Ressonância
Ilha no Espaço de Fase

Mecânica Clássica

The Physics of Chaos in Hamiltonian Systems

G. Zaslavsky, 2nd edition

2.1 Nonlinear Resonance and Chain of Islands

The perturbed motion of a system with one degree of freedom by Hamiltonian in action-angle variables is

$$H = H_0(I) + \epsilon V(I, \theta, t)\,, \tag{2.1.1}$$

where $\epsilon \ll 1$ is a small dimensionless parameter of perturbation and V is periodic in time with the period $T = 2\pi/\nu$. Therefore, V can be expanded in a double Fourier series in θ and t:

$$\begin{aligned} V(I, \theta, t) &= \frac{1}{2} i\epsilon \sum_{k,\ell} V_{k,\ell}(I) e^{i(k\theta - \ell\nu t)} + \text{c.c.} \\ V^*_{k,\ell} &= V_{-k,-\ell}\,, \end{aligned} \tag{2.1.2}$$

where c.c. designates the complex conjugate terms.

$$\dot{I} = -\epsilon\frac{\partial V}{\partial \theta} = -\frac{1}{2}\sum_{k,\ell} kV_{k,\ell}(I)e^{i(k\theta-\ell\nu t)} + \text{c.c.}$$

$$\dot{\theta} = \frac{\partial H}{\partial I} = \frac{dH_0}{dI} + \epsilon\frac{\partial V(I,\theta,t)}{\partial I} = \omega(I) + \frac{1}{2}\epsilon\sum_{k,\ell}\frac{\partial V_{k,\ell}(I)}{\partial I}e^{i(k\theta-\ell\nu t)} + \text{c.c.}\,, \tag{2.1.3}$$

where we introduce the frequency of oscillations for unperturbed motion:

$$\omega(I) = \frac{dH_0}{dI}\,. \tag{2.1.4}$$

The resonance condition implies that the following equation must be satisfied:

$$k\omega(I) - \ell\nu = 0 \tag{2.1.5}$$

where k and ℓ are integers. This means that we have to specify a pair of integers, (k_0, ℓ_0), and the corresponding value, I_0, such that (2.1.5) converts into the following identity:

$$k_0\omega(I_0) = \ell_0\nu\,. \tag{2.1.6}$$

As a rule, the values $(k_0, \ell_0; I_0)$ can be found in abundance. This is due largely to the system's nonlinearity, that is, to the dependence of $\omega(I)$ on I.

The way to deal with Eqs. (2.1.3) is to analyse certain simplified situations. First, we examine an isolated resonance (2.1.6) and ignore all other possible resonances. This means that in the equations of motion (2.1.3), only terms with $k = \pm k_0$, $\ell = \pm \ell_0$ should be retained which satisfy the resonance condition at $I = I_0$:

$$\begin{aligned} \dot{I} &= \epsilon k_0 V_0 \sin(k_0 \theta - \ell_0 \nu t + \phi) \\ \dot{\theta} &= \omega(I) + \epsilon \frac{\partial V_0}{\partial I} \cos(k_0 \theta - \ell_0 \nu t + \phi) \end{aligned} \tag{2.1.7}$$

where we set

$$V_{k_0,\ell_0} = |V_{k_0,\,\ell_0}| e^{i\phi} = V_0 e^{i\phi} \,. \tag{2.1.8}$$

It is also assumed that the value

$$\Delta I = I - I_0 \tag{2.1.9}$$

is small, that is, Eq. (2.1.6) is examined in the vicinity of the resonance value of the action I_0.

A list of the typical approximations is as follows:

(i) Set $V_0 = V_0(I_0)$ on the right-hand sides of Eqs. (2.1.7);

(ii) Expand the frequency $\omega(I)$ using (2.1.9):

$$\omega(I) = \omega_0 + \omega' \Delta I \tag{2.1.10}$$

where

$$\omega_0 = \omega(I_0)\,, \quad \omega' = d\omega(I_0)/dI\,; \tag{2.1.11}$$

(iii) Neglect the second term of the order of ϵ in the second equation of Eq. (2.1.7) for frequency.

Finally, we reduce system (2.1.7) to a simplified version:

$$\begin{aligned} \frac{d}{dt}(\Delta I) &= -\epsilon k_0 V_0 \sin\psi \\ \frac{d}{dt}\psi &= k_0 \omega' \Delta I \end{aligned} \tag{2.1.12}$$

where a new phase is introduced:

$$\psi = k_0\theta - \ell_0 \nu t + \phi - \pi\,. \tag{2.1.13}$$

The set of equations in (2.1.12) can be presented in a Hamiltonian form:

$$\frac{d}{dt}(\Delta I) = -\frac{\partial \bar{H}}{\partial \psi}\,; \quad \frac{d}{dt}\psi = \frac{\partial \bar{H}}{\partial(\Delta I)} \tag{2.1.14}$$

where

$$\bar{H} = \frac{1}{2}k_0\omega'(\Delta I)^2 - \epsilon k_0 V_0 \cos\psi\,. \tag{2.1.15}$$

$$\ddot{\psi} + \Omega_0^2 \sin\psi = 0\,, \tag{2.1.16}$$

where the "small amplitude" oscillation frequency is

$$\Omega_0 = (\epsilon k_0^2 V_0 |\omega'|)^{1/2}\,. \tag{2.1.17}$$

The value Ω_0 is also the frequency of phase oscillations. The formulas obtained in Section 1.4 and Appendix 1 for the nonlinear pendulum can be automatically applied to Hamiltonian (2.1.15).

Semi largura da ilha $L = 2\sqrt{\dfrac{\varepsilon V_0}{d\omega/dI}}$ $\omega = \dfrac{dH_0}{dI}$

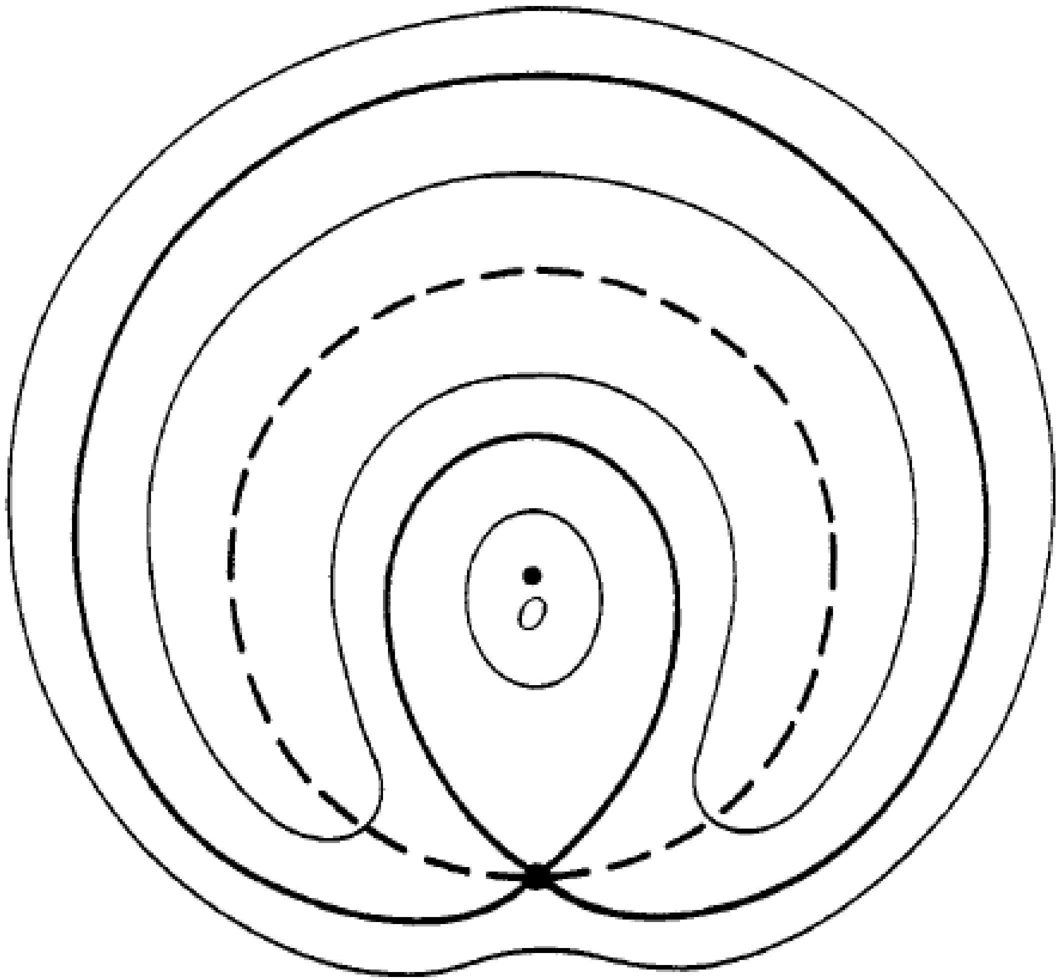


Fig. 2.1.1. The nonlinear resonance ($k_0 = \ell_0 = 1$) on the phase plane (p, q). The dashed line is the unperturbed trajectory with $I = I_0$. The thick line is a new separatrix of the phase oscillations.

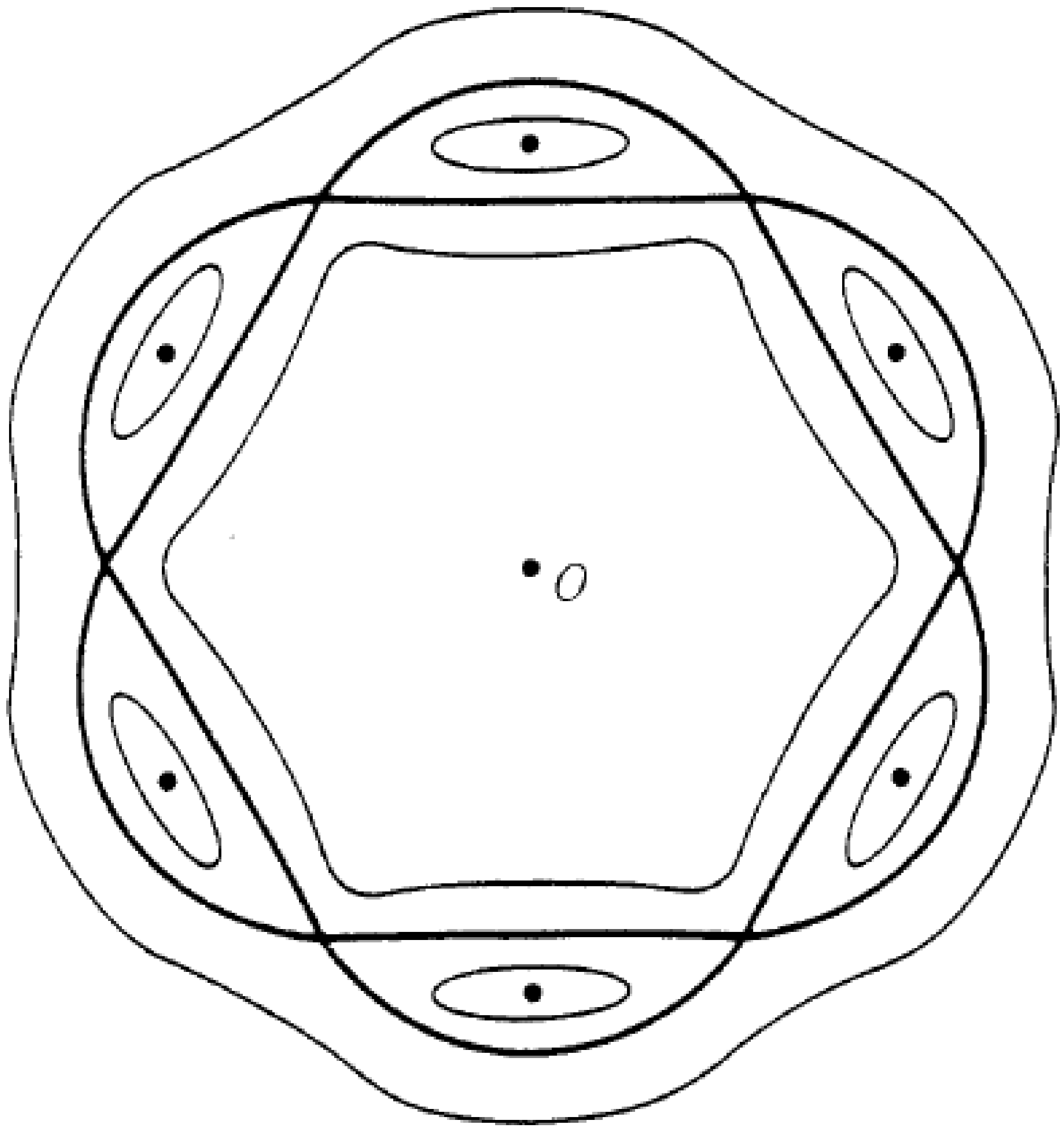


Fig. 2.1.2. Another version of the nonlinear resonance ($k_0 = 6,\ \ell_0 = 1$).

Pêndulo Forçado

$$H_0 = \frac{1}{2}\dot{x}^2 - \omega_0^2 \cos x$$

knowledge of unperturbed pendulum dynamics is required. The phase portrait of the pendulum (1.4.2) is shown in Fig. 1.4.2. Its main feature is the existence of a separatrix at the energy

$$H_0 = H_s = \omega_0^2 \tag{1.4.4}$$

with a corresponding solution for the co-ordinate

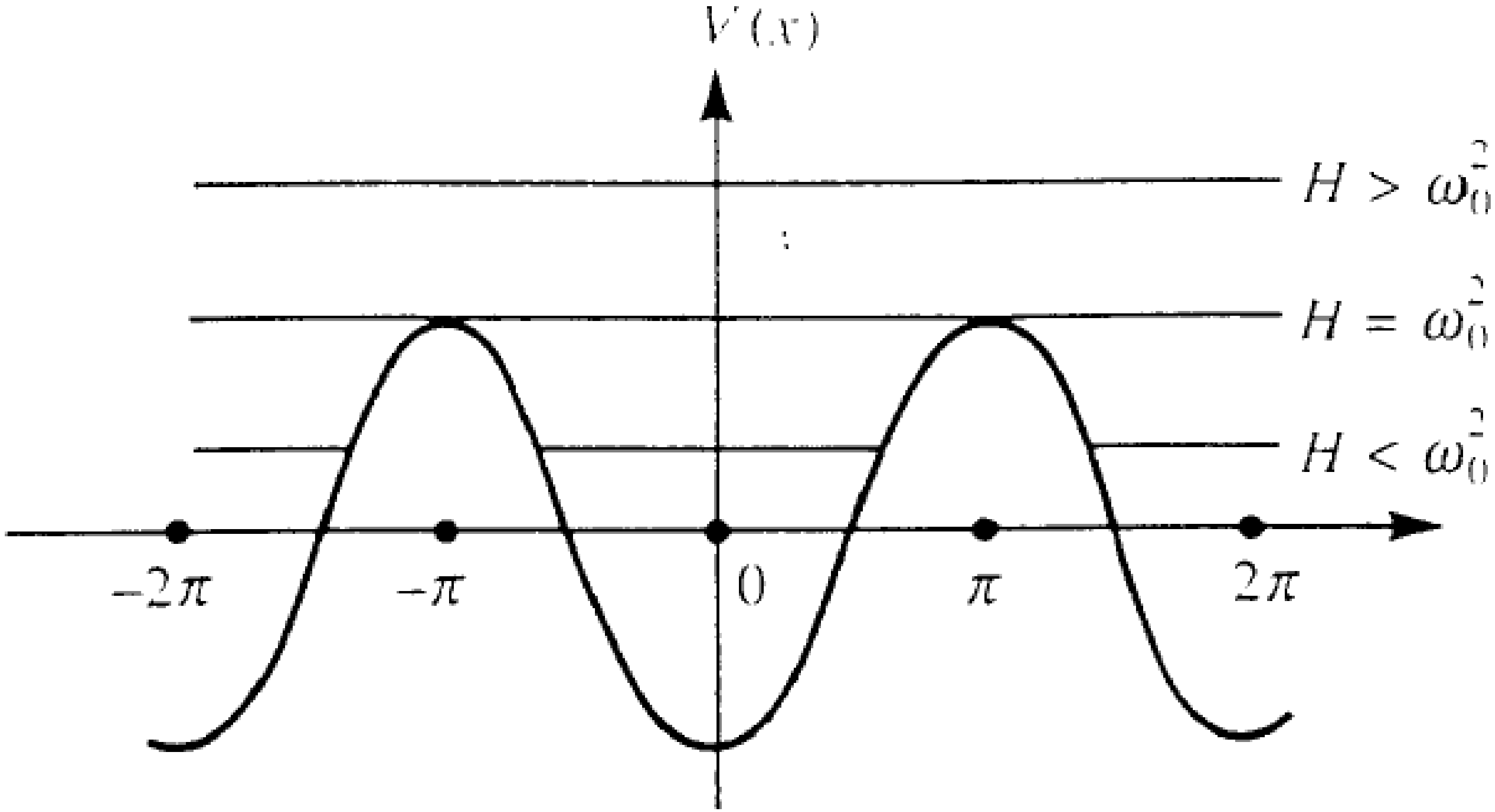
$$x = 4 \arctan \exp(\pm\omega_0 t) - \pi \tag{1.4.5}$$

(it has a form called a "kink") and for the momentum p or velocity if the mass is equal to one,

$$p = v = \dot{x} = \pm\frac{2\omega_0}{\cosh \omega_0 t}, \tag{1.4.6}$$

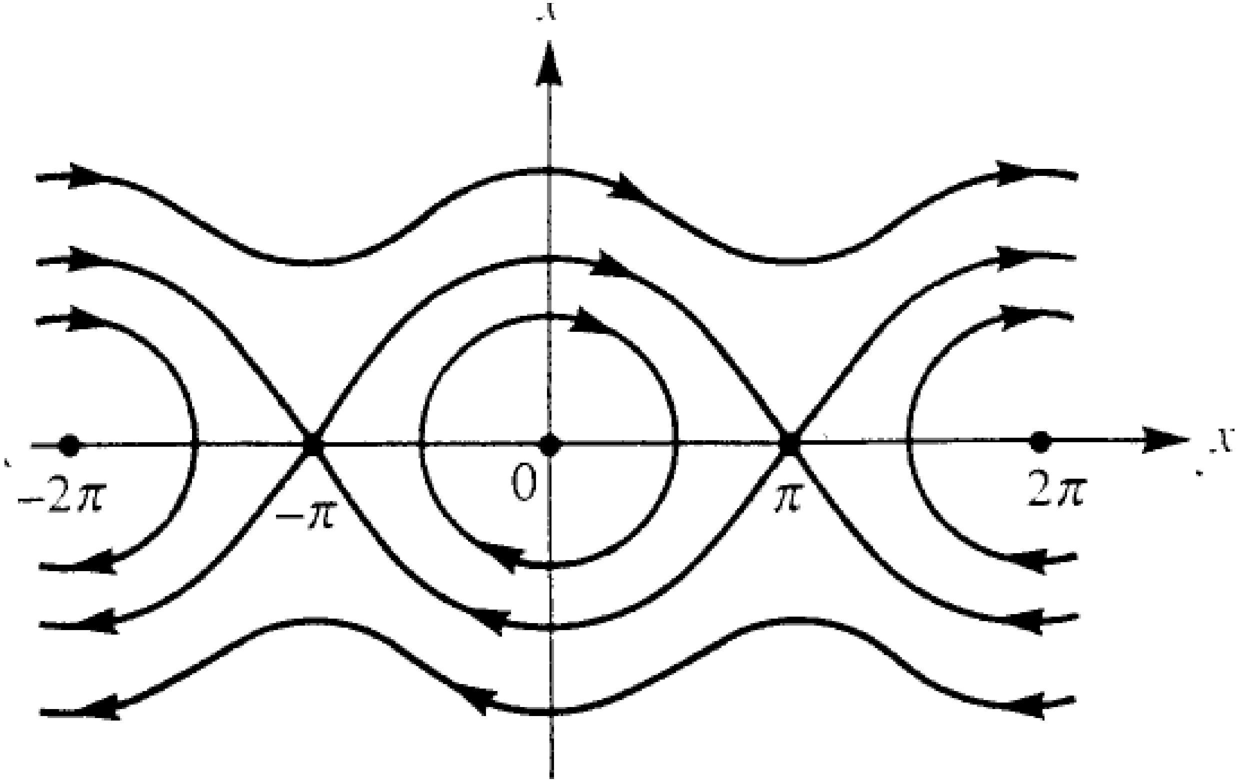
Here ω_0 is the frequency of small oscillations in the unperturbed pendulum with Hamiltonian

$$H_0 = \frac{1}{2}\dot{x}^2 - \omega_0^2 \cos x \,, \tag{1.4.2}$$



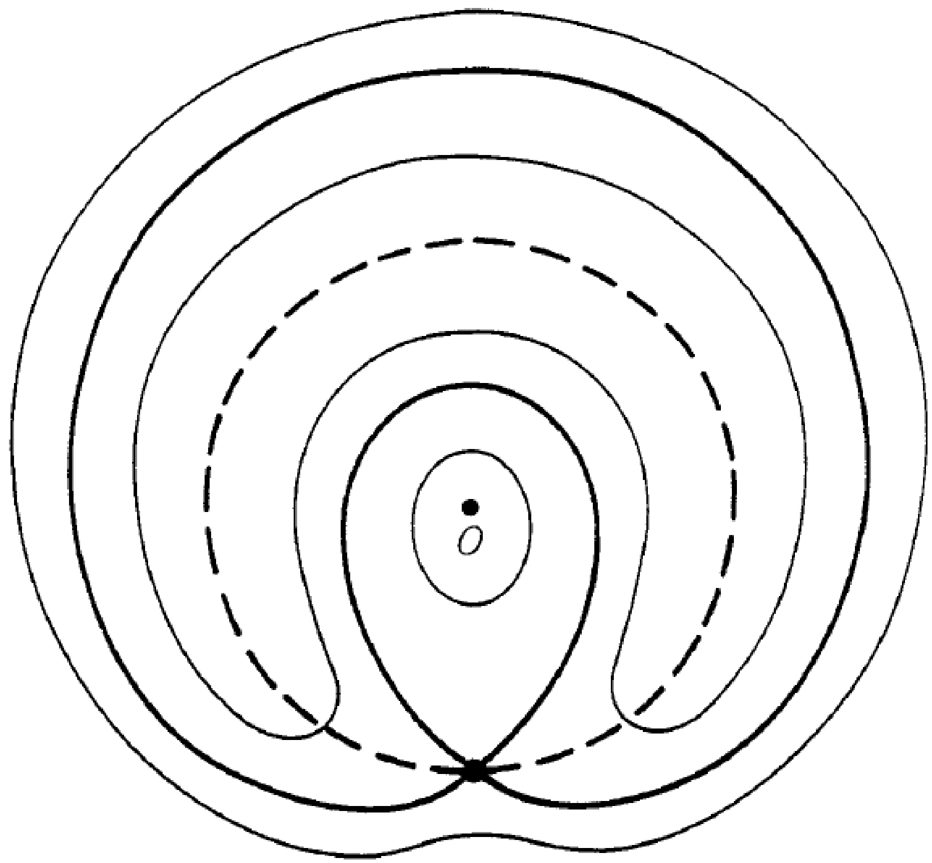
When considering the perturbed oscillators (4.1.9) and (4.1.11), the polar co-ordinates are

$$x = \rho \sin\phi\,; \quad \dot{x} = \omega_0 \rho \cos\phi\,. \tag{4.2.1}$$



(b)

Fig. 1.4.2. Unperturbed pendulum: (a) periodic potential; (b) the phase portrait.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is presented in Appendix 1 where the action-angle variables (I, θ) defined by the relations

$$
\begin{aligned}
I &= \frac{1}{2\pi} \oint p dx \\
\theta &= \frac{\partial S(x, I)}{\partial I} = \frac{\partial}{\partial I} \int^{x} dp\, x
\end{aligned}
\tag{1.4.7}
$$

with

$$
p = \pm[2(H_0 + \omega_0^2 \cos x)]^{1/2} \tag{1.4.8}
$$

are used.

The integration in the definition of I is performed over the period of motion $2\pi/\omega(I)$ where the nonlinear frequency

$$
\omega(I) = dH_0(I)/dI \tag{1.4.9}
$$

is introduced and the condition $H_0 < \omega_0^2$ is applied. In the case of $H_0 \geq \omega_0^2$, the integration in (1.4.7) is performed over the interval $x \in (0, 2\pi)$ for both positive and negative p. The details are found in Appendix 1.

1.4 Perturbed Pendulum

The perturbed pendulum is a typical model of the continuous equation of motion (no kicks) where one can still introduce a map. The Hamiltonian of the model is

$$H = \frac{1}{2}\dot{x}^2 - \omega_0^2 \cos x + \epsilon\omega_0^2 \cos(kx - \nu t)\,. \tag{1.4.1}$$

Here ω_0 is the frequency of small oscillations in the unperturbed pendulum with Hamiltonian

$$H_0 = \frac{1}{2}\dot{x}^2 - \omega_0^2 \cos x\,, \tag{1.4.2}$$

where ϵ is the small dimensionless parameter of perturbation and ν the frequency of perturbation. The Hamiltonian (1.4.1) corresponds to the pendulum (1.4.2) with rotating point of suspension.[4] The following

1.5 Perturbed Oscillator

For fairly small amplitudes of oscillations, the pendulum equation can be reduced to the linear oscillator equation and the problem (1.4.1) replaced by the perturbed oscillator problem with Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \omega_0^2 x^2) + \epsilon \frac{\omega_0^2}{k^2} \cos(kx - \nu t) \qquad (1.5.1)$$

and the equation of motion

$$\ddot{x} + \omega_0^2 x = \frac{1}{k} \epsilon \omega_0^2 \sin(kx - \nu t)\,. \qquad (1.5.2)$$

Problem (1.5.2) has another physical interpretation. It is equivalent to the motion of a particle in a constant magnetic field, B_0, and in the field of a plane wave travelling perpendicularly to the magnetic field. In this case, the equation of motion of a particle is written as

$$\ddot{\mathbf{r}} = \frac{e}{mc}[\dot{\mathbf{r}}, \mathbf{B}_0] + \frac{e}{m}\mathbf{E}_0 \sin(\mathbf{kr} - \nu t)\,, \tag{1.5.3}$$

where [,] means vector product. Assuming that $\mathbf{B}_0$ is directed along the z-axis, the vector $\mathbf{r}$ lies in the plane (x, y), and vectors $\mathbf{k}$ and $\mathbf{E}_0$ are directed along the x-axis (a longitudinal wave), it follows from (1.5.3) that

$$\begin{aligned} \ddot{x} &= \omega_0 \dot{y} + \frac{1}{k}\epsilon\omega_0^2 \sin(kx - \nu t)\,, \\ \ddot{y} &= -\omega_0 \dot{x}\,, \end{aligned} \tag{1.5.4}$$

where

$$\omega_0 = eB_0/mc\,, \quad \epsilon\omega_0^2 = eE_0 k/m\,. \tag{1.5.5}$$

From (1.5.4), it follows that the existence of an integral of motion is

$$\dot{y} + \omega_0 x = \text{const}\,. \tag{1.5.6}$$

turbed part as the kicked-pendulum case (1.3.1). As in (1.3.15), we can introduce the action-angle variables (I, ϕ):

$$\begin{aligned} \dot{x} &= (2\omega_0 I)^{1/2} \cos\phi\,, \\ x &= (2I/\omega_0)^{1/2} \sin\phi\,. \end{aligned} \tag{1.5.7}$$

Expressed via the following variables, the Hamiltonian (1.5.1) is

$$\begin{aligned} H &= \omega_0 I + \epsilon V(I, \phi; t)\,, \\ V(I, \phi; t) &= \frac{1}{k^2}\omega_0^2 \cos\left[k\left(\frac{2I}{\omega_0}\right)^{1/2} \sin\phi - \nu t\right]. \end{aligned} \tag{1.5.8}$$

The unperturbed part of the Hamiltonian, $H_0 = \omega_0 I$, does not satisfy the non-degeneracy condition (1.3.17). Therefore, in the case of a resonance

$$n\omega_0 = \nu\,, \tag{1.5.9}$$

where n is an integer, the amplitude of the oscillator increases strongly.

This problem can be described more specifically by using the models of perturbed pendulum and perturbed oscillator introduced in Sections 1.4 and 1.5. In considering the Hamiltonians

$$H = \frac{1}{2}\dot{x}^2 - \omega_0^2 \cos x + \epsilon\frac{\omega_0^2}{k^2}\cos(kx - \nu t) \tag{4.1.8}$$

for the perturbed pendulum and

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2 + \epsilon\frac{\omega_0^2}{k^2}\cos(kx - \nu t) \tag{4.1.9}$$

for the perturbed oscillator, their equations of motion are, respectively,

$$\ddot{x} + \omega_0^2 \sin x = \epsilon\frac{\omega_0^2}{k}\sin(kx - \nu t) \tag{4.1.10}$$

and

$$\ddot{x} + \omega_0^2 x = \epsilon\frac{\omega_0^2}{k}\sin(kx - \nu t)\,. \tag{4.1.11}$$

The effect of the perturbation for a linear oscillator is at its strongest in the resonant case

$$\nu = n_0 \omega_0 \, , \tag{4.1.12}$$

which has an integer n_0. Unless another condition is mentioned, this assumption is used below. The amplitude of a linear oscillator grows linearly with time until it is interrupted by nonlinearity. The latter is induced by the same perturbation.

When considering the perturbed oscillators (4.1.9) and (4.1.11), the polar co-ordinates are

$$x = \rho \sin \phi ; \quad \dot{x} = \omega_0 \rho \cos \phi . \tag{4.2.1}$$

The following expansion,

$$\cos(kx - \nu t) = \cos(k\rho \sin \phi - \nu t) = \sum_m J_m(k\rho) \cos(m\phi - \nu t) , \tag{4.2.2}$$

is also used, where J_m is the Bessel function. With the new variables described in (4.2.1), the Hamiltonian (4.1.9) becomes

$$H = \frac{1}{2}\omega_0^2 \rho^2 + \frac{1}{k^2}\epsilon\omega_0^2 \sum_m J_m(k\rho) \cos(m\phi - \nu t) . \tag{4.2.3}$$

A term with $m = n_0$ is then singled out:

$$H = \frac{1}{2}\omega_0^2\rho^2 + \frac{1}{k^2}\epsilon\omega_0^2 J_{n_0}(k\rho)\cos(m\phi - \nu t) + \frac{1}{k^2}\epsilon\omega_0^2 \sum_{m \neq n_0} J_m(k\rho)\cos(m\phi - \nu t)\,. \tag{4.2.4}$$

This is followed by the introduction of new action-angle variables:

$$I = \omega_0 \rho^2 / 2n_0 \,, \quad \theta = n_0 \phi - \nu t \,. \tag{4.2.5}$$

A new Hamiltonian is then written as

$$\tilde{H} = H - \nu I \,, \tag{4.2.6}$$

where H is expressed as a function of (I, θ). The use of a direct calculation ensures that the equations

$$\dot{I} = -\frac{\partial \tilde{H}}{\partial \theta} \,, \quad \dot{\theta} = \frac{\partial \tilde{H}}{\partial I} \tag{4.2.7}$$

are equivalent to the equation of motion (4.1.11). By substituting (4.2.5) in (4.2.4) and (4.2.6), this yields

$$\begin{aligned} \tilde{H} = (n_0 \omega_0 - \nu) I &+ \frac{1}{k^2} \epsilon \omega_0^2 J_{n_0}(k\rho) \cos\theta \\ &+ \frac{1}{k^2} \epsilon \omega_0^2 \sum_{m \neq n_0} J_m(k\rho) \cos\left[\frac{m}{n_0}\theta - \left(1 - \frac{m}{n_0}\right)\nu t\right] , \end{aligned} \tag{4.2.8}$$

where ρ is introduced to obtain a more compact notation. According to (4.2.5), ρ is

$$\rho = (2n_0 I/\omega_0)^{1/2} \,. \tag{4.2.9}$$

Thus, the expression $\tilde{H} = \tilde{H}(I, \theta; t)$ is the Hamiltonian with respect to the new canonical variables (I, θ). It can also be written as

$$\tilde{H} = \tilde{H}_0(I, \theta) + \tilde{V}(I, \theta; t) \,, \tag{4.2.10}$$

where in accordance with (4.2.8),

$$\begin{aligned} \tilde{H}_0(I, \theta) &= (n_0\omega_0 - \nu)I + \frac{1}{k^2}\epsilon\omega_0^2 J_{n_0}(k\rho)\cos\theta \,, \\ \tilde{V}(I, \theta; t) &= \frac{1}{k^2}\epsilon\omega_0^2 \sum_{m \neq n_0} J_m(k\rho)\cos\left[\frac{m}{n_0}\theta - \left(1 - \frac{m}{n_0}\right)\nu t\right] \end{aligned} \tag{4.2.11}$$

and expression (4.2.9) is used for ρ.

Turning now to the resonance case (4.1.12), the expression (4.2.11) for the Hamiltonian part $\tilde{H}_0$ takes the following form:

$$\tilde{H}_0 = \frac{1}{k^2}\epsilon\omega_0^2 J_{n_0}(k\rho)\cos\theta = \frac{1}{k^2}\epsilon\omega_0^2 J_{n_0}\left[k\left(\frac{2n_0 I}{\omega_0}\right)^{1/2}\right]\cos\theta\,. \quad (4.2.12)$$