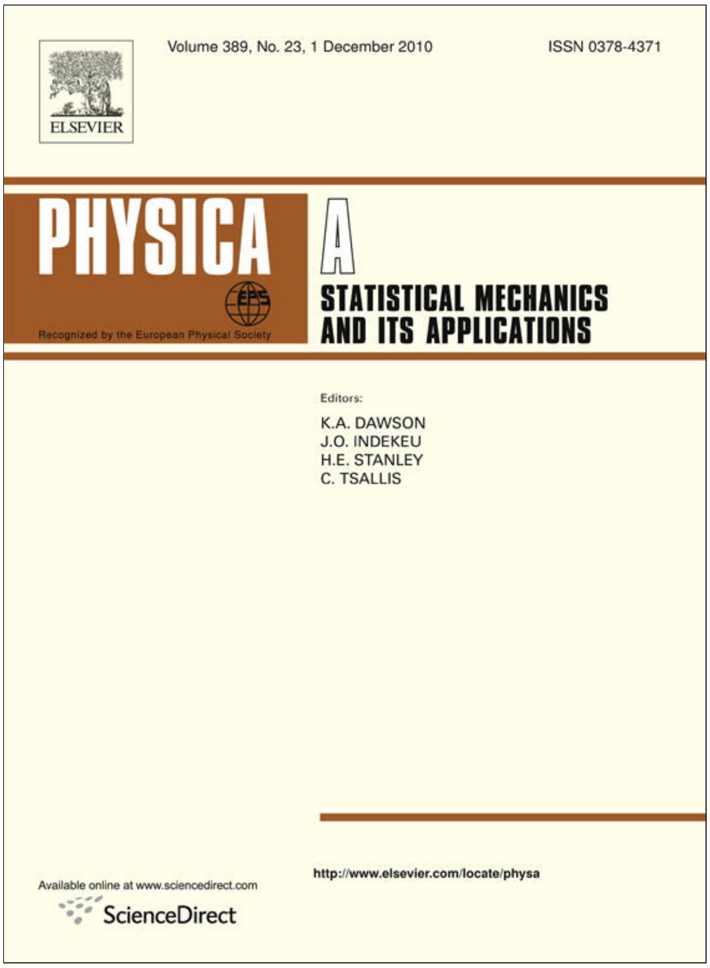
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Robust tori in a double-waved Hamiltonian model

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ABSTRACT

A Hamiltonian system perturbed by two waves with particular wave numbers can present robust tori, which are barriers created by the vanishing of the perturbed Hamiltonian at some defined positions. When robust tori exist, any trajectory in phase space passing close to them is blocked by emergent invariant curves that prevent the chaotic transport. Our results indicate that the considered particular solution for the two waves Hamiltonian model shows plenty of robust tori blocking radial transport.

1. Introduction

The effect of transport barriers in Hamiltonian systems is a subject of global interest in different branches of physics [1–3]. Horton introduced one type of Hamiltonian model with two waves, relevant for particle transport in plasma physics [4]. The Hamiltonian describes drift waves, originated by particles drift proportional to $\vec{E} \wedge \vec{B}$ in non-uniform plasmas, propagating in a magnetic toroidal field and an electric radial field. The model has been explored to describe the onset of stochasticity for test particles driven by these drift waves in tokamaks. The model has been applied in many works to investigate the influence of the equilibrium electric and magnetic fields on the radial transport and to analyze experimental results [5–7].

We observed that this model could present infinite robust tori (RT) which correspond to dynamical barriers that may appear in Hamiltonian systems [8–11]. In this work, we start with a Hamiltonian with only one wave in order to emphasize the abundance of RT, and then with the addition of another wave, these RT could be broken giving rise to anomalous radial transport. Our goal in this work is to present a particular solution for this wave Hamiltonian model that prevents the breaking of the RT, even if we add as many waves as we want in the perturbation. This is an important fact since the creation of barriers in Hamiltonian systems has been considered an important subject in several areas of physics especially in plasma confinement in tokamaks [2,7,12,13].

In this paper we are going to consider a Hamiltonian H consisting of an integrable term H_0 plus a perturbation H_1 in the form, $H(q, p, t) = H_0(q) + \varepsilon H_1(q, p, t)$, where ε is the perturbation parameter, q is the position of the particle and p is its associated momentum. In Section 2 we present the model and in Section 3 we conclude by presenting and discussing the results.

2. Two waves Hamiltonian model

In the present model, we identify the phase space (q, p) as being the physical space (x, y), where x and y are the radial and poloidal coordinates, respectively, (usually used for a large aspect ratio tokamak) that describe the particle position inside

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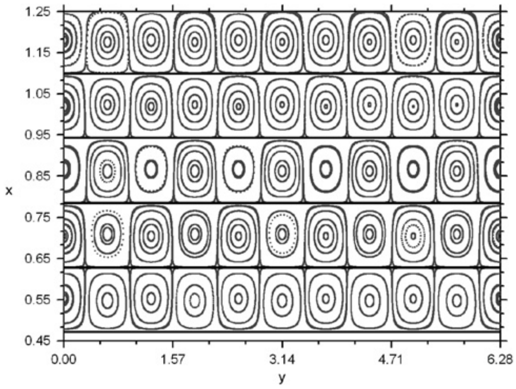

Fig. 1. Phase space for the Hamiltonian with only one wave $A_2 = 0$ and $\alpha = 0.76$.

the plasma [1]. Thus, the Hamiltonian in the canonical variables x and y is

$$H(x', y', t) = H_0(x') + \sum_n A_n \sin(k_{xn}x') \cos(k_{yn}y' - \omega_n t) \tag{1}$$

where A_n stands for the wave amplitudes, k_{xn}, k_{yn} are the wave numbers and ω_n is the wave frequency.

In order to create a dimensionless form for the Hamiltonian above, we perform a canonical transformation mediated by the generating function, $F(x, y', t) = x(y' - u_1.t)$, where $y = \frac{\partial F(x,y',t)}{\partial x} = y' - u_1 t,\ x' = \frac{\partial F(x,y',t)}{\partial y'} = x$ and $\frac{\partial F(x,y',t)}{\partial t} = -x\, u_1$. Hence, the new dimensionless Hamiltonian model with two waves is given by $H(x, y, t) = H(x', y', t) + \frac{\partial F(x,y',t)}{\partial t}$ and reads as

$$H(x, y, t) = H_0(x) - u_1 x + A_1 \sin(k_{x1}x) \cos(k_{y1}\, y) + A_2 \sin(k_{x2}x) \cos(k_{y2}\, (y - ut)) \tag{2}$$

where $u = (\omega_2/k_{y2}) - (\omega_1/k_{y1})$ is the difference of phase velocities between the two waves, $u_1 = (\omega_1/k_{y1})$ is the phase velocity of the first wave, A_1 and A_2 are the amplitudes of the first and the second wave, respectively, $(kx_1,\ ky_1)$ are the wave numbers for the first wave and $(kx_2,\ ky_2)$ are the wave numbers for the second wave. The Hamiltonian of Eq. (2) represents a particle under the action of a transversal wave propagating in the y-direction and oscillating in the x-direction.

We also consider the approach of the Ref. [1] for the unperturbed Hamiltonian, we take it as a monotonic function $H_0(x) = \alpha\, x$. This profile for H_0 creates only twist regions [2] in the phase space. This kind of H_0 has been used to simulate the motion of a test particle in a plasma medium with an applied constant electric field whose variation along the axis x is $E_r = \mathrm{d}H_0(x)/\mathrm{d}x$, and plays the role of the radial component of the electric field. The use of a linear unperturbed Hamiltonian does not play any decisive role in the appearance of RT in the perturbed system. Even though the RT that will appear are, in fact, present in the unperturbed system, the mechanism which makes them emerge is independent of the form of H_0, and this is the goal of this paper.

Initially, we take only one wave ($A_2 = 0$) in the Hamiltonian of Eq. (2) and we obtain

$$H(x, y, t) = (\alpha - u_1)x + A_1 \sin(k_{x1}x) \cos(k_{y1}\, y) \tag{3}$$

whose equations of motion are

$$\begin{aligned} \dot{x} &= A_1 k_{y1} \sin(k_{x1}x) \sin(k_{y1}\, y) \\ \dot{y} &= (\alpha - u_1) + A_1 k_{x1} \cos(k_{x1}x) \cos(k_{y1}\, y). \end{aligned} \tag{4}$$

We note that when $\sin(k_{x1}x) = 0$ the perturbation in Eq. (3) is null even for $A_1 \neq 0$, which is the necessary condition to have RT. From Eq. (4) we also observe that the motion in the x-direction vanishes, and we have $x =$ constant for any time leading to infinite RT in the positions $x = \frac{n\pi}{k_{x1}}$, for all integers n. However, in the y-direction the dynamics is not constant, as shown in Fig. 1. It is important to note that Kleva and Drake [14] have already defined a condition where the particles have been trapped by a single surface. Since the focus of their analysis was on the stochastic transport, no detail was developed on the relation between the wave numbers and the radial transport.

It is worthwhile to point out that if $\cos(k_{y1}\, y) = 0$, the perturbed Hamiltonian vanishes but the motion in the direction x and y continues to exist. The RT will appear when the perturbed Hamiltonian vanishes as well as at least one equation of motion [6].

3. Results and conclusions

In Fig. 1 we show the phase space for one wave Eq. (3) and we can observe that there are islands that look like *cells* flattened by straight lines. These lines are the barriers RT. As we are interested in analyzing the transport along the x-direction, interpreted as the radial transport [6,7], then the creation of robust barriers in some positions $x = constant$ will interfere significantly in the dynamical transport.

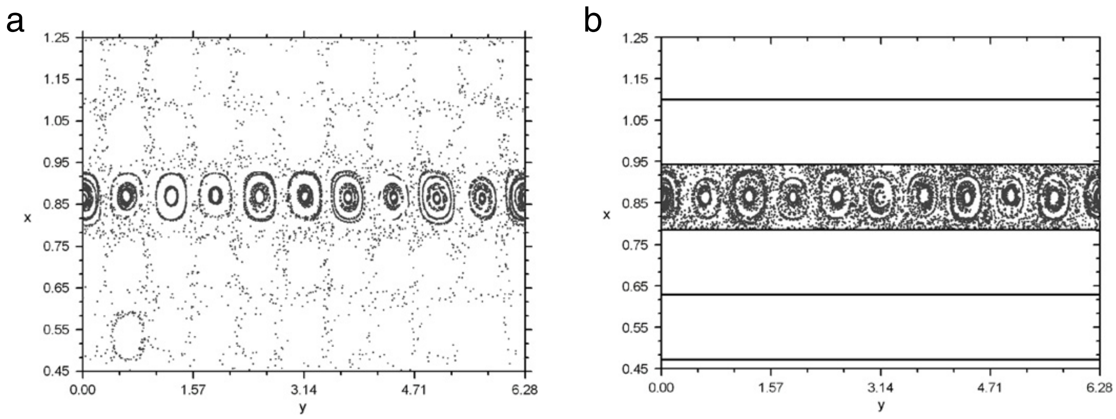

Fig. 2. Poincaré maps for the Hamiltonian with two waves Eq. (5): (a) for $k_{x1} \neq m.k_{x2}$, without RT; (b) for $k_{x2} = m.k_{x1}$ with RT.

As the Hamiltonian from Eq. (3) presents only one wave, the system is globally integrable. However, when we add another wave the integrability will be broken and chaos will be observed around the hyperbolic fixed points. For instance, when $H_0(x) = \alpha\, x$, Eq. (2) becomes

$$H(x, y, t) = (\alpha - u_1)x + A_1 \sin(k_{x1}x) \cos(k_{y1}y) + A_2 \sin(k_{x2}x) \cos(k_{y2}(y - ut)) \tag{5}$$

and we can observe that when $\sin(k_{x1}x) = 0 = \sin(k_{x2}x)$ the perturbation vanishes. Looking at the new equations of motion

$$\begin{aligned} \dot{x} &= A_1 k_{y1} \sin(k_{x1}x) \sin(k_{y1}y) + A_2 k_{y2} \sin(k_{x2}x) \sin(k_{y2}(y - ut)) \\ \dot{y} &= (\alpha - u_1) + A_1 k_{x1} \cos(k_{x1}x) \cos(k_{y1}y) + A_2 k_{x2} \cos(k_{x2}x) \cos(k_{y2}(y - ut)) \end{aligned} \tag{6}$$

we note that the motion in the x-direction can disappear if the wave numbers obey the condition $k_{x2} = m.k_{x1}$. If m is an integer number, RT will appear as in the integrable case, but if m is a non-integer number, only fewer RT will survive. The condition $\sin(k_{x1}x) = \sin(k_{x2}x) = 0$ should be satisfied to have robust tori. This means that, $x = \frac{n_1\pi}{k_{x1}} = \frac{n_2\pi}{k_{x2}}$ or simply $x = \frac{n_1}{n_2} = \frac{k_{x1}}{k_{x2}}$. Hence, when $n1$ and $n2$ are integers, the ratio $(n1/n2)$ will be an integer or rational, but the last equation can be still satisfied in both cases which allows us to be able to block the radial transport.

In Fig. 2, the initial conditions were given in the ranges $x \in [0.78, 0.94]$ and $y \in [0.0, 2\pi]$, and we present the following two different situations for the waves model of Eq. (5), the known case $k_{x2} \neq m.k_{x1}$ [1–4] and also the case $k_{x2} = m.k_{x1}$. The addition of the second wave breaks the integrability of the system and chaos may fill the phase space. The particles can move along the radial and poloidal coordinates making a chaotic web, as is shown in Fig. 2(a). We observe that there are not barriers for the radial transport developed by the particles. On the other hand, in Fig. 2(b) we show the Poincaré map for the particular case presented in this paper $k_{x2} = m.k_{x1}$ where m is an integer. RT, the straight lines, are again intact even after the addition of the second wave and there are no trajectories escaping along the phase space. As expected, RT blocked the radial diffusion.

The onset of chaos in the phase space takes place when we add a second wave and we investigate the dynamical transport rate by calculating the radial local diffusion coefficient from the orbits [15]. We consider the following equation for evaluating the diffusion [2],

$$D = \frac{1}{2tN} \sum_{i=1}^{N} [x_i(t) - x_i(0)]^2 \tag{7}$$

where $N = 1000$ is the total number of initial conditions distributed uniformly through the grid with $x : [0.50; 1.25]$ and $y : [0; 2\pi]$, and $t = 150$ is, in fact, the number of iterations for each initial condition. For each value of the wave number k_{x2} we estimate the radial diffusion coefficient D.

In Fig. 3(a) we show the dependence of the radial diffusion coefficient on the wave number k_{x2}. We used in the numerical simulations the following parameters: $k_{x1} = 20$; $k_{y1} = 3$. As we can see, the lowest values of the diffusion occur for $k_{x2} = m.k_{x1} = 20$ when $m = 1$ and for $k_{x2} = m.k_{x1} = 40$ when $m = 2$. The radial diffusion coefficient goes to zero when $k_{x2} = m.k_{x1}$ for all integer values of m and it is not zero but small for $k_{x2} = m.k_{x1}$ for non-integer m.

The behavior of the radial diffusion coefficient is explained in Fig. 3(b), which shows the percentage of intact remaining RT after the addition of the second wave, considering an initial amount of 40 RT in the phase space. For the wave numbers $k_{x2} = 20$ and $k_{x2} = 40$ all RT still exist, because the wave numbers satisfy the particular solution we present here, $k_{x2} = m.k_{x1}$ with m an integer. However, we can observe that a portion of RT also exists when m is a non-integer. The existence of RT affects directly the particles diffusion in the radial direction. For the particular solution introduced here, the radial diffusion coefficient is zero because all RT are preserved. However, for some intermediary values of k_{x2} some percentages of RT still remain intact, decreasing the radial transport.

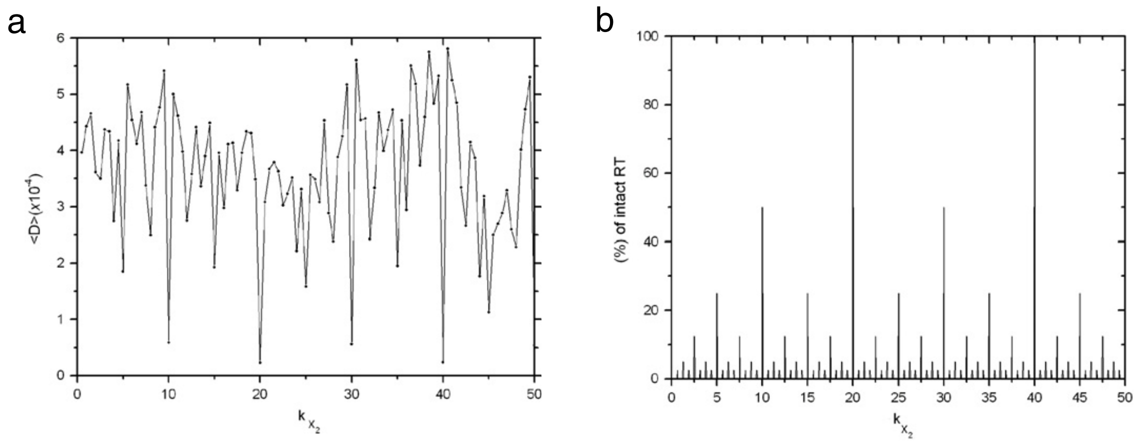

Fig. 3. Comparisons between (a) radial diffusion coefficient and (b) percentage of remaining RT.

We point out that the particular two waves solution presented in this paper can be extended to many waves and the non perturbed Hamiltonian H_0 does not influence the formation of RT along the x-direction. The relation is rewritten as: $k_{x(n+1)} = m_n.k_{xn}$ with n an integer ε $[1, (N-1)]$, where $(N-1)$ is the number of waves. The coefficients m_n have to obey the following condition: $\prod_{n=1}^{N-1} m_n =$ integer number. The multiplication of the coefficients m_n has to be an integer because all radial wave numbers k_{xn} have to be a multiple of the first radial wave number k_{x1}, to keep intact all RT in phase space.

Previous studies [2,3] have shown the importance of decreasing the radial transport induced by drift waves to improve the plasma confinement in tokamaks. It is also reported that similar Hamiltonians to the one presented in this paper have been used to study transport, but only few works were dedicated to control chaos in these systems [5,16]. Even though there is still not an effective way to control the wave numbers of the drift waves in tokamaks nor to measure the radial wave number k_{xn}, our contribution shows a direction to block the radial transport with the particular solution presented here for the two wave Hamiltonian model. If the control of the radial wave number is possible, then all RT will remain intact in the system blocking the radial diffusion. We emphasize that we are not proposing to suppress the modes, but to keep all of them to control the radial wave numbers. All waves of the system should have wave numbers that are multiples of the wave number of the first wave to block the radial diffusion.

Acknowledgements

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