Non-local couplings, synchronization, pattern formation, and all that

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Introduction

- It is important to characterize the complexity of spatial patterns in many fields of Science and Technology.
- Ordered and totally disordered spatial patterns are easier to characterize, but complex patterns pose challenges.
- Our work: we use recurrence-based quantities to characterize complex spatial patterns.
- Picture: landscape near Mount Roraima (northern tip of Brazil).
Complex spatial patterns and astrophysical plasmas

ultraviolet image (SOHO/ESA/NASA)

- textured appearance = solar plasma is constantly in (turbulent) motion
- as the Sun’s magnetic field lines get tangled, they create clusters of strong magnetic activity that push around the Sun’s surface plasma
- bright spots correspond to active regions
- solar prominence seen erupting at upper right
Complex spatial patterns and fusion plasmas


- temperature distribution at target plates (TEXTOR tokamak, Jülich, Germany)
- measurements give evaluation of heat flux at tokamak wall
- high temperature regions caused by plasma particles coming from the plasma core (large connection lengths)
- non-uniformity of distribution suggests fractal structures ("magnetic footprints")
Complex spatial patterns and technological plasmas

- AFM image of the surface of polymide (PI) films
- treated by $O_2$ inductively-coupled plasma
- goal: improvement of the adhesion strength between $Cr$ layer and the PI substrate
- bottom power in the ICP system controls the bombardment energy of oxygen ions in the plasma
- reactive etching of the PI surface leads to the increased morphological surface roughening
Difficulties in the characterization of complex spatial patterns

• complex profiles are not totally smooth or fractal
• they may contain clusters of spatially ordered interspersed with disordered (rough) region
• they defy usual methods of spatial characterization (RMS roughness or fractal analysis)
One-dimensional spatial patterns

- spatial profile is discretized with steplength $\Delta$: sites at $x_i = i\Delta$
- height at each site $h(i) = h(x = x_i)$
- spatial recurrence: two lattice points $i$ and $j$ have the same value, up to some precision: $|h(i) - h(j)| \approx \ell$
- spatial recurrence matrix elements

$$R_{ij} = \Theta(\ell - |h(i) - h(j)|),$$
Spatial recurrence plots

- graphical representation of the spatial recurrence matrix
- thresholded: \( R_{ij} = 1 \) (black pixel) or 0 (white pixel)
Structures in a spatial recurrence plot and quantification

- Horizontal or vertical structures $\rightarrow$ smooth plateaus or elevations with small slope
- Diagonal structures $\rightarrow$ spatially correlated points: sloped lines, curvy segments
- Principal diagonal line always exists by construction of $R_{ij}$
- Isolated points $\rightarrow$ parts of the profile with little or no spatial correlation with its neighbors
- white bands $\rightarrow$ existence of “defects” (irregular clusters)
- recurrence rate: probability of finding spatially recurrent point

\[ \text{REC} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} R_{ij}. \]

- laminarity: fraction of pixels belonging to horizontal structures in a spatial recurrence plot
Non-locally coupled chaotic maps

- non-local coupling: interaction strength decreases with the lattice distance as a power-law (with exponent $\alpha$)
- local dynamics: chaotic logistic map $f(x) = 4x(1 - x)$, with $x \in [0, 1]$
- $x_n^{(i)}$: state variable at time $n$ and spatial position $i = 1, 2, \ldots N$ (one-dimensional chain)
- periodic boundary conditions: $x_n^{(i)} = x_n^{(i\pm N)}$, random initial conditions $x_0^{(i)}$

$$x_{n+1}^{(i)} = (1 - \varepsilon)f(x_n^{(i)}) + \frac{\varepsilon}{\eta(\alpha)} \sum_{j=1}^{N'} \left[ f(x_n^{(i+j)}) + f(x_n^{(i-j)}) \right] \frac{1}{j^\alpha}$$

- normalization factor: $\eta(\alpha) = 2 \sum_{j=1}^{N'} j^{-\alpha}$
Spatial patterns and spatial recurrence plots

- $N = 1001$ maps with $\varepsilon = 1.0$ and (a) $\alpha = 2.0$, (b) $2.8$
- Top: spatial pattern at fixed time $n = 2001$
- Bottom: spatial recurrence plot with cutoff radius $\ell = 0.05$
Detecting synchronized patterns using spatial recurrences

- synchronized patterns
  \[ x_n^{(1)} = \ldots = x_n^{(N)} \]
- large laminarity corresponds to complete synchronization
- transition to non-synchronized behavior as \( \alpha \) increases (coupling becomes more local)
- \( \alpha \to \infty \): local (nearest-neighbor) coupling
- \( \alpha \to 0 \): global (all-to-all) coupling
Two-dimensional spatial patterns

- two-dimensional pattern $h_{ij}$ of $N$ sites
- at each site we take a square block of side $N'$ and compute the recurrence rate of $h_{ij}$ with all the other sites belonging to this block
- we obtain a recurrence-rate matrix with elements

$$REC_{ij} = \frac{1}{N'^2} \sum_{k=i}^{i+N'} \sum_{\ell=i}^{i+N'} \Theta (\ell - |h_{ij} - h_{k\ell}|) ,$$
Reaction-diffusion equations

- activator-inhibitor system

\[
\frac{\partial u}{\partial t} = f(u, v) + D_u \nabla^2 u
\]

\[
\frac{\partial v}{\partial t} = g(u, v) + D_v \nabla^2 v
\]

- \( u(r, t) \): local concentration of an activator species
- \( v(r, t) \): idem for an inhibitor species
- \( f(u, v) \) and \( g(u, v) \): local dynamics of the reaction (rate equations)
- local coupling with different diffusion coefficients \( D_u \) and \( D_v \)
- basic ingredients: mass conservation and Fick’s law (diffusion flux proportional to the concentration gradient)
**Reaction-diffusion equations in a square lattice**

- two-dimensional lattice with spacing $\Delta$
- discretized variables $k, j = 0, 1, \ldots (N - 1)$

\[
\begin{align*}
  u_{k,j}(t) &= u(x = k\Delta, y = j\Delta, t), \\
  v_{k,j}(t) &= v(x = k\Delta, y = j\Delta, t),
\end{align*}
\]

- discretized system of ODE’s

\[
\begin{align*}
  \frac{du_{k,j}}{dt} &= f(u_{k,j}, v_{k,j}) + \frac{D_u}{4} (u_{k+1,j} + u_{k-1,j} + u_{k,j+1} + u_{k,j-1} - 4u_{k,j}) \\
  \frac{dv_{k,j}}{dt} &= g(u_{k,j}, v_{k,j}) + \frac{D_v}{4} (v_{k+1,j} + v_{k-1,j} + v_{i,k+1} + v_{i,k-1} - 4v_{k,j})
\end{align*}
\]
Reaction-diffusion equations with nonlocal coupling

- diffusion flux does not depend only on the local gradients but on a wider vicinity of each point
- yields a (non-linear) integro-differential equation

\[
\frac{\partial u}{\partial t} = f(u, v) + D_u \int d^2 r' \sigma(r, r') u(r', t),
\]

\[
\frac{\partial v}{\partial t} = g(u, v) + D_v \int d^2 r' \sigma(r, r') v(r', t)
\]

- \(\sigma(r, r')\) is a non-local interaction kernel
- if the coupling is mediated by a third (locally) diffusing substance then \(\sigma(r) \sim K_0(r)\) in the fast relaxation limit (Kuramoto and Nakao)
- another example is a power-law kernel: \(\sigma(r) \sim r^{-\alpha}\)
Reaction-diffusion equations in a square lattice with powerlaw coupling

\[
\frac{du_{k,j}}{dt} = f(u_{k,j}, v_{k,j}) - D_u u_{k,j} + \frac{D_u}{\kappa(\alpha)} \sum_{r=-N'}^{N'} \sum_{\ell=-N'}^{N'} u_{k+r,j+\ell} \frac{1}{(r^2 + \ell^2)^{\alpha/2}}
\]

\[
\frac{dv_{k,j}}{dt} = g(u_{k,j}, v_{k,j}) - D_v v_{k,j} + \frac{D_v}{\kappa(\alpha)} \sum_{r=-N'}^{N'} \sum_{\ell=-N'}^{N'} v_{k+r,j+\ell} \frac{1}{(r^2 + \ell^2)^{\alpha/2}}
\]

- \( N' = (N - 1)/2 \), with \( N \) odd,
- starred sums: we exclude from them the terms with \( r = \ell = 0 \)
- normalization factor

\[
\kappa(\alpha) = \sum_{r=-N'}^{N'} \sum_{\ell=-N'}^{N'} \frac{1}{(r^2 + s^2)^{\alpha/2}}
\]
Local dynamics of activator-inhibitor system

- Meinhardt and Gierer model [RMP 66, 1481 (1994)]

\[ f(u, v) = 0.01 \left( \frac{u^2}{v} - u \right) \quad g(u, v) = 0.02 \left( u^2 - v \right) \]

- activator undergoes an auto-catalytic reaction with inhibition and degradation
- inhibitor also increases with activator and suffers degradation
- two equilibria: \((0, 0)\) (unstable) and \((1, 1)\) (asymptotically stable)
- these are also equilibria for the coupled system
- Turing instability: a formerly stable spatially homogeneous pattern becomes inhomogeneous as the diffusion coefficients are changed
- pattern formation occurs after linearly unstable modes suffer saturation (due to the nonlinearity of the local dynamics)
Spatial patterns in the locally coupled case ($\alpha = 1000$)

- $101 \times 101$ cells, $D_u = 0.016$, $D_v = 0.2$, activator concentration in colorscale
- Left: spatial pattern (fixed time): “zebra” patterns (fingering - lateral inhibition)
- Right: recurrence-rate matrix (blocks of size $N' = 20$): $REC \sim 0.25 - 0.29$, not spatially homogeneous
Spatial patterns in the locally coupled case ($\alpha = 1000$)

- 101 × 101 cells, $D_u = 0.005$, $D_v = 0.2$
- Left: spatial pattern (fixed time): spot “leopard” patterns (fingering - lateral inhibition)
- Right: recurrence-rate matrix: $REC \sim 0.74 - 0.81$, locally recurrent regions
Spatial patterns in the intermediate coupled case ($\alpha = 1$)

- $101 \times 101$ cells, $D_u = 0.015$, $D_v = 0.2$
- Left: spatial pattern (fixed time): regular (sinusoidal) pattern
- Right: recurrence-rate matrix: $REC \sim 0.3 - 0.9$, lower-concentration background more recurrent than crests
Spatial patterns in the intermediate coupled case ($\alpha = 1$)

- 101 $\times$ 101 cells, $D_u = 0.005$, $D_v = 0.2$
- Left: spatial pattern (fixed time): spots again but with higher concentration (squeezing effect)
- Right: recurrence-rate matrix: capable to detect very subtle differences in spatial patterns due to localized structures
Spatial patterns produced by mammographic images

- scanned mammographic images (LAPIMO, USP, S. Carlos)
- transformed into integer grey-level matrices → lossless image files with 65,000 levels
- very sensitive to differences in grey-levels → detection of structures barely visible to the naked eye (tumours)
Conclusions

- complex spatial patterns have coexistent ordered and disordered parts
- we developed recurrence-based numerical diagnostics for characterization of complexity in spatial patterns (1D and 2D)
- they are good indicators of synchronized states
- spatio-temporal dynamical systems: patterns in reaction-diffusion equations (activator-inhibitor)
- recurrence-based diagnostics are very sensitive to pattern changes
- analysis of images: structures in digital mammographic images
Thank you very much.

Publications