa\alpha}\,\frac{\nu}{\nu_M}}$

But... what are the quantities: α, κ, and ν_M?

They are the "height of free-fall": $\alpha = \frac{1}{2} g_x \tau_A^2$

... in the Alfvén time: $\tau_A = \frac{a}{v_A}$

defined from the Alfvén speed: $v_A = \frac{a\left|B'_{0k}\right|}{\sqrt{\mu_0 \rho_0}}$...

corrected by the factor: $\kappa = (\ln \rho_0^2)'$...

... and the magnetic viscosity: $\nu_M = \frac{v_A^2 \tau_D}{3}$

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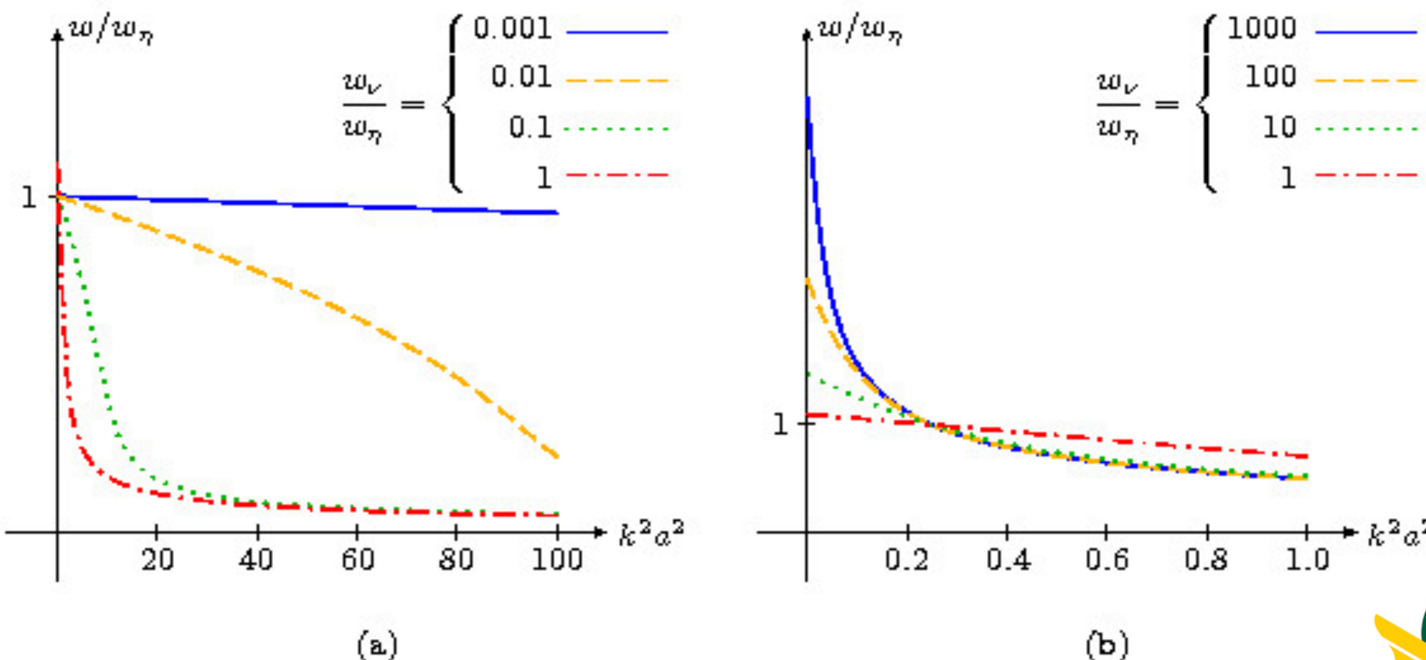
Limiting results

$$w_{\upsilon} \ll w_{\eta} \Rightarrow w \cong w_{\eta}$$

$$w_{\eta} \ll w_{\upsilon} \Rightarrow w \cong \sqrt{w_{\eta} w_{\nu}}$$



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Energy conservation

- In the neighborhood of the rational surface gravitational work ≅ variation of fluid kinetic energy

$$v_{1x}\rho_1 g_x \cong \gamma\rho_0\left(v_{1x}^2 + v_{1k}^2\right)$$

LHS: trivial, just recall $\rho_1 = -\frac{\rho_0}{\gamma} v_{1x}$

RHS: first, $\nabla \cdot \vec{v}_1 = v_{1x}' + ikv_{1k} = 0$

second, $v_{1x}' \cong -\frac{1}{wa} v_{1x}$

hence, $v_{1k} \cong -i\frac{1}{wka} v_{1x}$

then, $v_{1x}^2 + v_{1k}^2 \cong \left(1 - \frac{1}{w^2k^2a^2}\right) v_{1x}^2$

Since $w \ll 1$, we get $v_{1x}^2 + v_{1k}^2 \cong -\frac{1}{w^2k^2a^2} v_{1x}^2$

Put it all together and you will find...

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… how to evaluate the time growth rate

$$\frac{\rho_0' g_x}{\gamma} = \frac{\gamma \rho_0}{w^2 k^2 a^2}$$

leading to the general dispersion relation

$$\gamma^2 \tau_A^2 = \kappa \alpha w^2 k^2 a^2$$

which shows that $\gamma > 0$ (the equilibrium is unstable)

when $\kappa\alpha > 0$ (particles "fall down" in the gravitational field)

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Scaling laws of the time growth rate

❑ $$w \cong w_\eta \Rightarrow \gamma^2\tau_A^2 \cong \left[1 + (1 + k^2\delta_e^2)\frac{\gamma\tau_D}{k^2a^2}\right]\frac{k^2a^2}{\gamma\tau_D}\kappa^2\alpha^2$$

(a) $$\gamma\tau_D \ll k^2a^2 \Rightarrow \gamma \cong (ka)^{2/3}(\kappa\alpha)^{2/3}(\tau_D\tau_A^2)^{-1/3} \Rightarrow \gamma{\sim}\eta^{1/3}$$

(b) $$k^2a^2 \ll \gamma\tau_D \Rightarrow \gamma \cong (k\delta_e)(\kappa\alpha)\tau_A^{-1} \Rightarrow \gamma{\sim}n_e^{-1/2}$$

❑ $$w \cong \sqrt{w_\eta w_\nu} \Rightarrow \gamma^2\tau_A^2 \cong (\kappa\alpha)(ka)^2\sqrt{\left[1 + (1 + k^2\delta_e^2)\frac{\gamma\tau_D}{k^2a^2}\right]\frac{\nu}{\nu_M}}$$

(a) $$\gamma\tau_D \ll k^2a^2 \Rightarrow \gamma \cong (\kappa\alpha)^{1/2}(ka)\left(\frac{\nu}{\nu_M}\right)^{1/4}\tau_A^{-1} \Rightarrow \gamma{\sim}(\eta\nu)^{1/4}$$

(b) $$k^2a^2 \ll \gamma\tau_D \Rightarrow \gamma \cong (\kappa\alpha)^{2/3}(k^2a\delta_e)^{2/3}\left(\frac{\tau_D}{\tau_A^4}\right)^{1/3}\left(\frac{\nu}{\nu_M}\right)^{1/3} \Rightarrow \gamma{\sim}\left(\frac{\nu}{n_e}\right)^{1/3}$$

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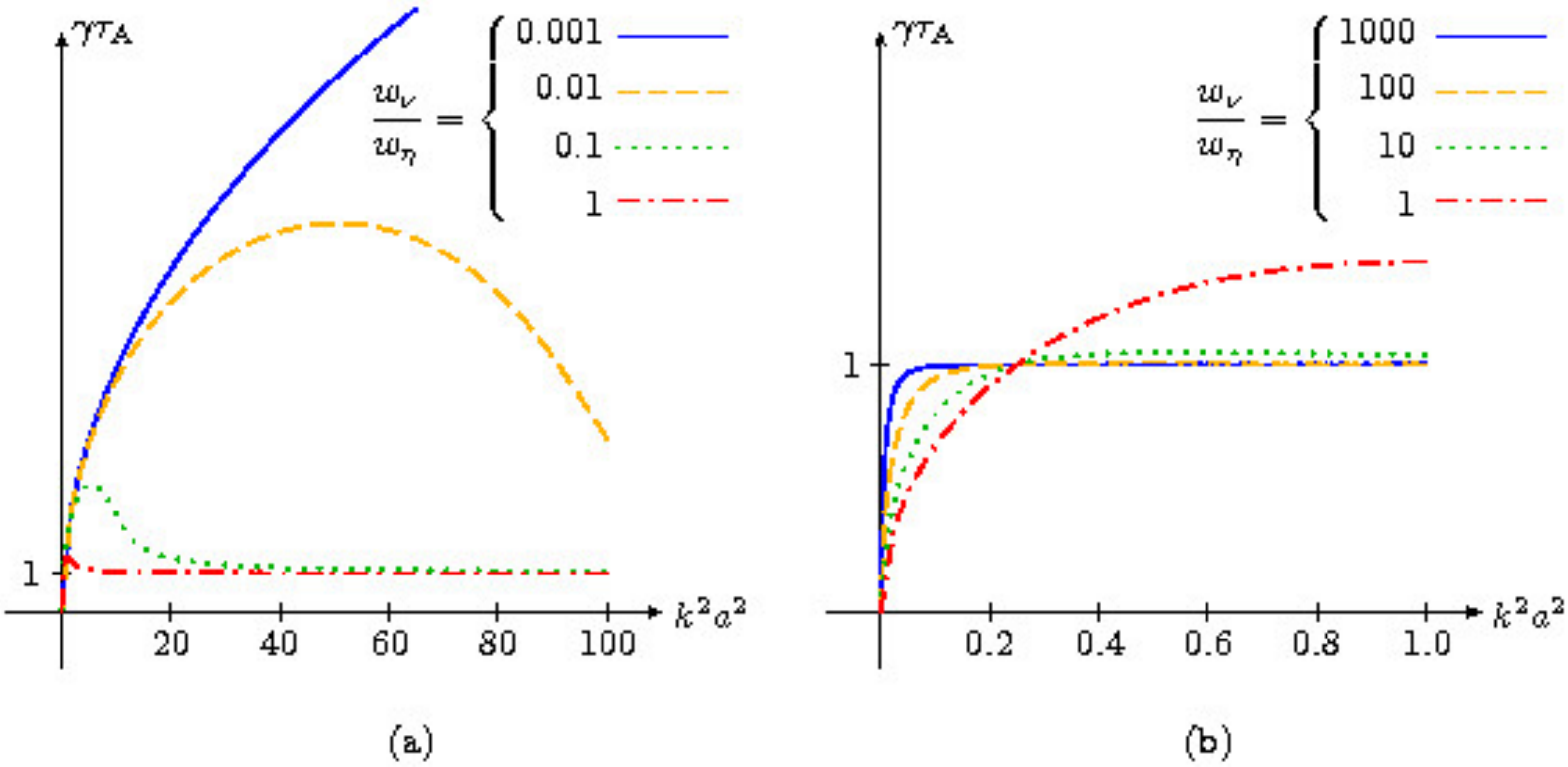
General behavior of dispersion relation



The End.

Thank you for your attention!

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