I – Equações Diferenciais

Referência Principal: *Caos*

N.F. Ferrara, C. P. C. Prado

Editora Edgard Blücher (1997)

Sistema de Equações Diferenciais

$$dx/dt = \dot{x} = f(x, y)$$

$$dy/dt = \dot{y} = g(x, y)$$

Equações de primeira ordem

Sistema autônomo

Trajetória no Espaço de Fase

$$\dot{r} = br(1 - r)$$

$$\dot{\theta} = 1$$

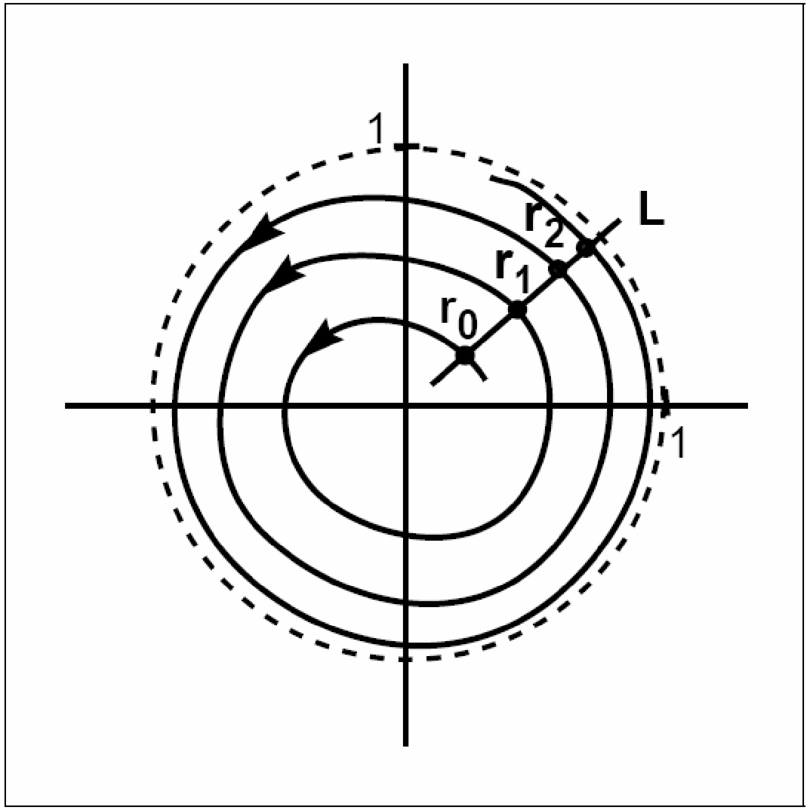


Figure 11.20 Poincaré map for a limit cycle of a planar system.
The system $\frac{dr}{dt} = 0.2r(1 - r)$, $\frac{d\theta}{dt} = 1$, has a limit cycle $r = 1$. The line segment L is approximately perpendicular to this orbit. Successive images $r_1, r_2, \ldots$, of initial point r_0 under the Poincaré map converge to $r = 1$.

Mapa de Poincaré:
$\{r_0, r_1, r_2, \ldots\}$

Atrator:
ciclo limite com r = 1

Chaos
Alligood et al.

Sistema Não Autônomo

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F/m \cos(\omega t)$$

Introduzindo as variáveis y, z

$$z = t \quad \rightarrow \dot{z} = 1$$

$$y = \dot{x}$$

obtemos

$$\dot{x} = y$$

$$\dot{y} = F/m \cos(\omega z) - \gamma y - \omega_0^2 x$$

$$\dot{z} = 1$$

Sistema autnômo

Exemplo de Sistema Integrável: pêndulo simples

$$m\, l^2\, \ddot{\theta} \;+\; m\, g\, l\, sen\theta \;=\; 0 \quad \rightarrow$$

$$\dot{\theta} \;=\; \varphi$$

$$\dot{\varphi} \;=\; -{}^{g}\!/_{l}\; sen\theta$$

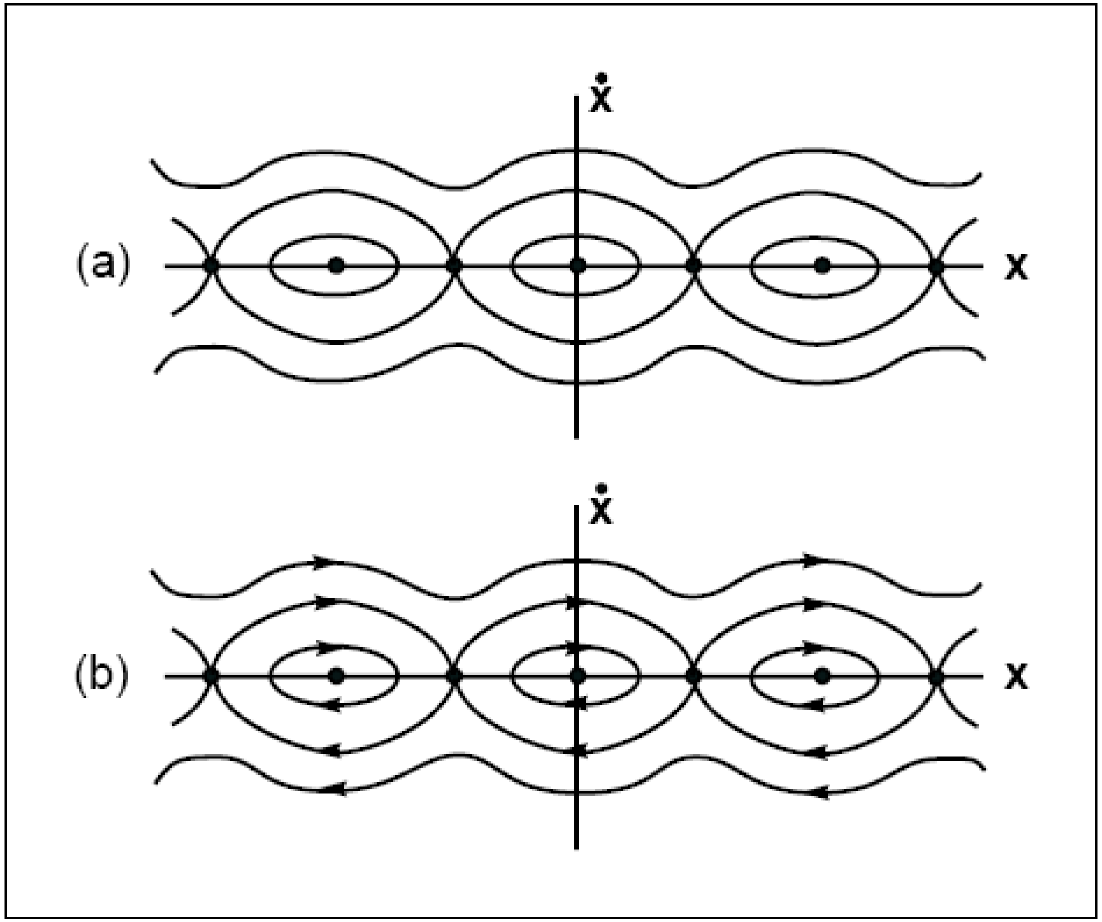
Equação das trajetórias no espaço de fase :

$$d\theta/d\varphi \;=\; \frac{\varphi}{(-{}^{g}\!/_{l}\,)}\, sen\,\theta \quad \rightarrow \quad -{}^{g}\!/_{l}\; sen\theta\, d\theta \;=\; \varphi\; d\;\varphi$$

$$\therefore\; \varphi^2 \;-\; a\cos\theta = C \quad ; \;\; a = 2\,{}^{g}\!/_{l} \qquad C \;=\; cte$$

$$(x \equiv \theta)$$

Espaco de Fase
Pêndulo Simples



Uma curva para
cada valor de C

Figure 7.14 Solution curves of the undamped pendulum.
(a) Level curves of the energy function. (b) The phase plane of the pendulum. The solutions move along level curves; equilibria are denoted by dots. The variable x is an angle, so what happens at x also happens at $x + 2\pi$. As a results, (a) and (b) are periodic in x with period 2π.

Oscilador Harmônico Amortecido

$$\ddot{x} + \gamma \dot{x} + \omega_0^{\,2} x = 0$$

$$\dot{x} = y$$

$$\dot{y} = -\gamma y - \omega_0^{\,2} x \quad \rightarrow \quad {dy}/{dx} = \frac{-\gamma y - \omega_0^{\,2} x}{y}$$

$$x(t) = Ae^{p_+ t} + Be^{p_- t}$$

$$p_\pm = \frac{1}{2}[-\gamma \pm \sqrt{\Delta}] \qquad \Delta = \gamma^2 - 4\omega_0^2$$

$$x(t) = a \exp[-{1}/{2}\,\gamma t \, \cos({\Delta}/{2\,t} + \alpha)]$$

a , α *dependem da condição inicial*

Ponto Fixo

$$dx/dt = \dot{x} = f(x^*, y^*) = 0$$

$$dy/dt = \dot{y} = g(x^*, y^*) = 0$$

$$O \ \ ponto \ \ P(x^*, y*) \ é \ um \ \ ponto \ \ fixo$$

Estabilidade do Ponto Fixo

Ponto P é assintóticamente estável se

$$\lim_{t \to \infty} (x, y) \to (x^*, y^*)$$

P é um atrator

Ponto P é estável se

$$\lim_{t \to \infty} (x, y) \approx (x^*, y^*)$$

Caso contrário é instável

Estabilidade Estrutural

As soluções da equação do pêndulo ideal

$$\ddot{x} + \omega_0^2 x = 0$$

são alteradas se considerarmos atrito $(\gamma \neq 0)$

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

Exemplos de Pontos Fixos

Referência Principal: *Chaos*

K. Alligood, T. D. Sauer, J. A. Yorke

Springer (1997)

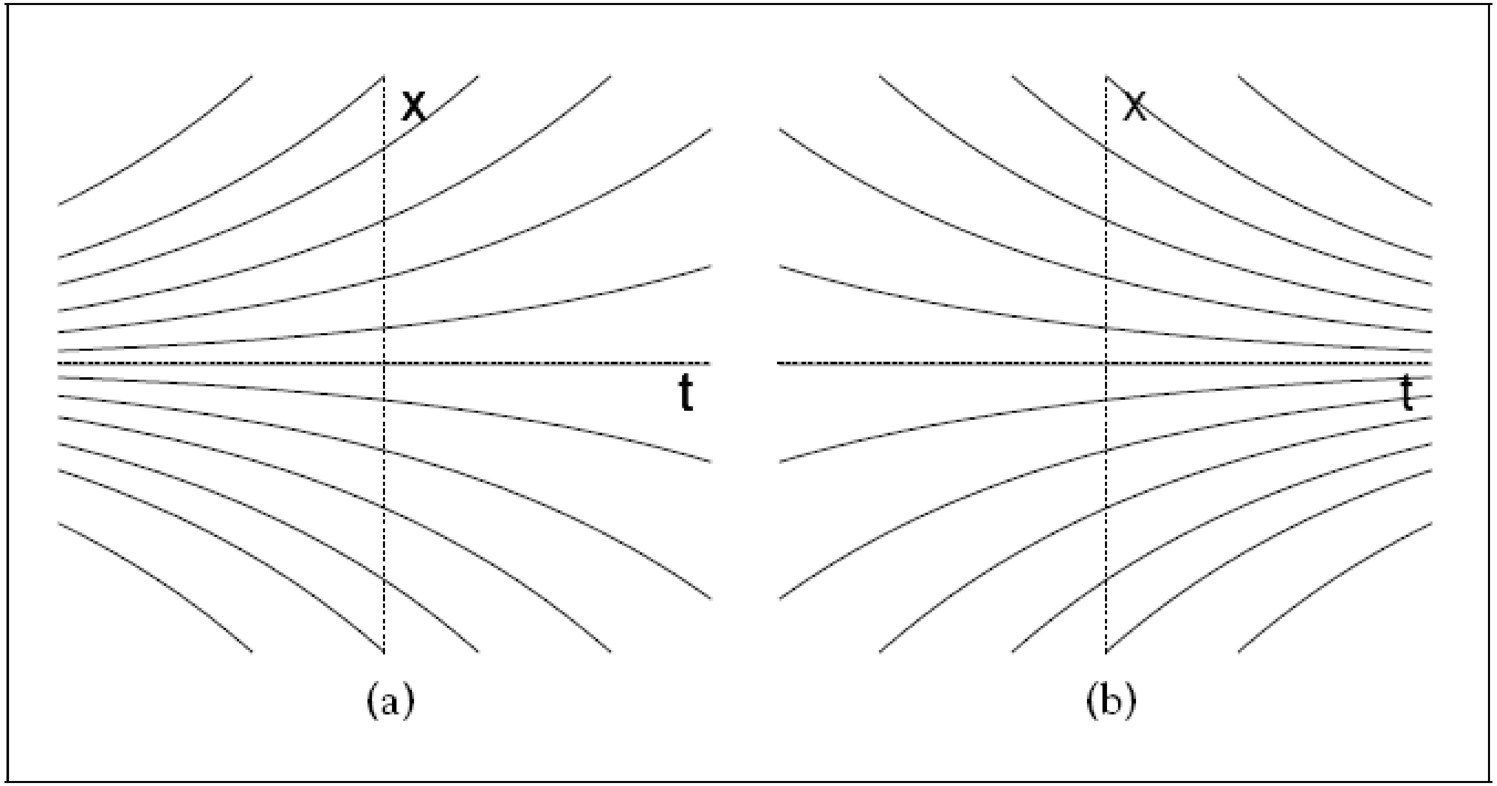


Figure 7.1 The family of solutions of $\dot{x} = ax$.
(a) $a > 0$: exponential growth (b) $a < 0$: exponential decay.

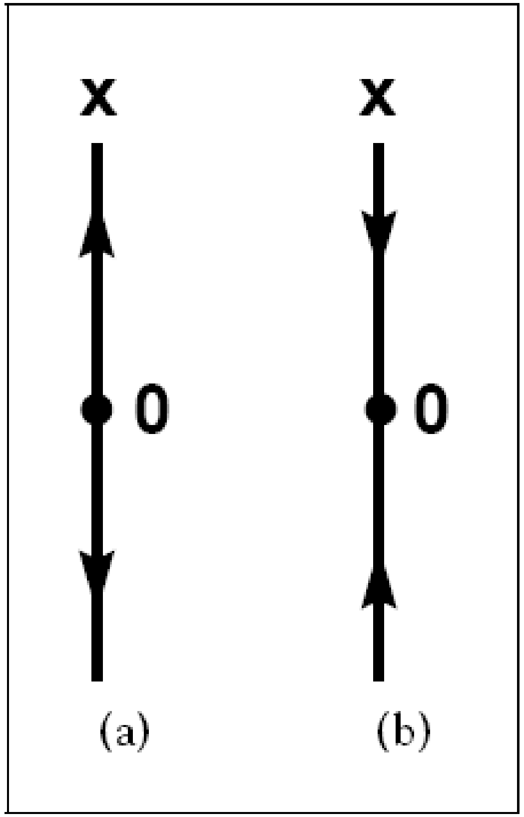


Figure 7.2 Phase portraits of $\dot{x} = ax$.
Since x is a scalar function, the phase space is the real line $\mathbb{R}$. (a) The direction of solutions is away from the equilibrium for $a > 0$. (b) The direction of solutions is toward the equilibrium for $a < 0$.

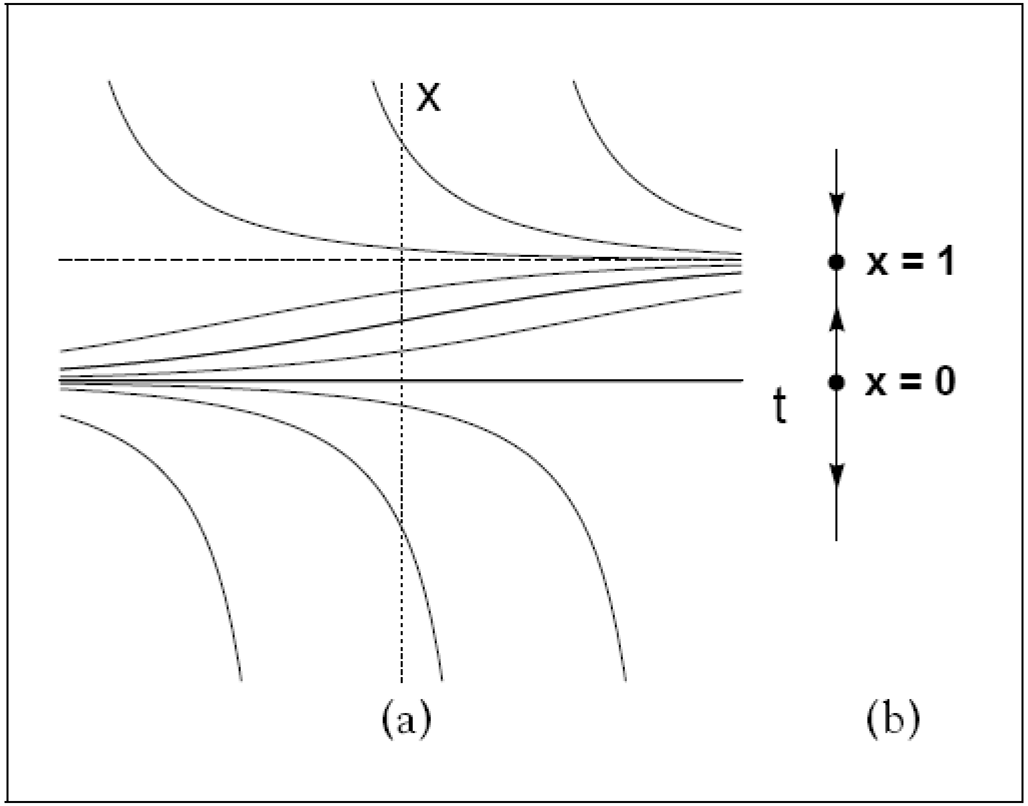


Figure 7.3 Solutions of the logistic differential equation.
(a) Solutions of the equation $\dot{x} = x(1 - x)$. Solution curves with positive initial conditions tend toward $x = 1$ as t increases. Curves with negative initial conditions diverge to $-\infty$. (b) The phase portrait provides a qualitative summary of the information contained in (a).

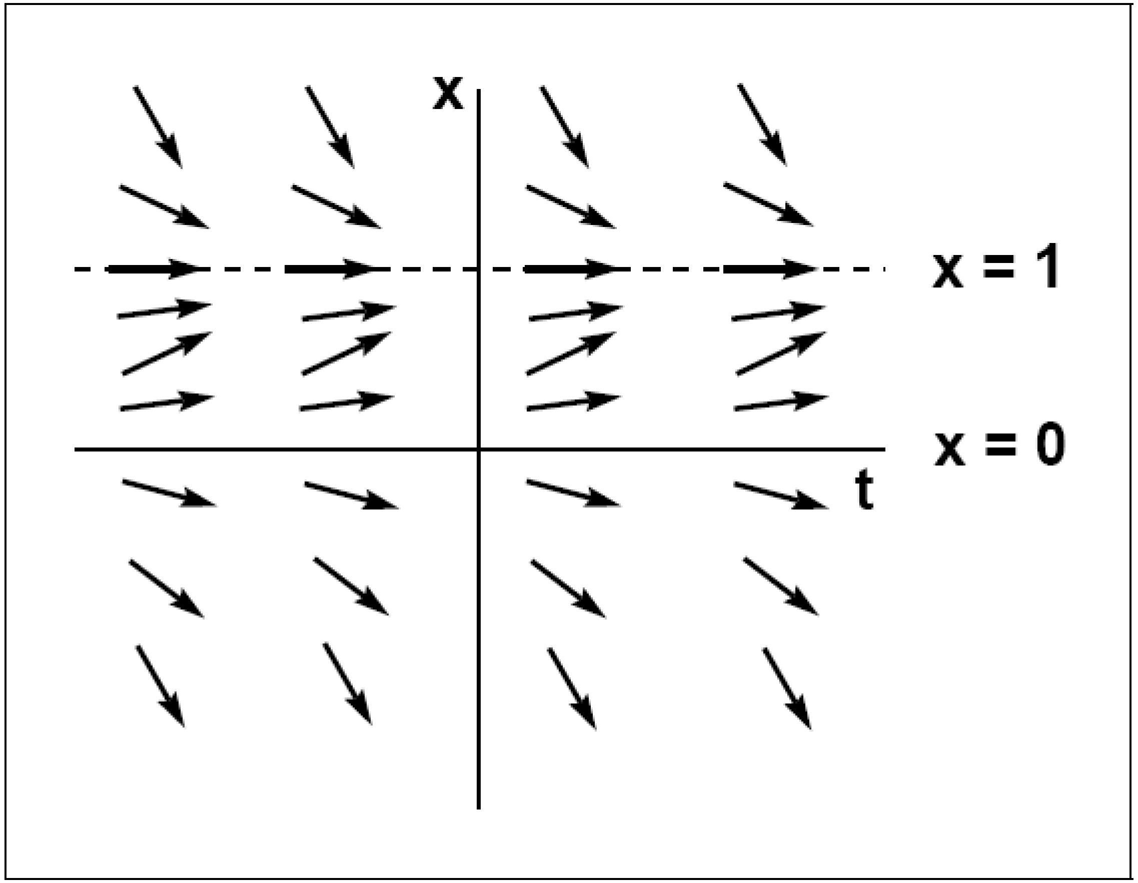


Figure 7.4 Slope field of the logistic differential equation.
At each point (t, x), a small arrow with slope $ax(1 - x)$ is plotted. Any solution must follow the arrows at all times. Compare the solutions in Figure 7.3.

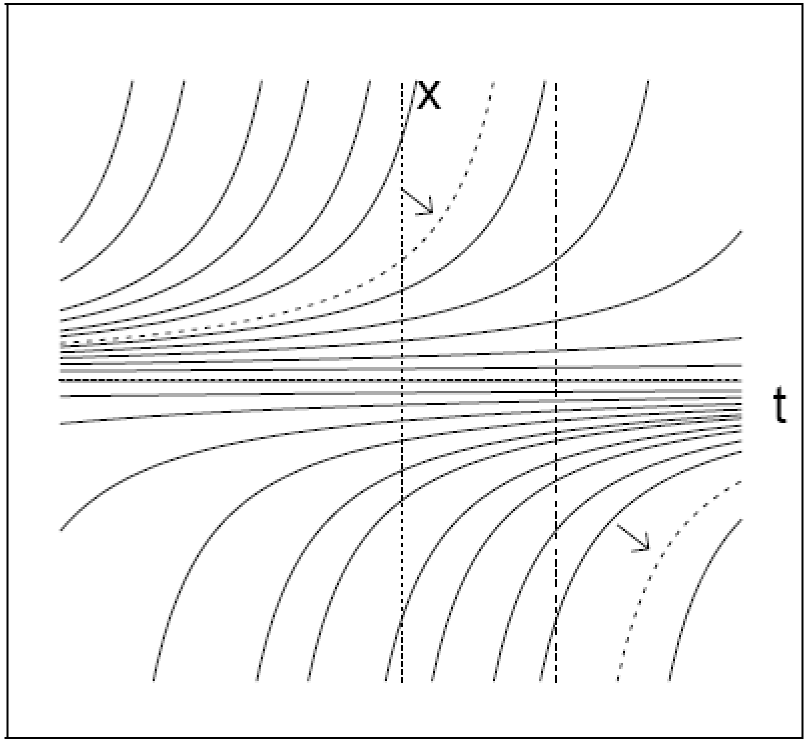


Figure 7.5 Solutions that blow up in finite time.
Curves shown are solutions of the equation $\dot{x} = x^2$. The dashed curve in the upper left is the solution with initial value $x(0) = 1$. This solution is $x(t) = 1/(1 - t)$, which has a vertical asymptote at $x = 1$, shown as a dashed vertical line on the right. The dashed curve at lower right is also a branch of $x(t) = 1/(1 - t)$, one that cannot be reached from initial condition $x(0) = 1$.

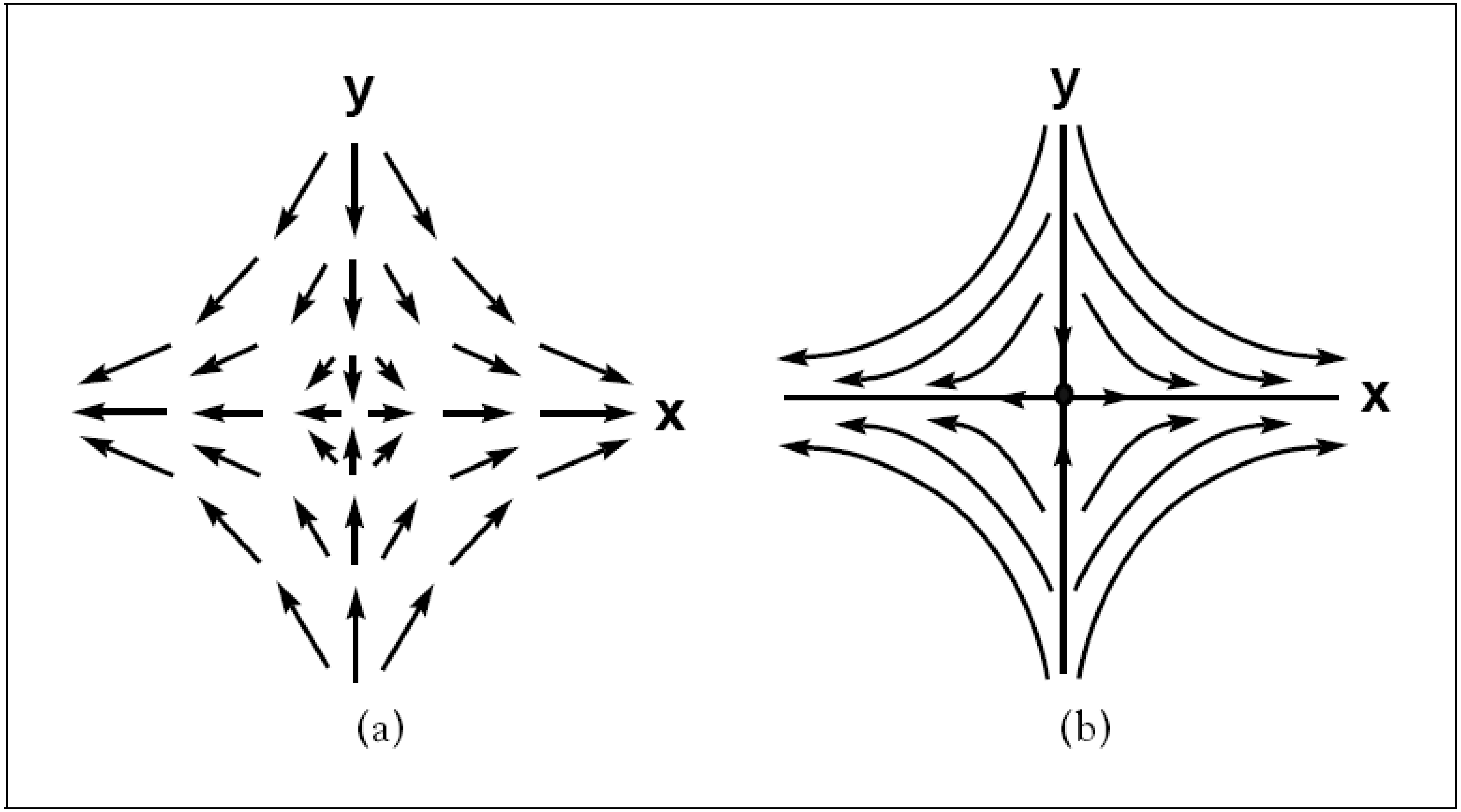


Figure 7.6 Vector field and phase plane for a saddle equilibrium.
(a) The vector field shows the vector $(\dot{x}, \dot{y})$ at each point (x, y) for (7.14). (b) The phase portrait, or phase plane, shows the behavior of solutions. The equilibrium $(x, y) \equiv (0, 0)$ is a saddle. The time coordinate is suppressed in a phase portrait.

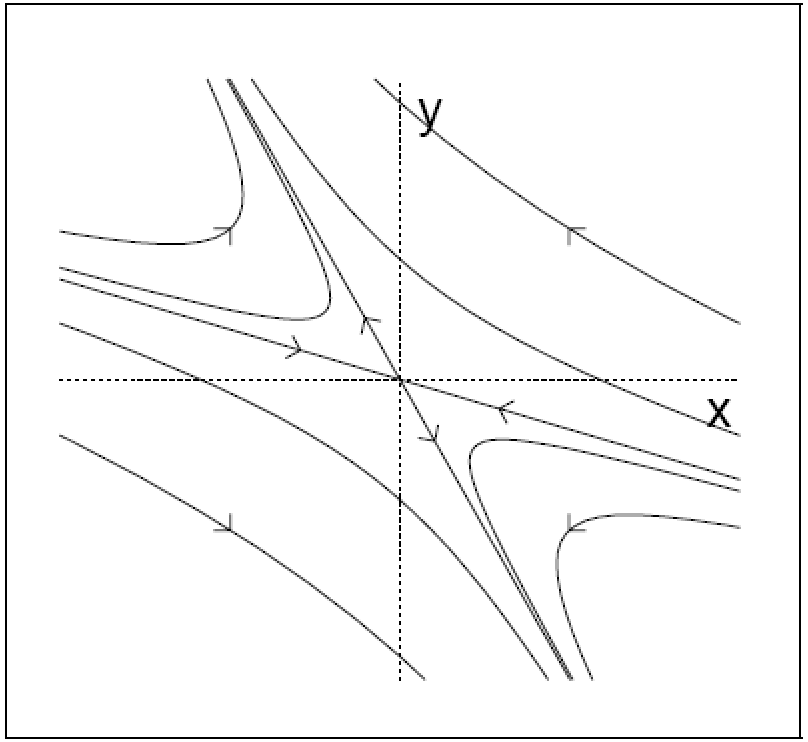


Figure 7.7 Phase plane for a saddle equilibrium.
For (7.16), the origin is an equilibrium. Except for two solutions that approach the origin along the direction of the vector $(3, -1)$, solutions diverge toward infinity, although not in finite time.

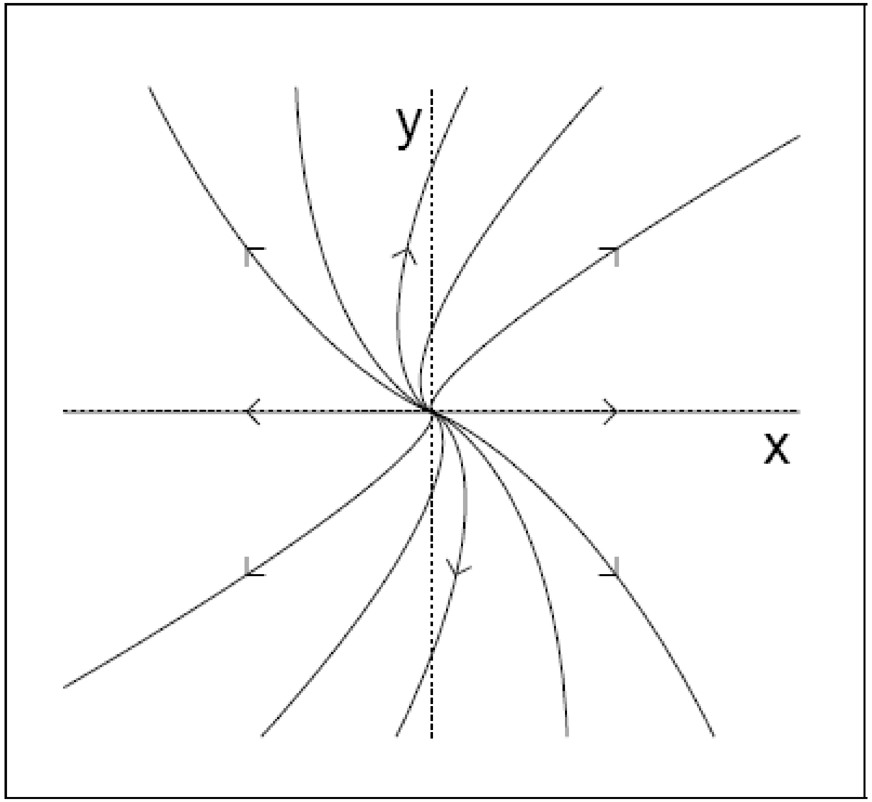


Figure 7.8 Phase plane for Equation (7.17).
The coefficient matrix **A** for this system has only one eigenvector, which lies along the *x*-axis. All solutions except for the equilibrium diverge to infinity.

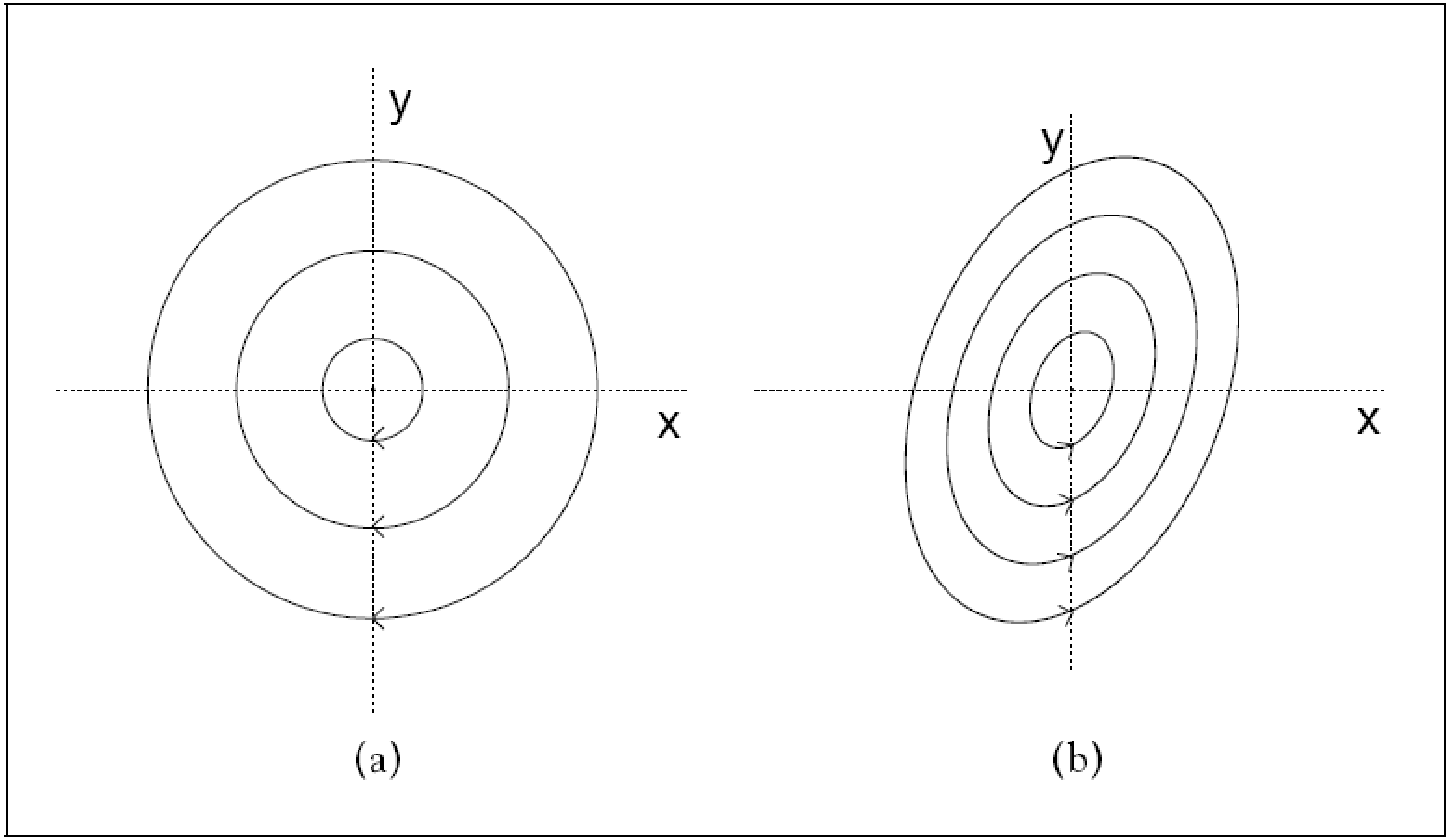


Figure 7.9 Phase planes for pure imaginary eigenvalues.
(a) In (7.18), the eigenvalues are $\pm i$. All solutions are circles around the origin, which is an equilibrium. (b) In (7.23), the eigenvalues are again pure imaginary. Solutions are elliptical. Note that for this equilibrium, some points initially move farther away, but not too far away. The origin is (Lyapunov) stable but not attracting.

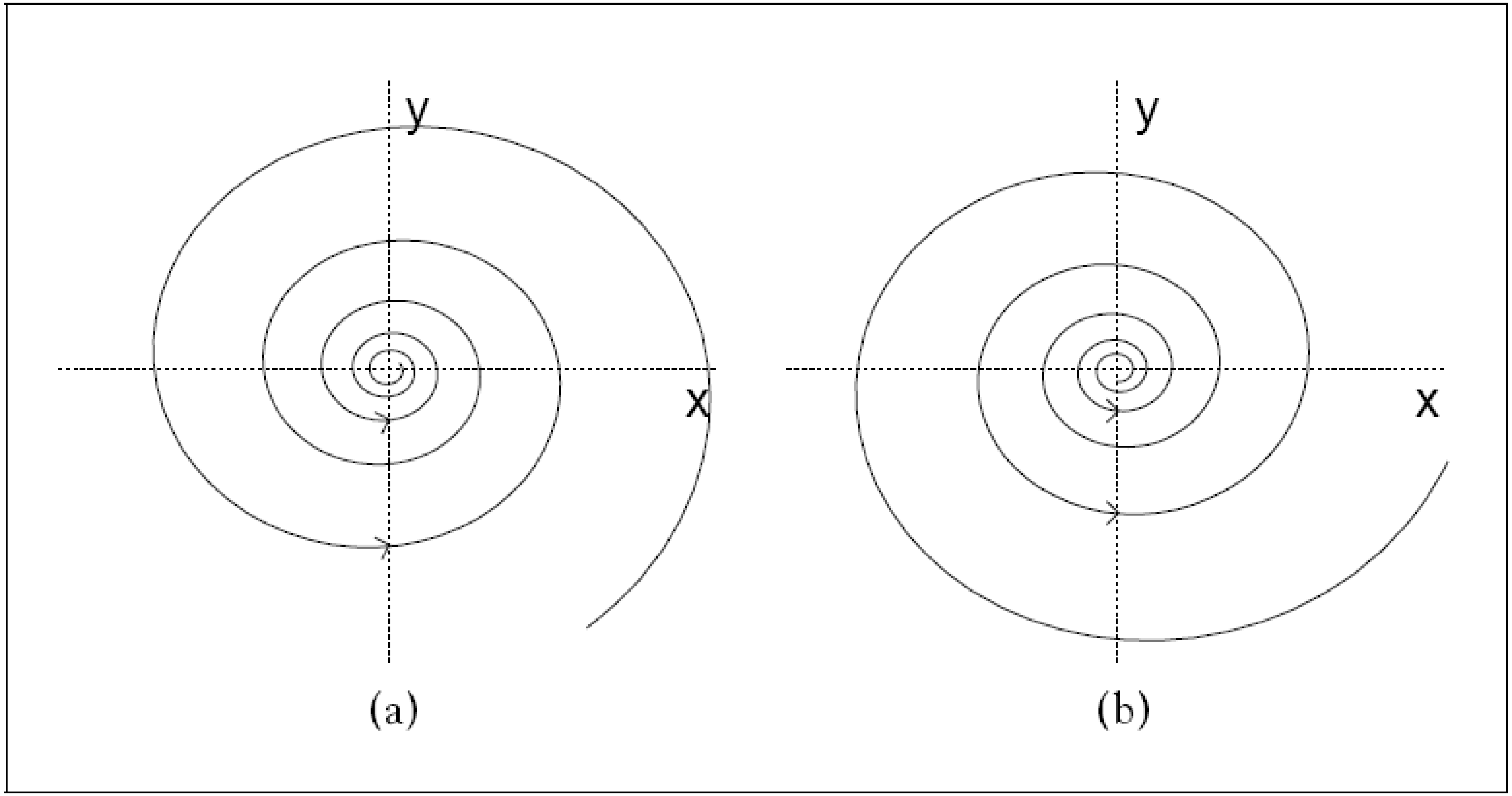


Figure 7.10 Phase planes for complex eigenvalues with nonzero real part. (a) Under (7.21), trajectories spiral in to a sink at the origin. The eigenvalues of the coefficient matrix A have negative real part. (b) For (7.22), the trajectories spiral out from a source at the origin.

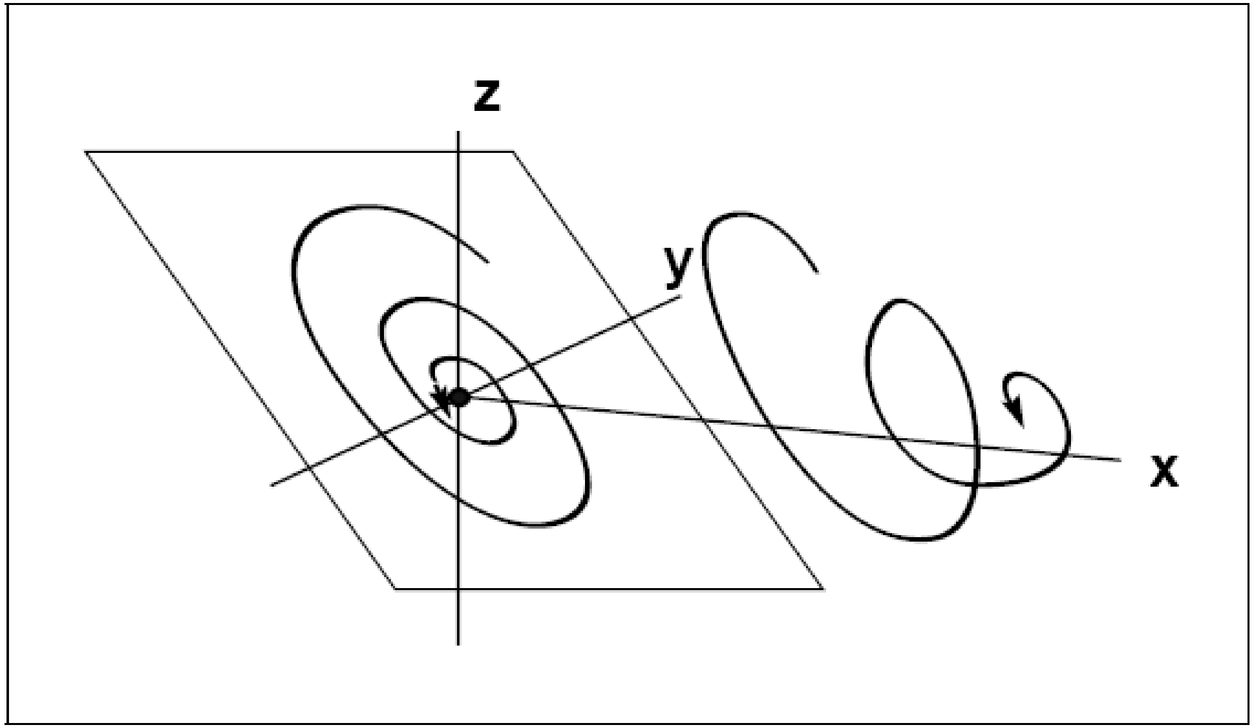


Figure 7.11 A three-dimensional phase portrait.
In Example 7.13, the origin $(0, 0, 0)$ is a saddle equilibrium. Trajectories whose initial values lie in the plane move toward the origin, and all others spiral away along the x-axis.

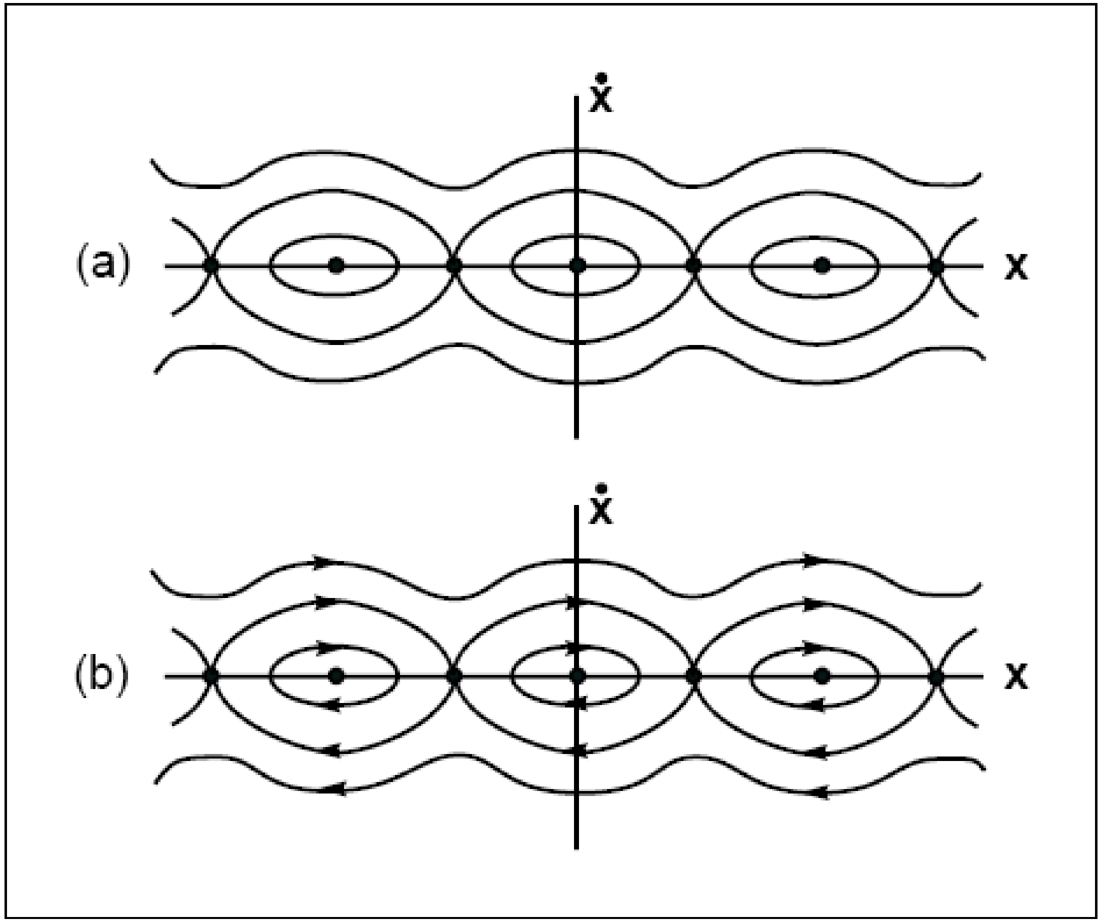


Figure 7.14 Solution curves of the undamped pendulum.
(a) Level curves of the energy function. (b) The phase plane of the pendulum. The solutions move along level curves; equilibria are denoted by dots. The variable x is an angle, so what happens at x also happens at $x + 2\pi$. As a results, (a) and (b) are periodic in x with period 2π.

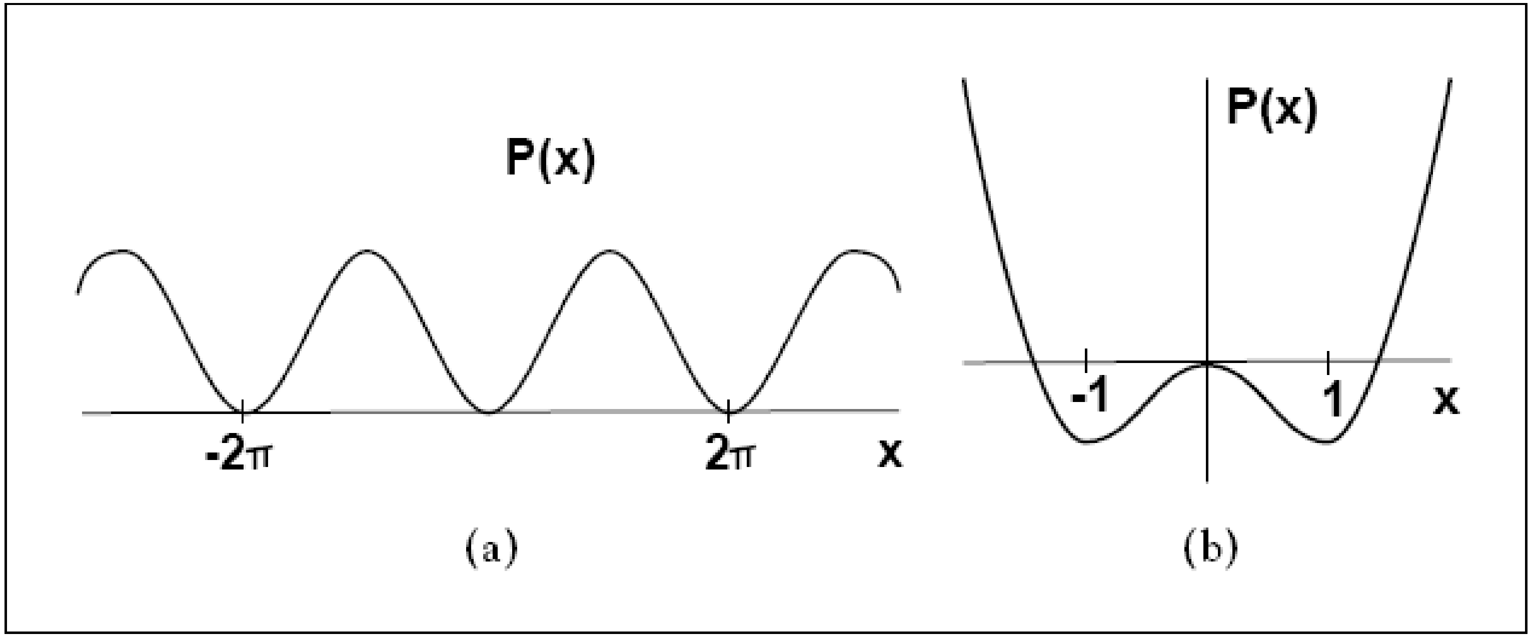


Figure 7.15 Potential energy functions.
(a) The potential function for the pendulum is $P(x) = 1 - \cos x$. There are infinitely many wells. (b) The double-well potential $P(x) = x^4/4 - x^2/2$.

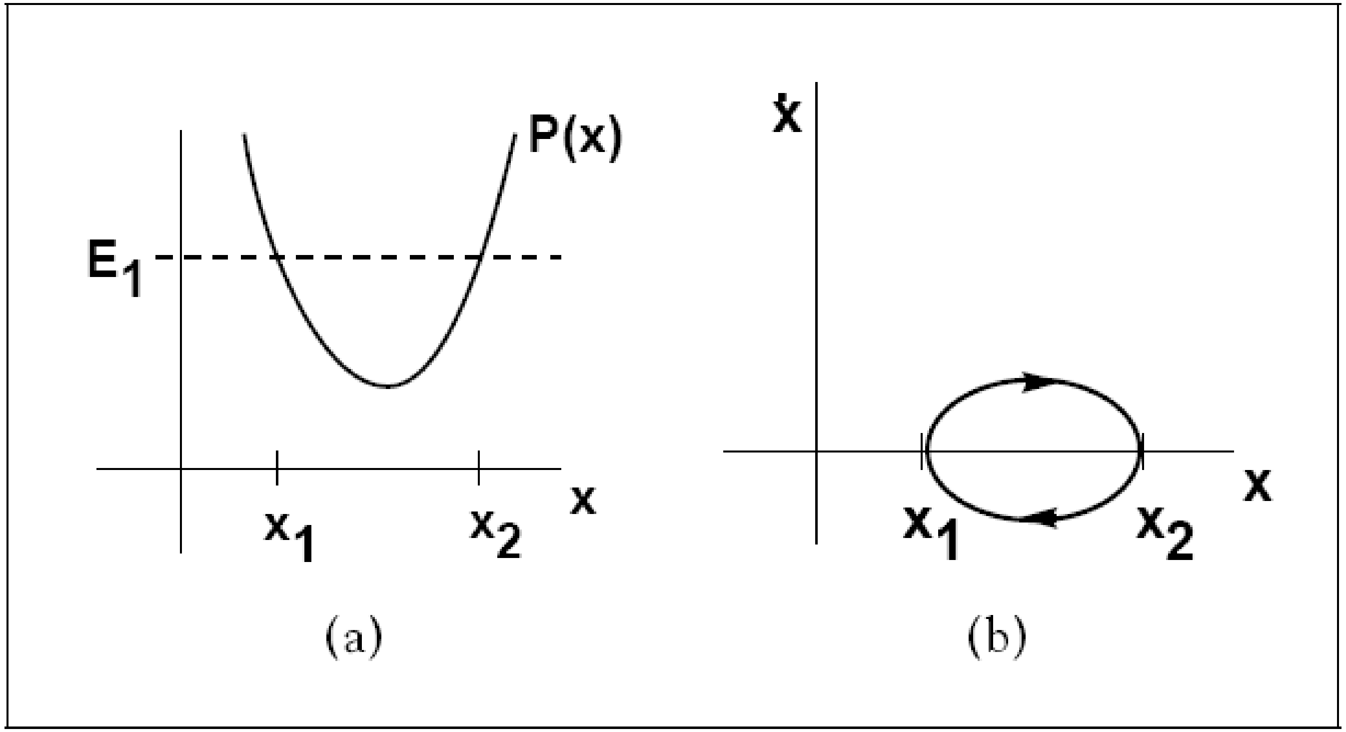


Figure 7.16 Drawing phase plane curves from the potential.
(a) Graph of the potential energy function $P(x)$. Each trajectory of the system is trapped in a potential energy well. The total energy $\dot{x}^2/2 + P(x)$ is constant for trajectories. As a trajectory with fixed total energy E_1 tries to climb out near x_1 or x_2, the kinetic energy $\dot{x}^2/2 = E_1 - P(x)$ goes to zero, as the energy E converts completely into potential energy. (b) A periodic orbit results: The system oscillates between positions x_1 and x_2.

Exemplos de Órbitas Periódicas e Conjuntos Limites

Referência Principal: *Chaos*

K. Alligood, T. D. Sauer, J. A. Yorke

Springer (1997)

Dimensão do espaço de fase limita as formas do comportamento assintótico das soluções dos sistemas autônomos.

Na linha, soluções limitadas convergem para um ponto de equilíbrio. No plano, elas convergem a ciclos limites.

No plano não há soluções caóticas.

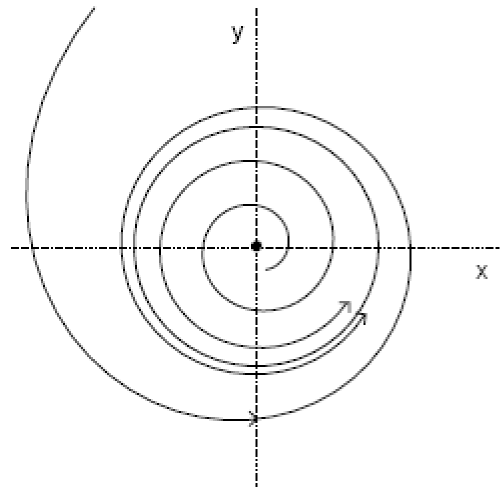
Teorema da curva de Jordan $\rightarrow$ Teorema de Poincaré-Bendixson

No espaço tridimensional pode haver caos.

Exemplo:

$$\dot{r} = r(1 - r)$$

$$\dot{\theta} = 8$$



Ponto de equilíbrio instável: (0, 0)

r = 1 órbita periódica $\rightarrow$ ciclo limite estável

$$r = \frac{c\,e^{t}}{c\,e^{t} - 1} \qquad \theta = 8\,t + d$$

Exemplo:

$$\dot{r} = -r(1 - r)$$

$$\dot{\theta} = 8$$

Ponto de equilíbrio estável: $(0,0)$

$r = 1$ órbita periódica instável $\rightarrow$ ciclo limite instável

Exemplo

$$\dot{x} = x(a - x), \quad a > 0$$

Pontos de equilíbrio: $x = 0$ e $x = a$

Para $x_0 > 0$, $\omega(x_0) = \{a\}$

Para $x_0 = 0$, $\omega(x_0) = \{0\}$

Para $x_0 < 0$, $\omega(x_0) = \{\ \}$: conjunto vazio

Exemplo

$$\dot{r} = r(a - r)$$

$$\dot{\theta} = b$$

Origem é ponto de equilíbrio instável

$$\omega(0) = \{0\}$$

$$\omega(r_0, \theta_0) = \{r = a\} \; ; \; r_0 \neq 0$$

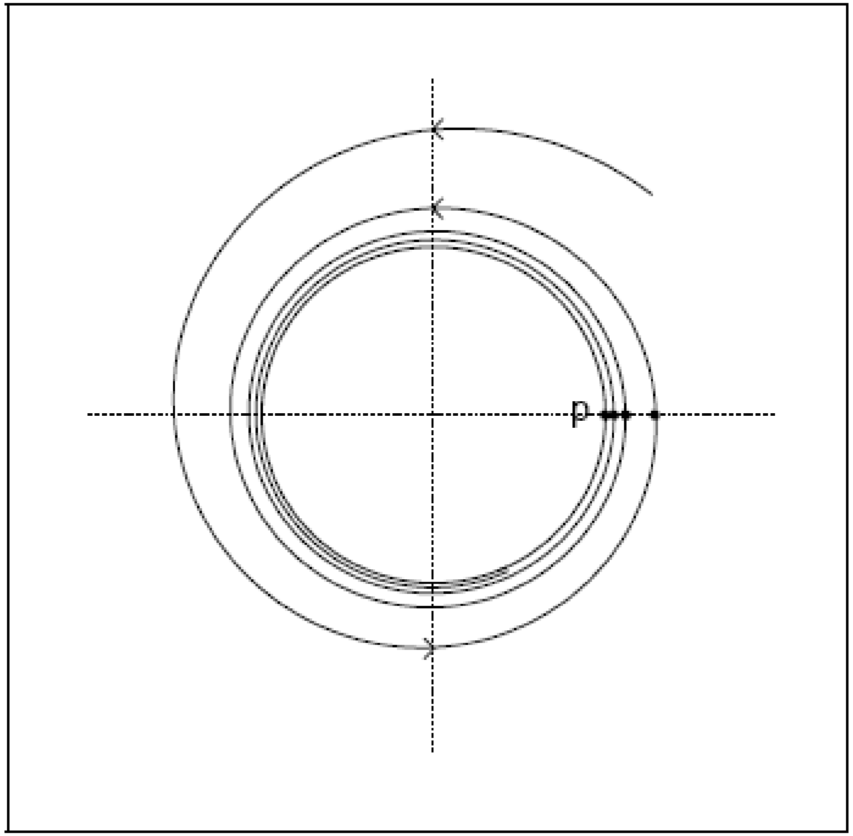


Figure 8.1 The definition of an ω-limit set.
The point **p** is in the ω-limit set of the spiraling trajectory because there are points $\mathbf{F}(t_1, \mathbf{v}_0)$, $\mathbf{F}(t_2, \mathbf{v}_0)$, $\mathbf{F}(t_3, \mathbf{v}_0)$ . . . of the trajectory, indicated by dots, that converge to **p**. The same argument can be made for any point in the entire limiting circle of the spiral solution, so the circle is the ω-limit set.

Exemplo

$$\dot{r} = r(a - r)$$

$$\dot{\theta} = \mathrm{sen}^2\theta + (r - a)^2$$

$$\text{Pontos de equilíbrio}: \begin{cases} (a, 0) \\ (a, \pi) \\ (0, 0) \end{cases}$$

$$\omega(0) = \{0\}$$

$$\omega(r_0, \theta_0) = \{r = a\} \; ; \quad \forall \; r_0 \neq 0$$

$$\omega(a, \theta_0) = (a, \; \theta = 0, \pi)$$

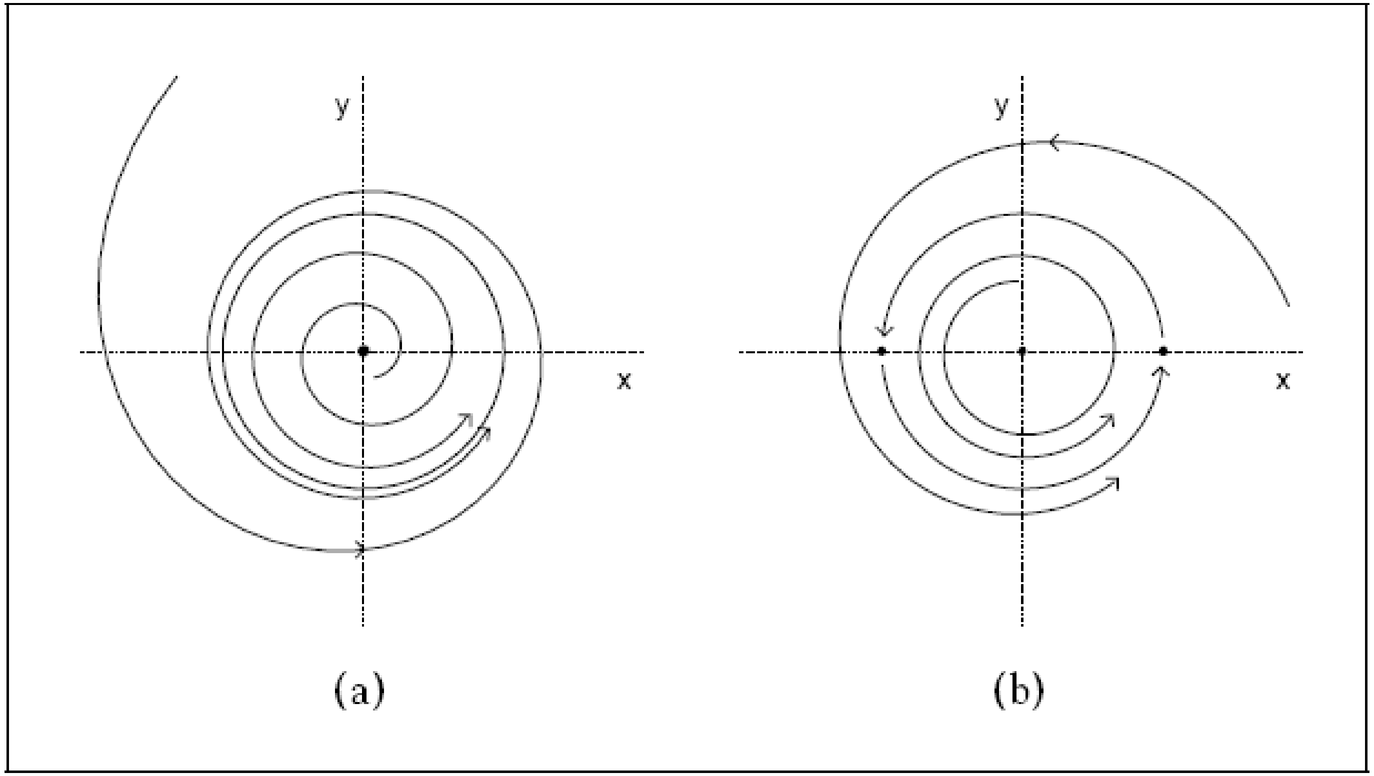


Figure 8.3 Examples of ω-limit sets for planar flows.
(a) The phase plane for Example 8.5 shows the circle $r = a$ as an attracting periodic orbit of system (8.3). The origin is an unstable equilibrium. The ω-limit set of every trajectory except the equilibrium is the periodic orbit. (b) The phase plane for Example 8.6 looks very similar to the phase plane in (a), except that in this example there are no periodic orbits. There are three equilibria: the origin and the points $(a, 0)$ and (a, π). Every other point on the circle $r = a$ is on a solution called a connecting arc, whose α- and ω-limit sets are the equilibria. The ω-limit set of each nonequilibrium solution not on the circle is the circle $r = a$.

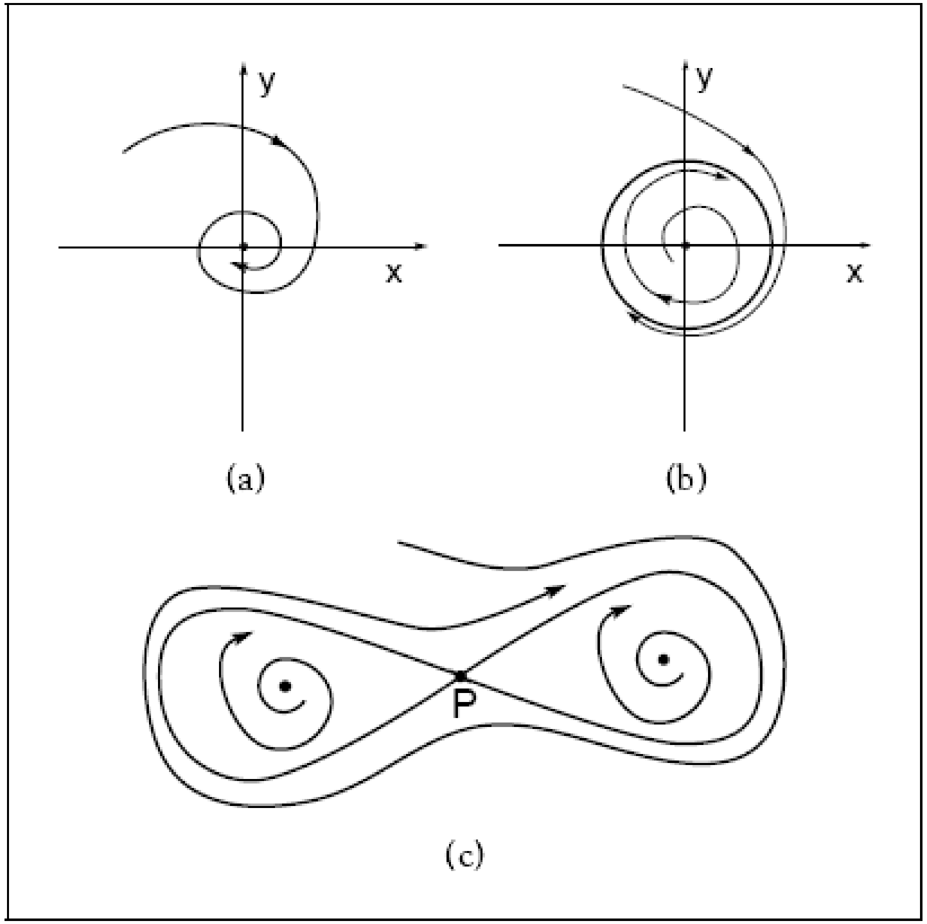


Figure 8.4 Planar limit sets.
The three pictures illustrate the three cases of the Poincaré-Bendixson Theorem. (a) The limit set is one point, the origin. (b) The limit set of each spiraling trajectory is a circle, which is a periodic orbit. (c) The limit set of the outermost trajectory is a figure eight. This limit set must have an equilibrium point P at the vertex of the "eight". It consists of two connecting arcs plus the equilibrium. Trajectories on the connecting arcs tend to P as $t \longrightarrow \infty$ and as $t \longrightarrow -\infty$.

Duas freqências incomensuráveis formam um torus T^2

Movimento preenche uma superfície toroidal
em um volume tridimensional.

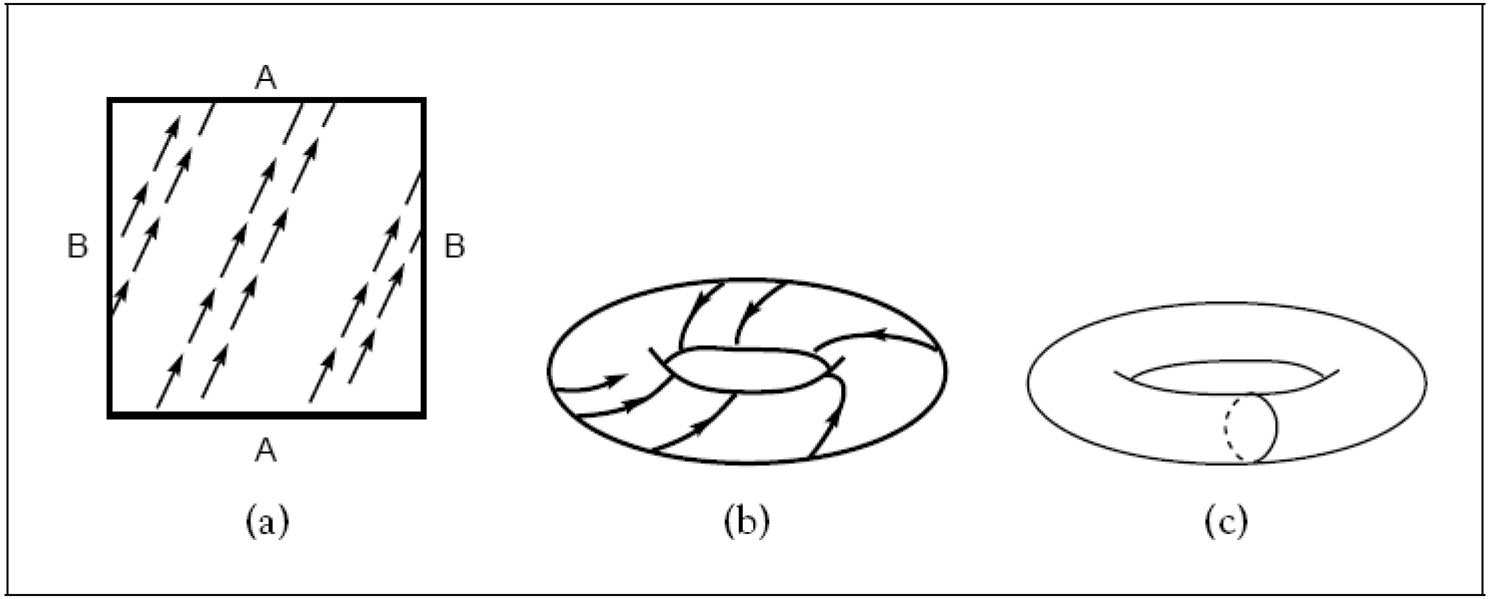


Figure 8.9 A dense orbit on the torus.