Scaling laws in two-dimensional discrete mappings

Juliano A. de Oliveira

UNESP – câmpus de São João da Boa Vista

Reunião sobre Dinâmica Não Linear - Grupo Controle de Oscilações

USP – São Paulo - April - 2018

Universidade Estadual Paulista “Júlio de Mesquita Filho” - UNESP

Created in 1976, the Unesp has 34 units in 24 cities, being 22 in the Interior; one in the State Capital - São Paulo; and one on the São Paulo Coast, in São Vicente.

Collaborators

- Prof. Tit. Edson D. Leonel - UNESP - Rio Claro
- Célia M. Kuwana - UNESP - CAPES - Rio Claro
- Hans M. J. de Mendonça - UNESP – FAPESP - Rio Claro
- Bárbara P. Carneiro - UNESP – CAPES - Rio Claro
- Anderson A. Silva - UNESP – Rio Claro
- Felipe de Castro Geraldo - UNESP - São João da Boa Vista
- Rodrigo Perre - UNESP - São João da Boa Vista
- Larissa C. N. Ramos - UNESP - FAPESP - São João da Boa Vista
- Leonardo T. Montero - UNESP - CNPq - São João da Boa Vista
- Anderson H. Curcio - UNESP - CNPq - São João da Boa Vista
- Vitor Diungaro - UNESP - CNPq - Rio Claro

Main Goals

- Define a family of two-dimensional Hamiltonian mappings;
- Introduce dissipation on the systems;
- Build the phase spaces;
- Investigate the deviation of the average along of the chaotic orbits;
- Find critical exponents to define universality classes.

We consider a family of two-dimensional Hamiltonian mappings given by

(1)
$$T: \begin{cases} J_{n+1} = |J_n - \epsilon \sin(\theta_n)| \\ \theta_{n+1} = \left[\theta_n + \frac{a}{J_{n+1}^{\gamma}}\right] \mod (2\pi) \end{cases}$$

where J corresponds to action variable, θ corresponds to angle variable, a, ϵ and γ are control parameters.

$\gamma = 1$ we recover the Fermi-Ulam model.
we recover the corrugated waveguide.

$\gamma = 3/2$ we recover the Kepler map.

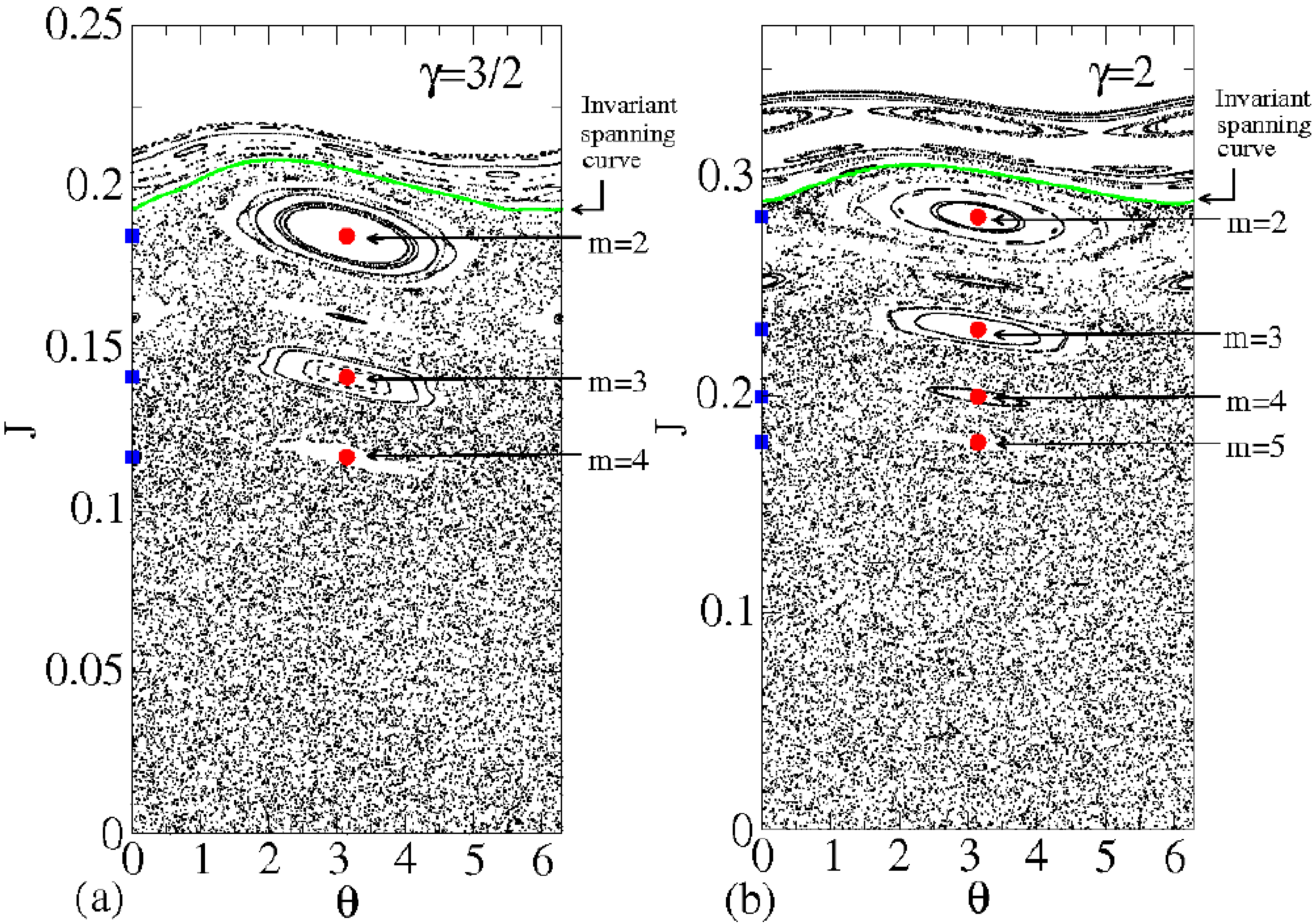
$\gamma = 2$ include relevant applications for plasma physics.

$\gamma = -1$ we recover the bouncer model.
Chirikov map.

PHYSICAL REVIEW E **81**, 046212 (2010)

Finding critical exponents for two-dimensional Hamiltonian maps

Juliano A. de Oliveira, R. A. Bizão, and Edson D. Leonel



$a = 1$, $\epsilon = 10^{-2}$ and (a) $\gamma = 3/2$ and (b) $\gamma = 2$.

The fixed points may be obtained by matching the conditions: $J_{n+1} = J_n = J$ and $\theta_{n+1} = \theta_n = \theta + m$, where $m = 1, 2, 3, \ldots$.

They are classified using the eigenvalues given by

(2) $$det(Jac - \lambda I) = 0,$$

where I is identity matrix and Jac is Jacobian matrix written as

(3) $$Jac = \begin{pmatrix} \frac{\partial\theta_{n+1}}{\partial\theta_n} & \frac{\partial\theta_{n+1}}{\partial J_n} \\ \frac{\partial J_{n+1}}{\partial\theta_n} & \frac{\partial J_{n+1}}{\partial J_n} \end{pmatrix}$$

The elements of the Jacobian matrix are given by

$$j_{11} = \frac{\partial\theta_{n+1}}{\partial\theta_n} = 1 + 2\pi\epsilon a\gamma cos(2\pi\theta_n)(r)^{-(\gamma+1}$$
$$j_{12} = \frac{\partial\theta_{n+1}}{\partial J_n} = a\gamma(r)^{-(\gamma+1)};$$
$$j_{21} = \frac{\partial J_{n+1}}{\partial\theta_n} = -2\pi\epsilon cos(2\pi\theta_n);$$
$$j_{22} = \frac{\partial J_{n+1}}{\partial\theta_n} = 1,$$

where the auxiliary variable is $r = J_n - \epsilon sin(2\pi\theta_n)$

The determinant of the Jacobian matrix is $Det\ Jac = 1$.

So we obtained the eigenvalues written as

$$\lambda = \frac{(j_{11}+j_{22}) \pm \sqrt{(j_{11}+j_{22})^2 - 4}}{2} \tag{4}$$

TABLE I. Fixed points and their classification for different values of γ.

γ	Fixed point 1	Fixed point 2	Elliptic	Hyperbolic
$\frac{2}{3}$	$[0,(\frac{a}{m})^{3/2}]$	$[\frac{1}{2},(\frac{a}{m})^{3/2}]$	$0<m<(\frac{6\ 2^{19/78}}{b\pi})^{2/5}$	$m>(\frac{6\ 2^{19/78}}{b\pi})^{2/5}$
$\frac{2}{5}$	$[0,(\frac{a}{m})^{5/2}]$	$[\frac{1}{2},(\frac{a}{m})^{5/2}]$	$0<m<(\frac{20\sqrt{2}}{b\pi})^{2/5}$	$m>(\frac{20\sqrt{2}}{b\pi})^{2/5}$
$\frac{1}{2}$	$[0,(\frac{a}{m})^{2}]$	$[\frac{1}{2},(\frac{a}{m})^{2}]$	$0<m<(\frac{16}{b\pi})^{1/3}$	$m>(\frac{16}{b\pi})^{1/3}$
$\frac{3}{4}$	$[0,(\frac{a}{m})^{4/3}]$	$[\frac{1}{2},(\frac{a}{m})^{4/3}]$	$0<m<(\frac{16\ 2^{1/3}}{3b\pi})^{1/2}$	$m>(\frac{16\ 2^{1/3}}{3b\pi})^{1/2}$
$\frac{3}{5}$	$[0,(\frac{a}{m})^{5/3}]$	$[\frac{1}{2},(\frac{a}{m})^{5/3}]$	$0<m<(\frac{20\ 2^{2/3}}{3b\pi})^{3/8}$	$m>(\frac{20\ 2^{2/3}}{3b\pi})^{3/8}$
$\frac{4}{5}$	$[0,(\frac{a}{m})^{5/4}]$	$[\frac{1}{2},(\frac{a}{m})^{5/4}]$	$0<m<(\frac{5\ 2^{1/4}}{b\pi})^{4/9}$	$m>(\frac{5\ 2^{1/4}}{b\pi})^{4/9}$
1	$(0,\frac{a}{m})$	$(\frac{1}{2},\frac{a}{m})$	$0<m<(\frac{4}{b\pi})^{1/2}$	$m>(\frac{4}{b\pi})^{1/2}$

To estimate analytically the position of the first invariant spanning curve - FISC we suppose that

(5) $$J_{n+1} \cong J^* + \Delta J_{n+1},$$

where J^* is a typical value along of the invariant tori and ΔJ_{n+1} is a small perturbation of J_{n+1}.

A connection with standard map give us

$$T : \begin{cases} J_{n+1} = J_n + \left(\frac{2\pi a \epsilon \gamma}{J^{*(1+\gamma)}}\right) sin(\theta_n) \\ \theta_{n+1} = [\theta_n + J_{n+1}] \mod(2\pi) \end{cases}$$

(6)

where

(7) $$K_{eff} = \left(\frac{2\pi a \epsilon \gamma}{J^{*(1+\gamma)}}\right)$$

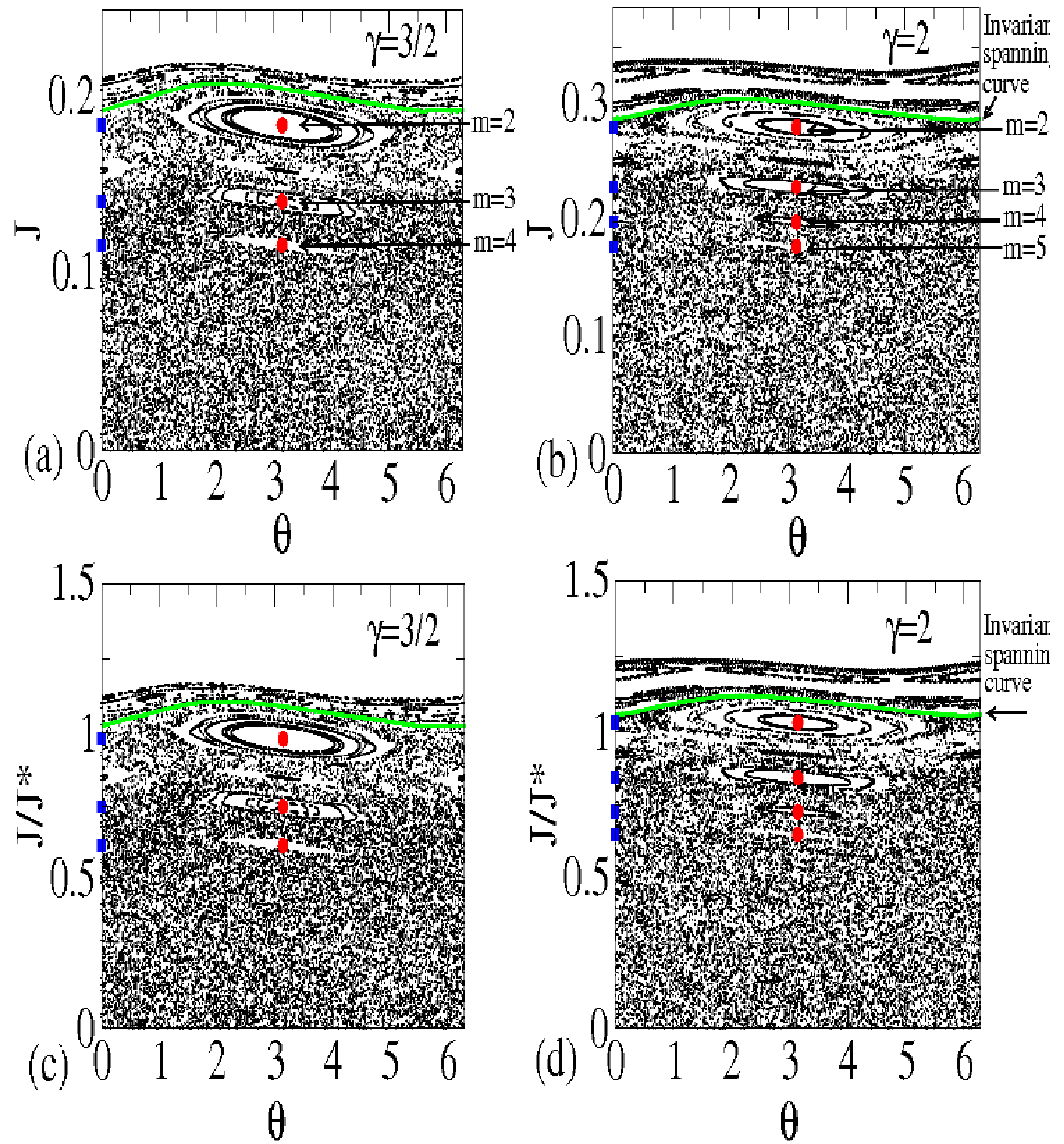
Table 1
Evaluation of K_{eff} at the first invariant spanning curve. The lowest (highest) value of K_{eff} corresponds to the maximum (minimum) value of the variable y^* on the invariant spanning curve.

γ	a	ϵ	K_{eff}
2/5	1	10^{-4}	0.952–1.003
2/5	1	10^{-3}	0.898–0.993
2/5	10	10^{-4}	1.000–1.011
2/5	10	10^{-3}	0.961–0.979
1/2	1	10^{-4}	0.964–0.995
1/2	1	10^{-3}	0.886–0.946
1/2	10	10^{-4}	0.968–0.975
1/2	10	10^{-3}	0.968–0.983
3/4	1	10^{-4}	0.971–0.985
3/4	1	10^{-3}	0.913–0.946
3/4	10	10^{-4}	1.004–1.008
3/4	10	10^{-3}	0.969–0.979

Therefore, since the transition from local to global chaos occurs at $K_eff \approx 0.9716...$ the location of the FISC is given by

$$J^* \cong \left(\frac{2\pi a\epsilon\gamma}{0.9716\ldots}\right)^{1/(1+\gamma)} . \quad (8)$$

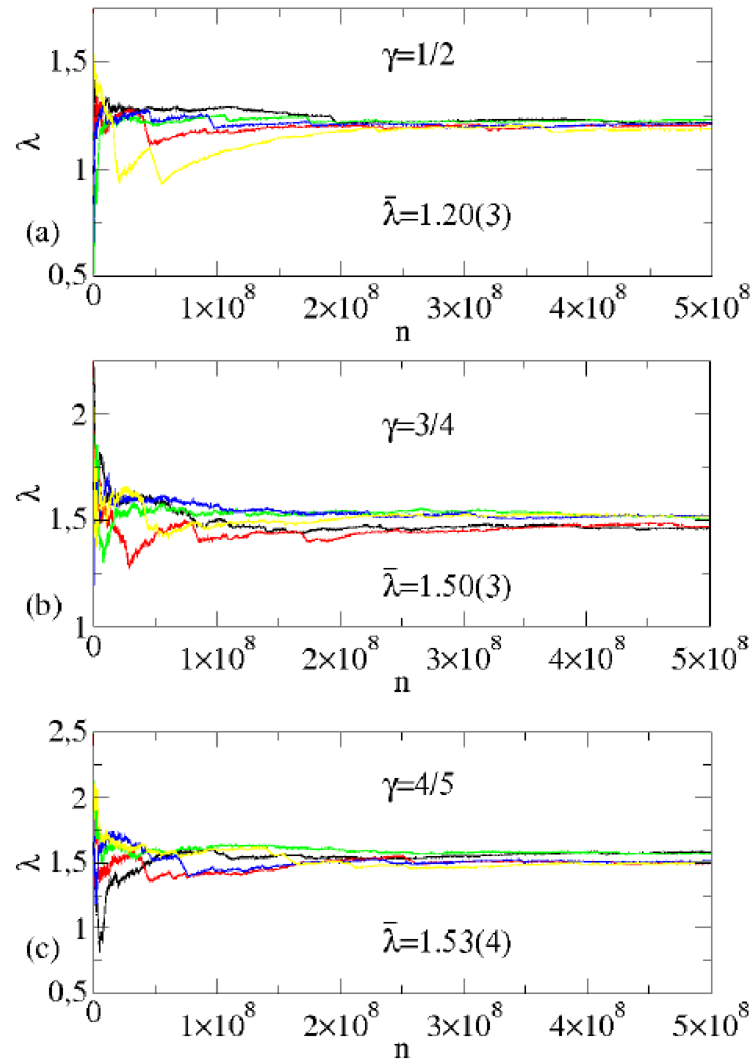
From equation above we conclude that the size of the chaotic sea is proportional to $(2\pi a\epsilon\gamma)^{1/(1+\gamma)}$.



The Lyapunov exponents are defined as

(9) $$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \ln |\Lambda_j^{(n)}|, \quad j = 1, 2,$$

where $\Lambda_j^{(n)}$ are the eigenvalues of the matrix $M = \Pi_{i=1}^{n} Jac_i(J, \theta)$ where Jac_i is the Jacobian matrix of the mapping evaluated along the orbit.



λ for $a = 2$, $\epsilon = 10^{-4}$ and: (a) $\gamma = 1/2$; (b) $\gamma = 3/4$ and; (c) $\gamma = 4/5$.

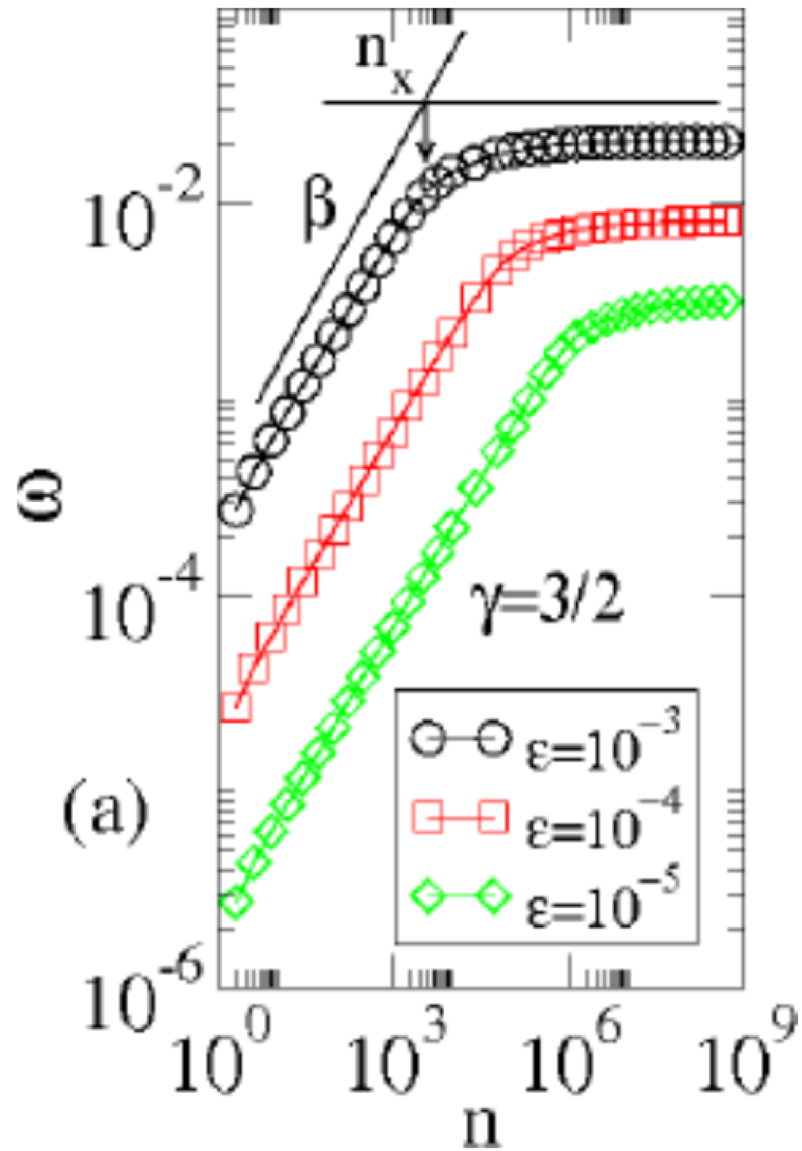
The average quantity to be explored is the deviation of the average $\bar{J}$ for chaotic orbits, denoted as ω. It is defined as

$$\omega(n, a, \epsilon) = \frac{1}{M}\sum_{i=1}^{M}\sqrt{\overline{J_i^2}(n, a, \epsilon) - \overline{J_i}^2(n, a, \epsilon)}\ , \tag{10}$$

where M corresponds to an *ensemble* of different initial conditions $J_i \in (0, 1)$ randomly chosen for a fixed $J_0 = 10^{-3}\epsilon$ and $\bar{J}_i$ is given by

$$\overline{J_i}(n, a, \epsilon) = \frac{1}{n}\sum_{j=1}^{n} J_{j,i}\ . \tag{11}$$

OBS: The behavior of $\bar{J}$ shows the same properties of ω.



$a = 1$ and $\omega = 3/2$.

The curves start growing for small n till a crossover iteration number n_x and after go to regime of convergence. Thus we can suppose that:

- (i) For $n \ll n_x$, ω grows according to a power law of the type

  (12) $$\omega \propto \left(n\epsilon^2\right)^{\beta},$$

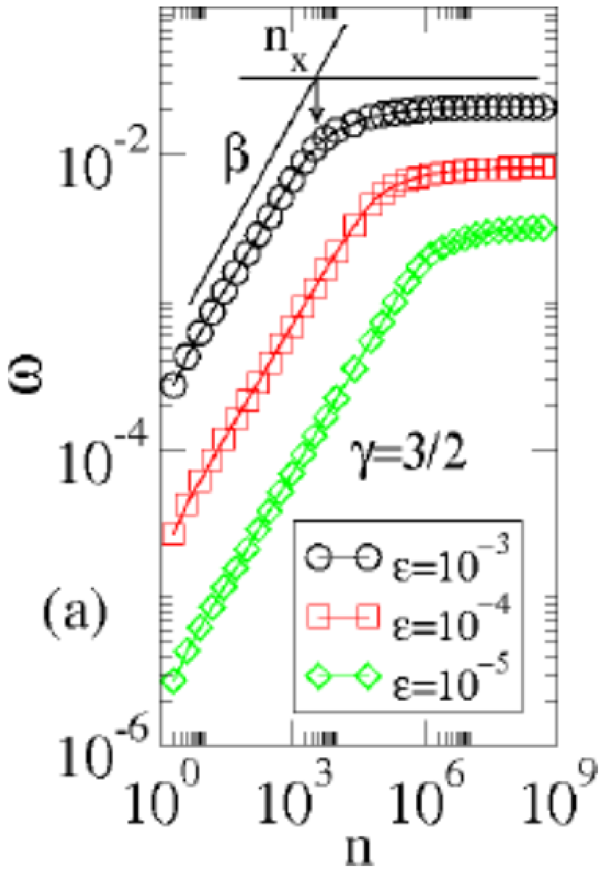
  where β is a critical exponent;
- (ii) For large n, say $n \gg n_x$, the behavior of ω is

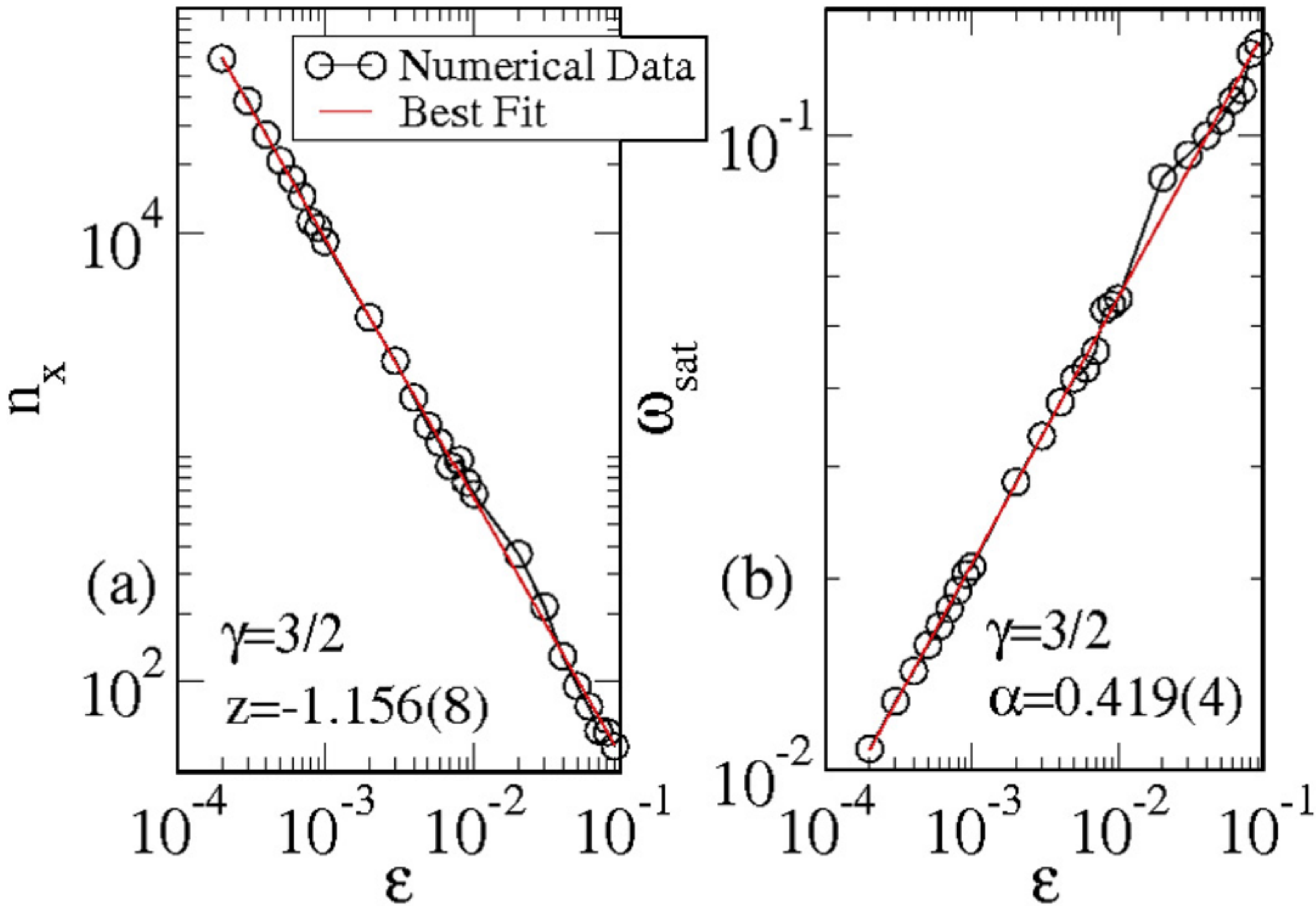
  (13) $$\omega \propto \epsilon^{\alpha},$$

  where α is critical exponent;
- (iii) The crossover n_x, that characterizes the transition of the growing regime for the saturation is

  (14) $$n_x\epsilon^2 \propto \epsilon^z,$$

where z is a dynamical exponent.





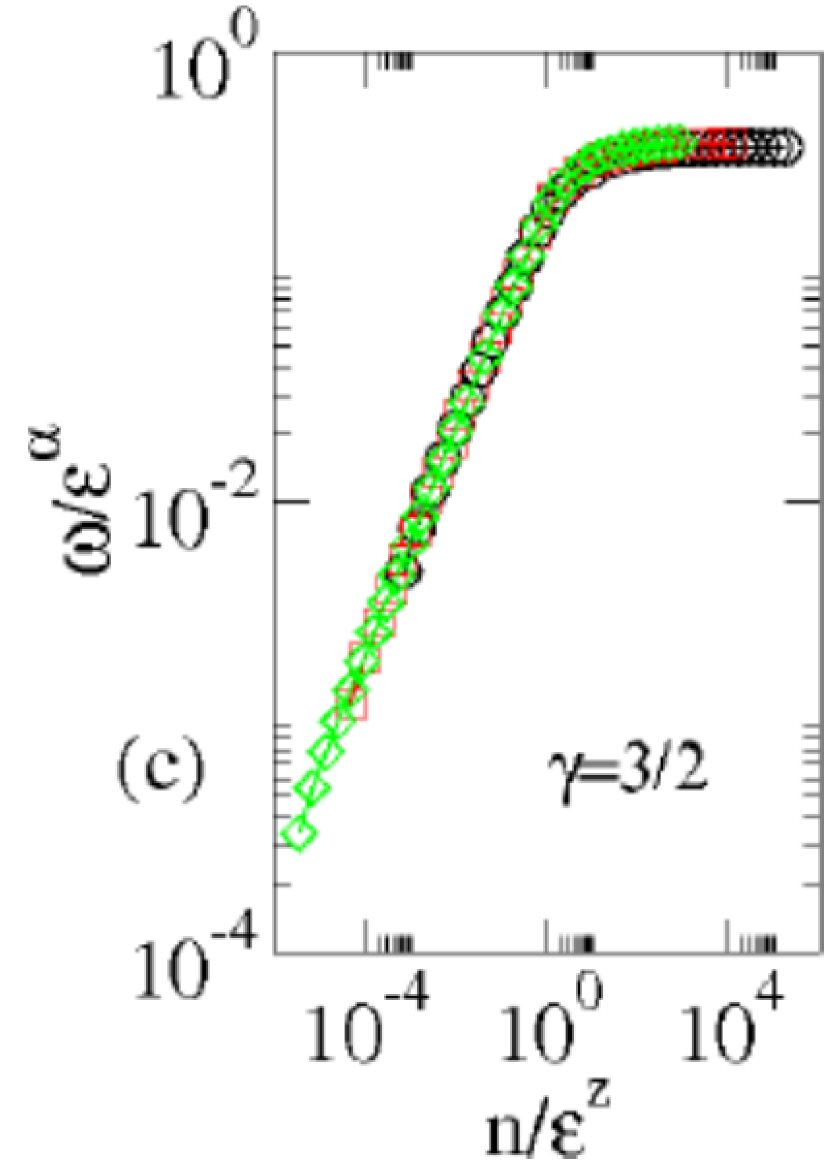
$\gamma = 3/2$ and $a = 1$ fixed for: (a) $n_x \times \epsilon$, (b) $\omega_{sat} \times \epsilon$.

Now we can describe ω in terms of a scaling function of the type

$$\omega(n\epsilon^2, \epsilon) = l\,\omega\left(l^{a_1} n\epsilon^2, l^{b_1}\epsilon\right), \quad (15)$$

where l is a scaling factor, a_1 and b_1 are scaling exponents must be related to the critical exponents β, α and z.

$$z = \frac{\alpha}{\beta} - 2 \quad \text{scaling law.}$$



$a = 1$ and $\omega = 3/2$.

We saw that the size of the chaotic sea varies as the control parameter γ varies. Consequently, average properties of the dynamics are also dependent on the position of the invariant spanning curves.

We suppose that $\bar{J}$ is dependent on the position of the lowest invariant spanning curve, which leads us to conclude that

$$\alpha \cong \frac{1}{(1+\gamma)} \ . \tag{16}$$

Using scaling arguments one can show that $z = \alpha/\beta - 2$, therefore leading to

$$z \cong \left[\frac{1}{\beta(1+\gamma)} - 2\right] \ . \tag{17}$$

To validate the equations above we show the comparasion of the critical exponents to follow.

γ	$1/(1+\gamma)$	α	β	$1/[\beta(1+\gamma)]-2$	z
$3/7$	$7/10$	$0.696(4)\approx 7/10$	$0.487(7)$	-0.563	$-0.57(2)$
$4/9$	$9/13$	$0.710(2)\approx 9/13$	$0.484(5)$	-0.570	$-0.58(1)$
$1/2$	$2/3$	$0.673(2)\approx 2/3$	$0.488(4)$	-0.634	$-0.641(7)$
$3/5$	$5/8$	$0.59(1)\approx 5/8$	$0.488(5)$	-0.719	$-0.68(2)$
$2/3$	$3/5$	$0.607(1)\approx 3/5$	$0.489(7)$	-0.773	$-0.757(4)$
$5/7$	$7/12$	$0.5893(9)\approx 7/12$	$0.491(5)$	-0.812	$-0.808(5)$
$3/4$	$4/7$	$0.588(3)\approx 4/7$	$0.488(6)$	-0.829	$-0.817(8)$
$4/5$	$5/9$	$0.563(1)\approx 5/9$	$0.489(7)$	-0.863	$-0.858(6)$
1	$1/2$	$0.518(4)\approx 1/2$	$0.495(6)$	-0.989	-1

Table 1. Comparison of the critical exponent $1/(1+\gamma)$ and α, $1/[\beta(1+\gamma)]-2$ and z. The range considered was $\epsilon \in \left[10^{-5}, 10^{-3}\right]$.

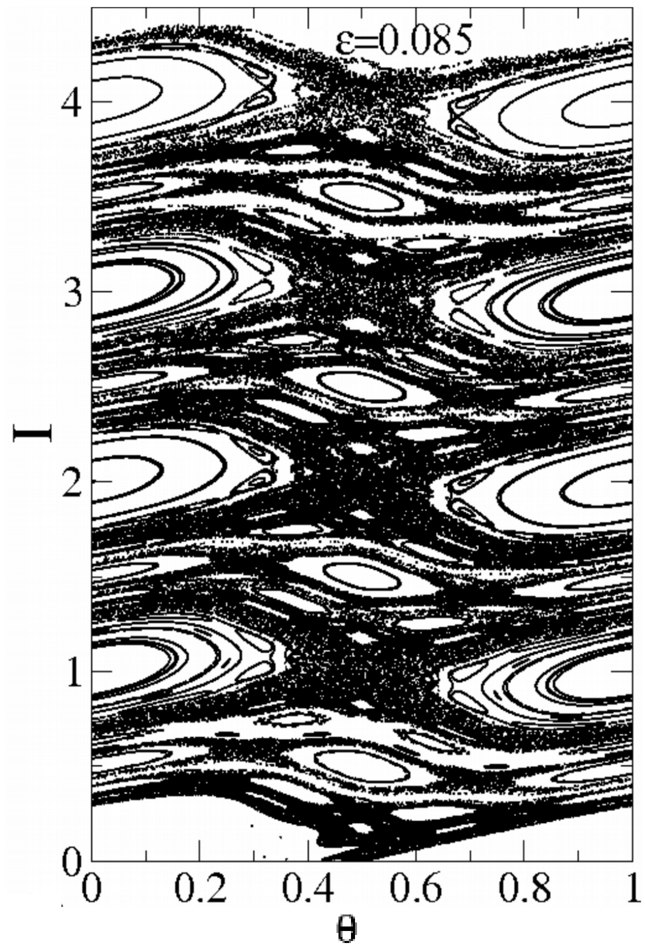
Physics Letters A 379 (2015) 1808–1815

A dynamical phase transition for a family of Hamiltonian mappings: A phenomenological investigation to obtain the critical exponents

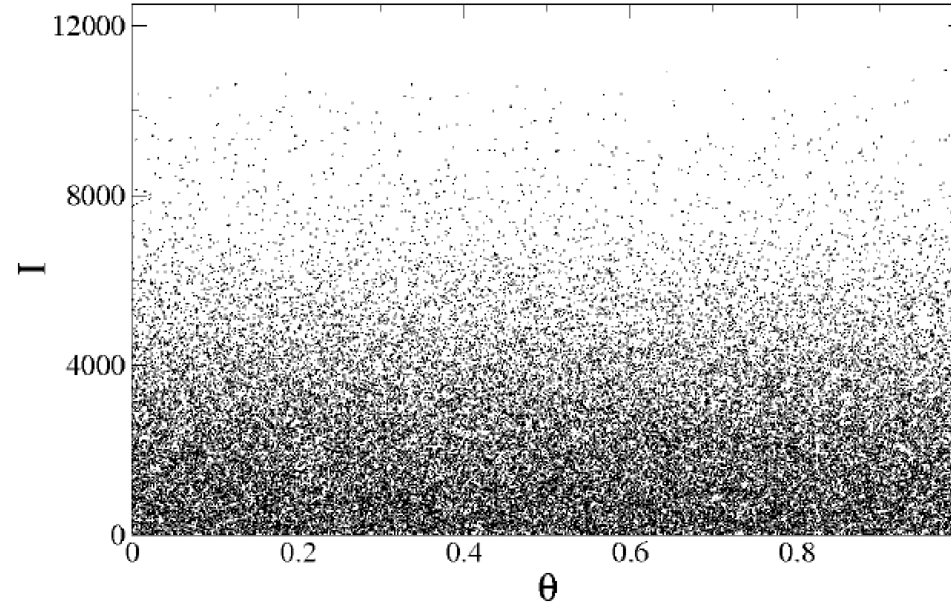
Edson D. Leonel [a,b,c,*], Julia Penalva [a], Rivânia M.N. Teixeira [c], Raimundo N. Costa Filho [c], Mário R. Silva [d], Juliano A. de Oliveira [e]

The mapping for ***dissipative systems***

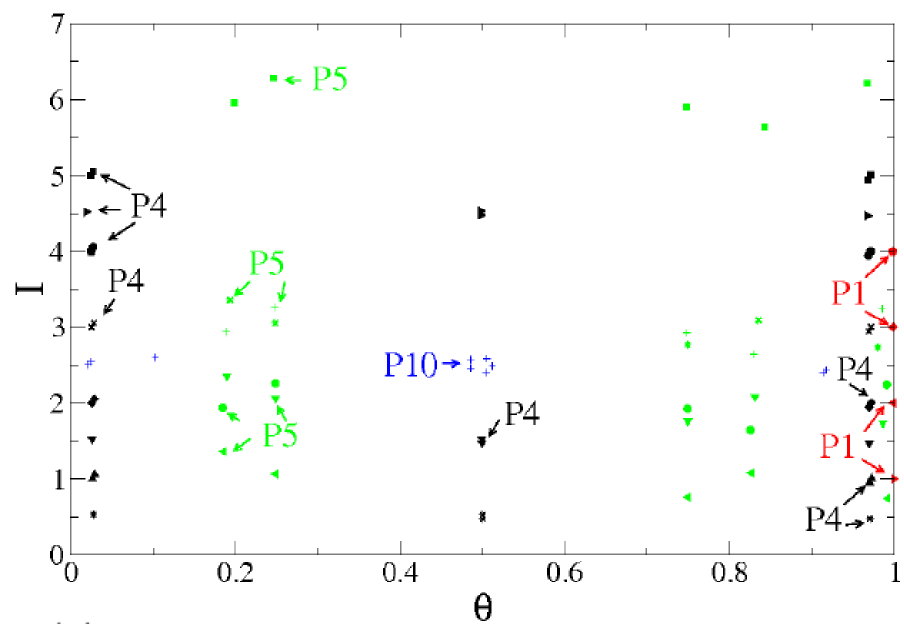
$$T: \begin{cases} I_{n+1} = |\delta I_n - (1+\delta)\epsilon \sin(2\pi\theta_n)| \\ \theta_{n+1} = \left[\theta_n + I^{\gamma}_{n+1}\right] \mathrm{mod} 1, \end{cases}$$



(a) $\gamma = 1, \delta = 1$ (conservative case) $\epsilon = 0.085$



(b) $\gamma = 1$, $\delta = 0.999$ and $\epsilon = 100$



(c) $\gamma = 1$, $\delta = 0.999$ and $\epsilon = 0.16$

An analytical argument for approaching orbits to the attractors is

$$I_1 = |\delta \ I_0 - (1+\delta)\epsilon \sin(2\pi\theta_0)|,$$
$$I_2 = |\delta^2 \ I_0 - (1+\delta)\epsilon \left[\delta \sin(2\pi\theta_0) + \sin(2\pi\theta_1)\right]|$$

A general expression can be written as

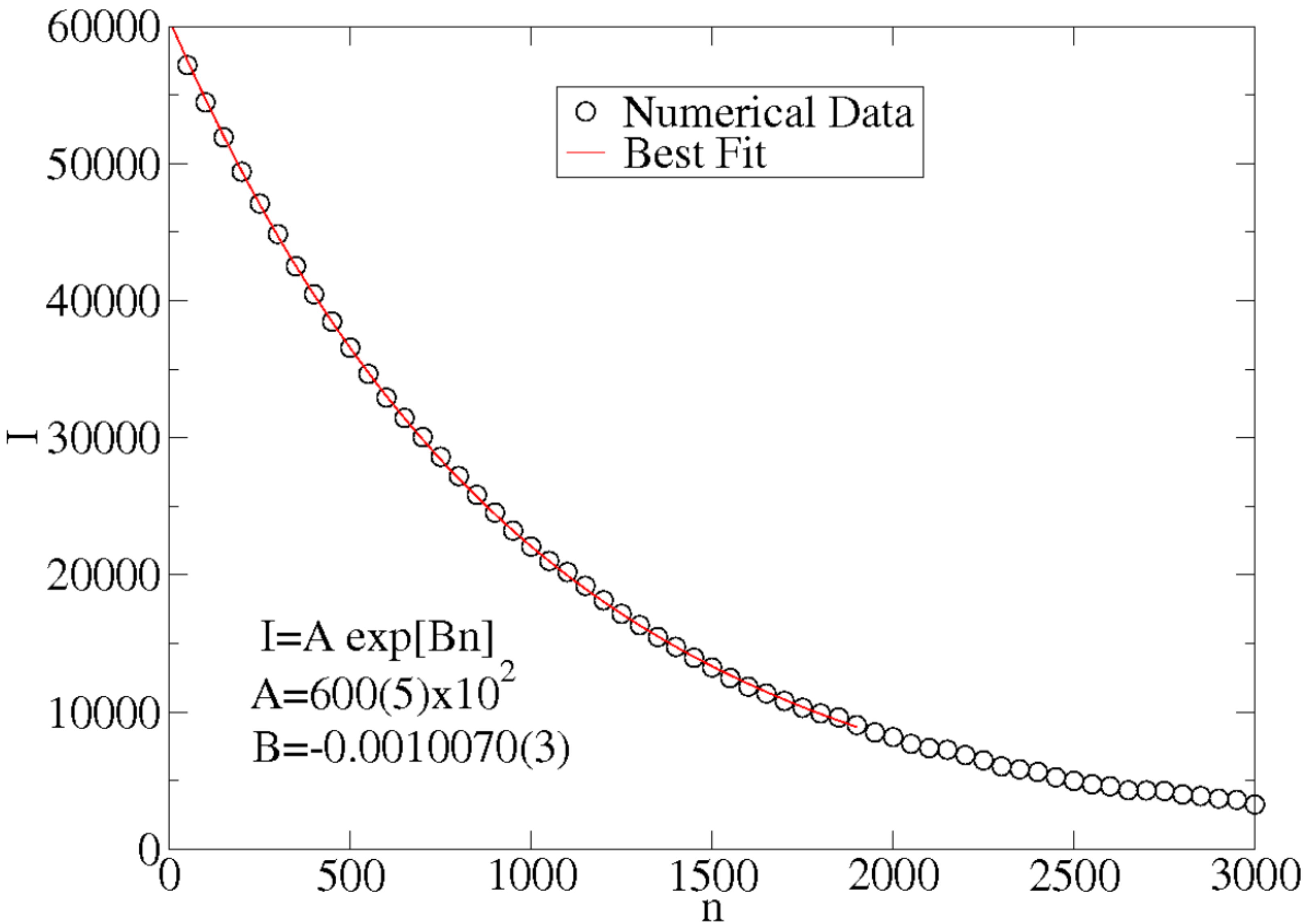
$$I_n = \delta^n I_0 - (1+\delta)\,\epsilon \sum_{i=1}^{n} \delta^{n-i} \sin(2\pi\theta_{i-1}).$$

Expanding the first term in powers of n, we obtain

$$I_n \simeq I_0 \left[\ln(\delta)n + \frac{1}{2!}\ln(\delta)^2 n^2 + \frac{1}{3!}\ln(\delta)^3 n^3 + \frac{1}{4!}\ln(\delta)^4 n^4 + \ldots\right]$$

and recover the definition of an exponential function given as

$$I_n = I_0 \ e^{ln(\delta)n}$$



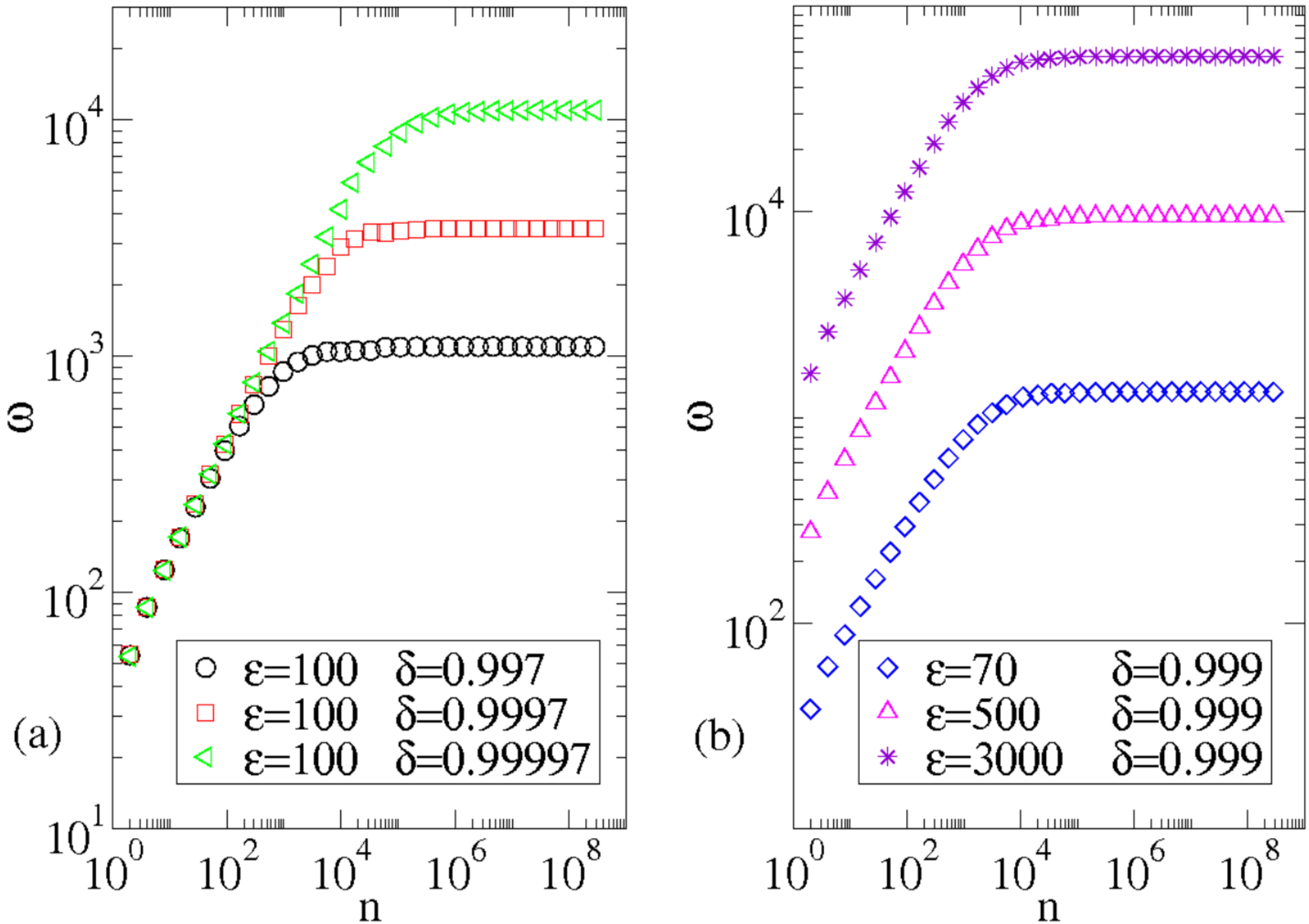
Behaviour of $I \times n$ using $\epsilon = 100,\ \delta = 0.999,\ I_0 = 60000$ and $\gamma = 2$.

Deviation of the average $\bar{I}$ for chaotic attractors defined as

$$\omega(n,\epsilon,\delta) = \frac{1}{M}\sum_{i=1}^{M}\sqrt{\overline{I_i^2}(n,\epsilon,\delta) - \overline{I_i}^2(n,\epsilon,\delta)}\ ,$$

where M corresponds to an ensemble of different initial conditions $\theta_i \in (0,1)$ randomly chosen for a fixed $I_0 = 10^{-3}\epsilon$ and $\bar{I}_i$ is given by

$$\overline{I_i}(n,\epsilon,\delta) = \frac{1}{n}\sum_{j=1}^{n} I_{j,i}\ .$$



ω as a function of n for $\gamma = 2$, and: (a) δ and; (b) ϵ. We considered an ensemble of $B = 5000$ different initial conditions.

From behavior of ω we suppose that:

- (i) For $n \ll n_x$, ω grows according to a power law of the type

$$\omega \propto \left(n\epsilon^2\right)^{\beta},$$

where β is an exponent;

- (ii) For large n, say $n \gg n_x$, the behaviour of ω is

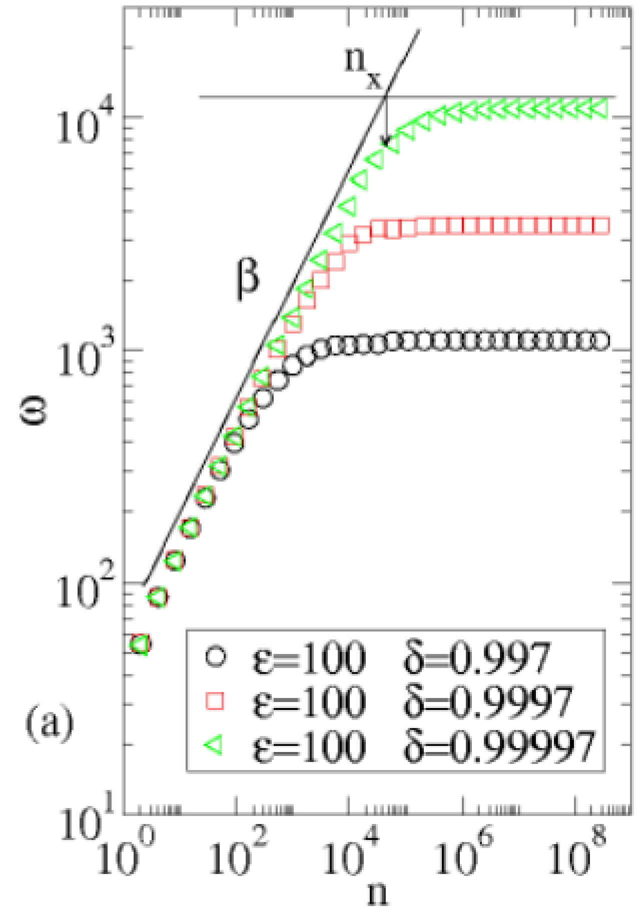
$$\omega \propto (1-\delta)^{\alpha_1}\epsilon^{\alpha_2},$$

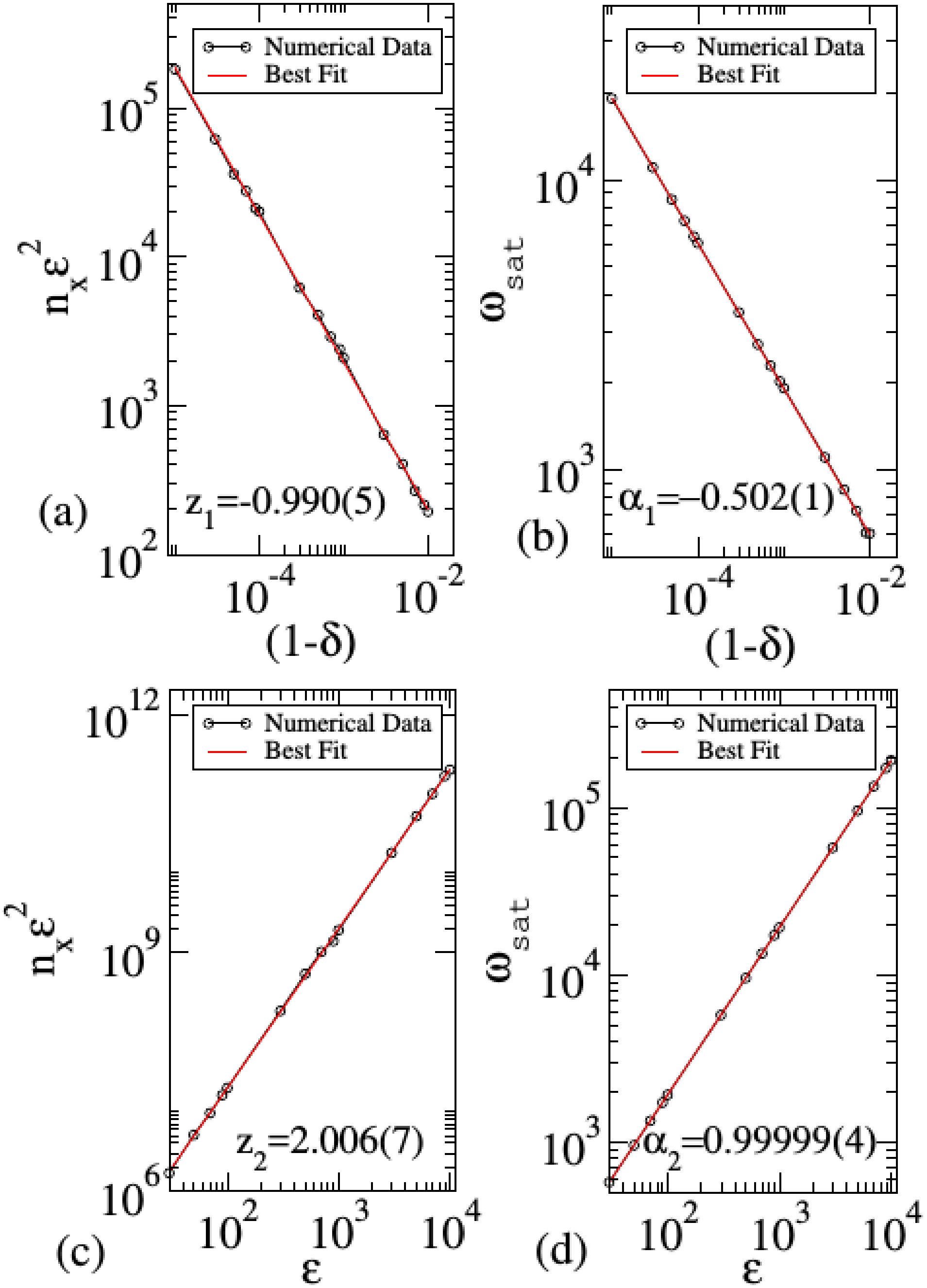
where α_1 and α_2 are critical exponents;

- (iii) The crossover n_x, that characterises the transition of the growing regime for the saturation is

$$n_x\epsilon^2 \propto (1-\delta)^{z_1}\epsilon^{z_2},$$

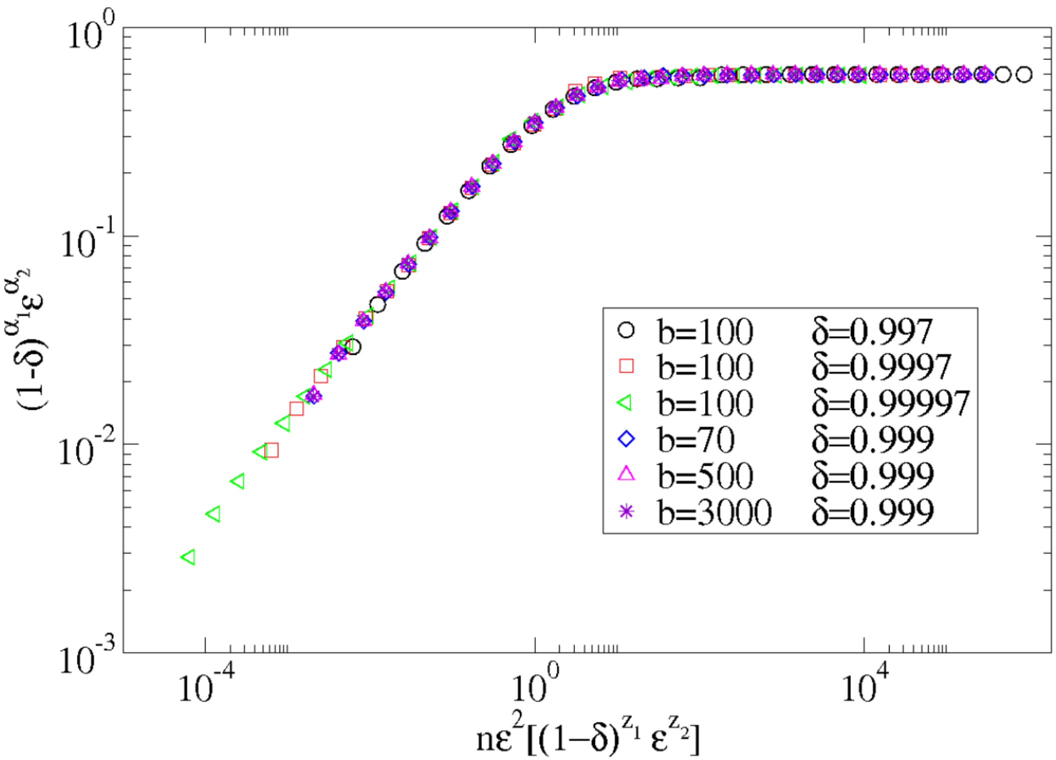
where z_1 and z_2 are called as dynamical exponents.





n_x and ω_{sat} for $\gamma = 2$ as function of:
(a,b) $(1-\delta)$ for $\epsilon = 100$ and; (c,d) ϵ for $\delta = 0.999$.

The behaviour of ω is scaling invariant with respect to the control parameters.



Overlap of different curves of ω onto a single plot.

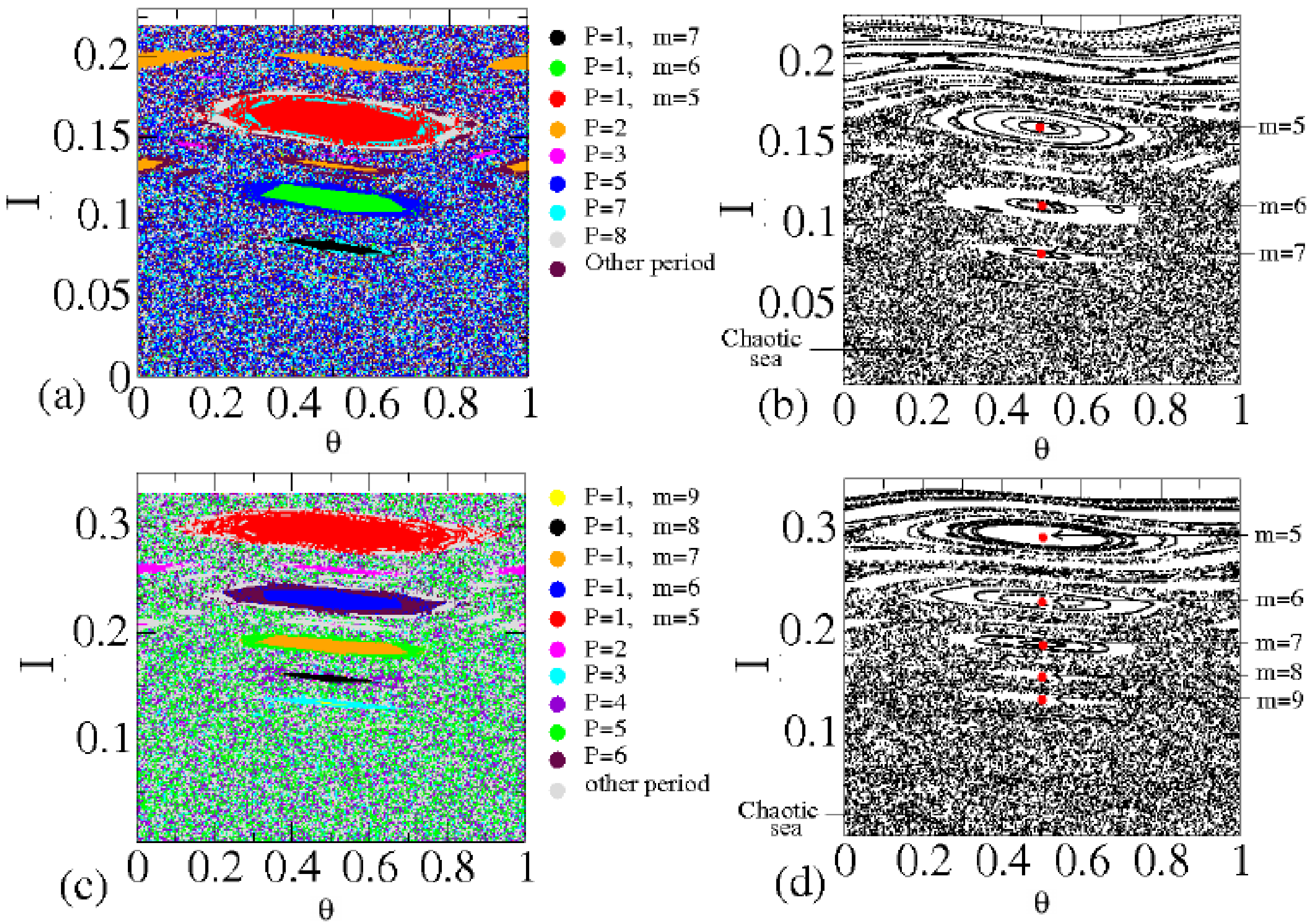
Table 1. Scaling exponents obtained for $\epsilon = 100$ and $\delta \in [0.99, 0.99999]$

γ	β	α_1	z_1
3/5	0.496(6)	−0.508(1)	−1.028(4)
3/4	0.496(4)	−0.5005(3)	−0.989(4)
4/5	0.496(5)	−0.5010(2)	−0.989(4)
1	0.494(5)	−0.496(2)	−0.988(6)
2	0.496(5)	−0.502(1)	−0.990(5)

Table 2. Scaling exponents obtained for the range $\epsilon \in [10, 10^3]$ and $\delta = 0.999$

γ	β	α_2	z_2
3/5	0.489(5)	0.9975(7)	1.987(9)
3/4	0.492(5)	0.9970(9)	1.977(4)
4/5	0.493(3)	0.9996(1)	1.995(3)
1	0.494(4)	1.0008(4)	1.997(5)
2	0.491(5)	0.999 99(4)	2.006(7)

Basin of attraction for the attractors and the phase space for the non-dissipative system.



a) and (c) basin of attraction for the periodic attractors for $\delta = 10^{-3}$. (b) and (d) phase space for the non-dissipative case. We use $\epsilon = 10^{-2}$ and: (a) and (b) $\gamma = 1/2$.

International Journal of Bifurcation and Chaos, Vol. 22, No. 10 (2012) 1250248
JULIANO A. DE OLIVEIRA* and EDSON D. LEONEL

Comments

- We have found critical exponents to define universality classes for a family of two dimensional Hamiltonian mappings;

- We checked the effects and consequences of dissipation in the dynamics of the system;

Acknowledgments

Thanks Brazilian agencies

- FAPESP (2014/18672-8);
- CNPq (421254/2016-5) and (311105/2015-7);
- CAPES;
- Pró-reitoria de pesquisa da Unesp.

Thank you very much