# Cenário de Ruelle-Takens via Quase-Periodicidade

Iberê L. Caldas

Abril de 2009

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- I- Cenário de Landau
- II- Mapa do Círculo
- III- Cenário de Ruelle-Takens
- IV- Janelas Periódicas

# I- Cenário de Landau

Landau e Lifshitz, Mecânica dos Fluidos: Oscilação turbulenta de F seria resultante de bifurcações sucessivas que dão origem às frequências $f_i$

$F = F(f_1 t, f_2 t, f_3 t, \ldots fkt) \rightarrow$ Torus k

Cenário: $T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow .. \rightarrow T_k$

*Cenário nunca verificado experimentalmente*

# II- Mapa (Unidimensional) do Círculo

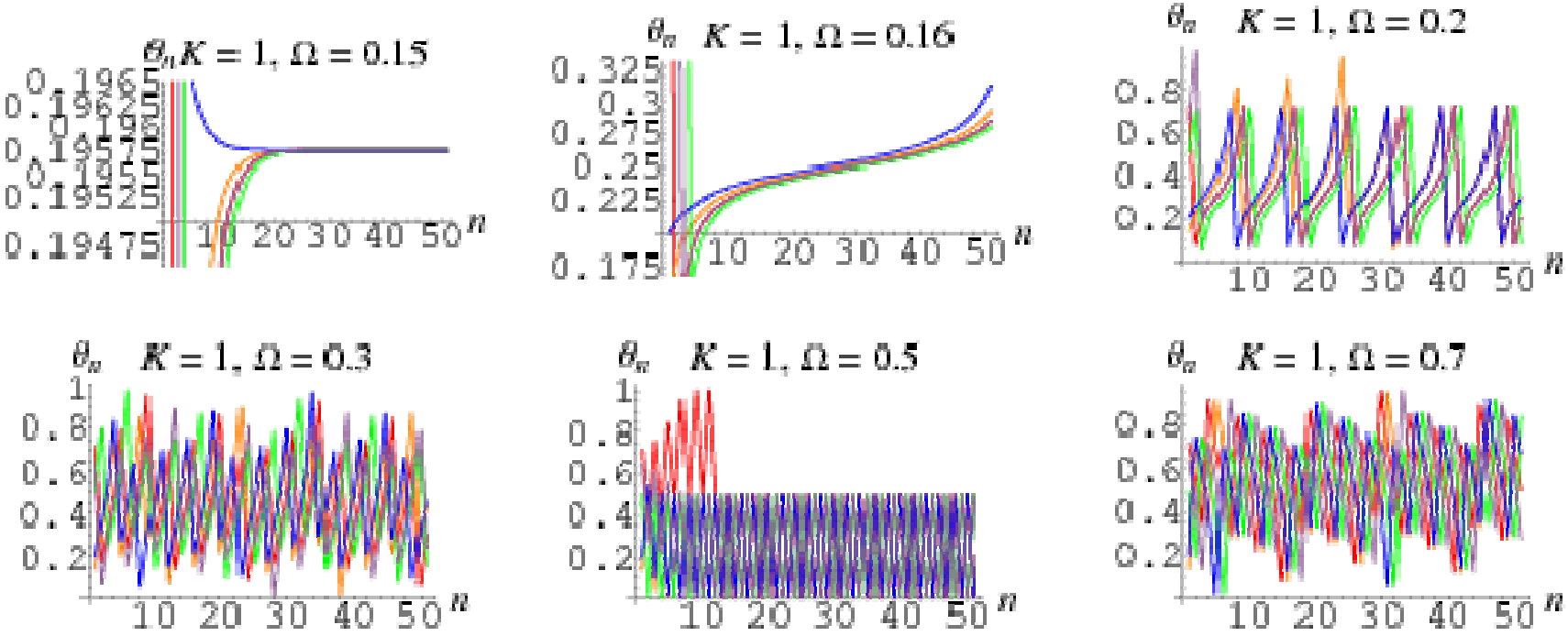
# Mapa do Círculo

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n).$$

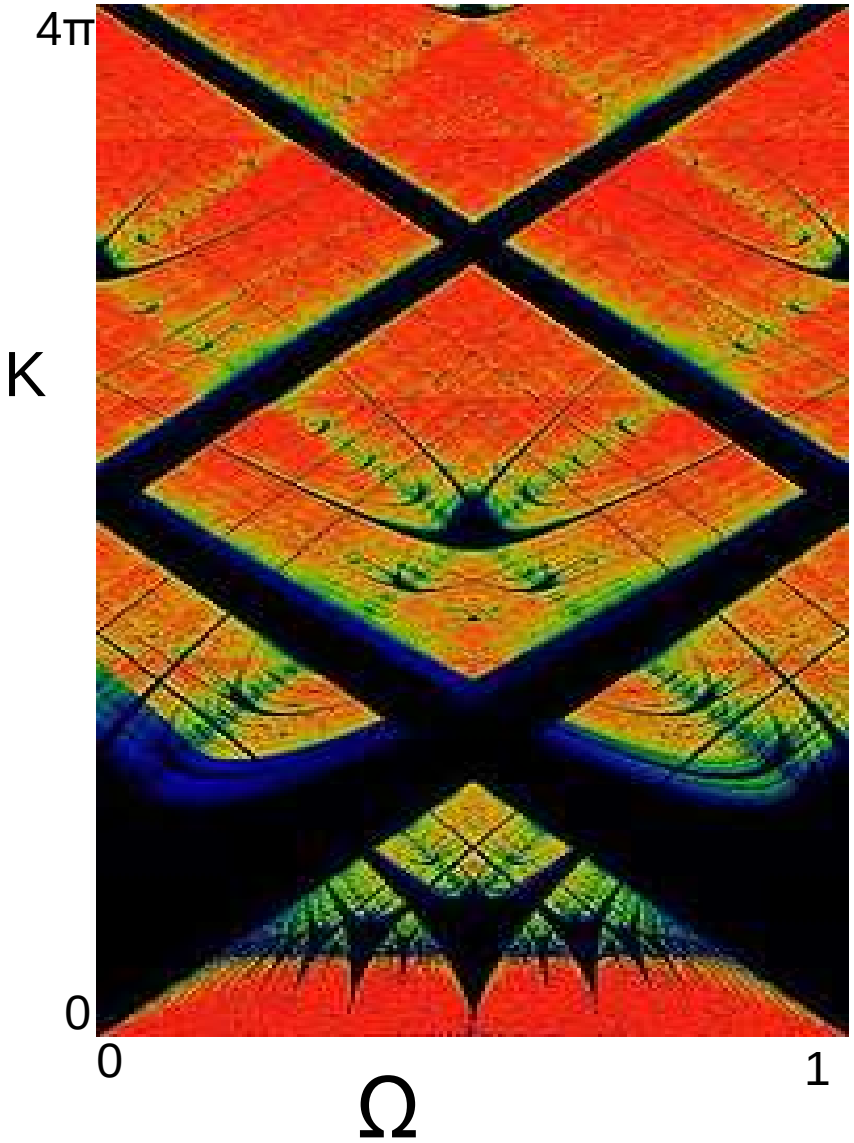
K: constante de acoplamento
Ω: frequência de forçamento

K = 0 → rotação
Ω = n/m → órbitas quase-periódicas
Ω = irracional → órbitas periódicas

# Exemplos de Séries Temporais



# Espaço dos Parâmetros




Preto: órbitas periódicas

K~ 0 → linguas de Arnold

Ω = n/m → órbitas periódicas
Ω = irracional → órbitas quase-periódicas

Multiestabilidade
(coexistência de atratores)

# Número de Rotação

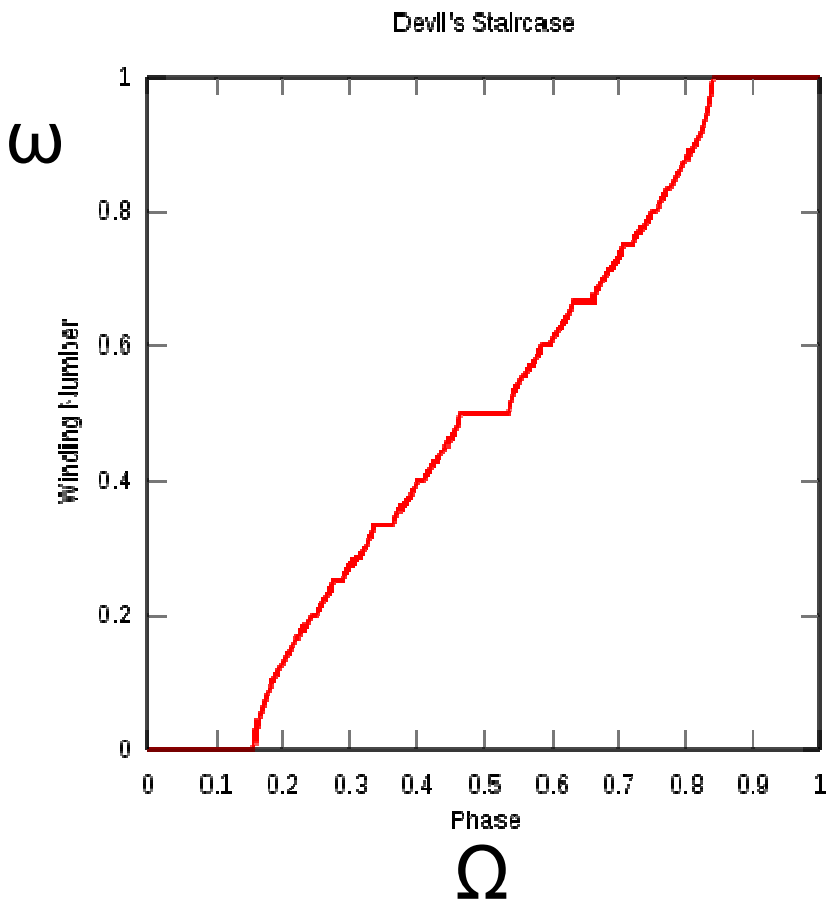
$$\omega = \lim_{n \to \infty} \frac{\theta_n}{n}.$$

Órbitas regulares
ω = racional = n/m → órbitas periódicas
ω = irracional → órbitas quase-periódicas

Órbitas caóticas → ω não converge

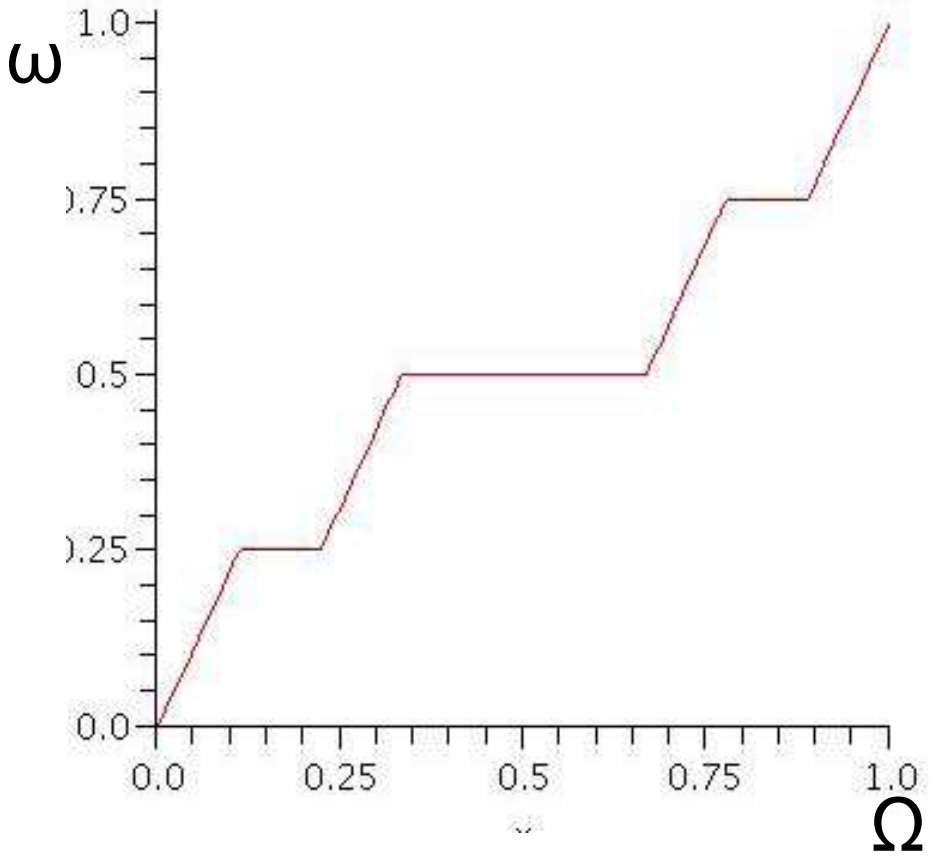
# A Escada do Diabo

Órbitas periódicas
→ Ω = cte.



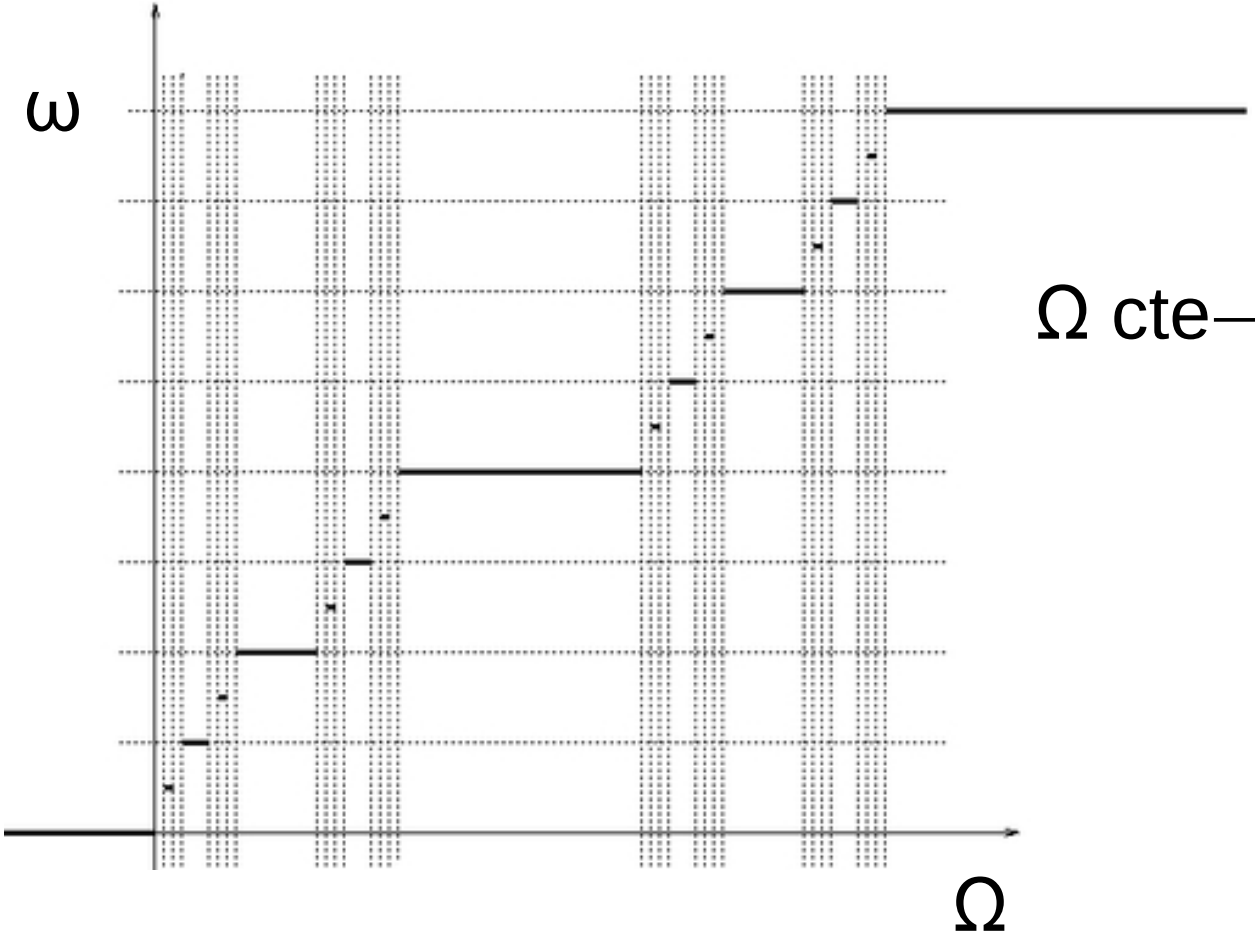

Número de rotação ω x frequência Ω
para K = 1

# Escada do Diabo (Fractal)




Número de rotação X frequência
(amplitude fixa)

# Escada do Diabo (Fractal)

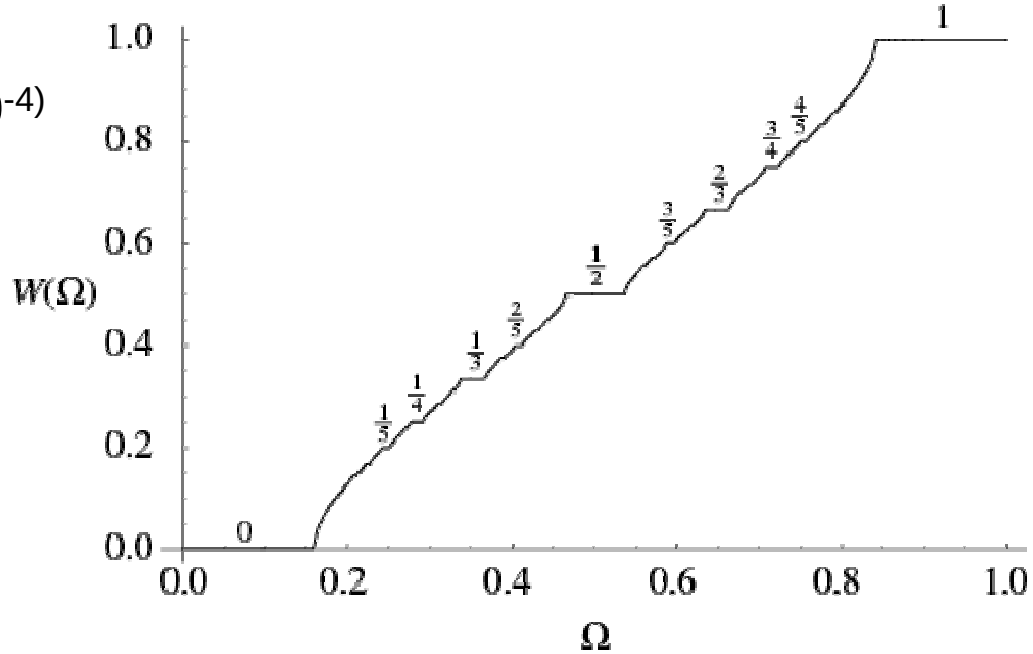



Ω cte→ órbita periódica

Número de rotação X frequência (amplitude fixa)

# Número de Rotação de Algumas Órbitas Periódicas

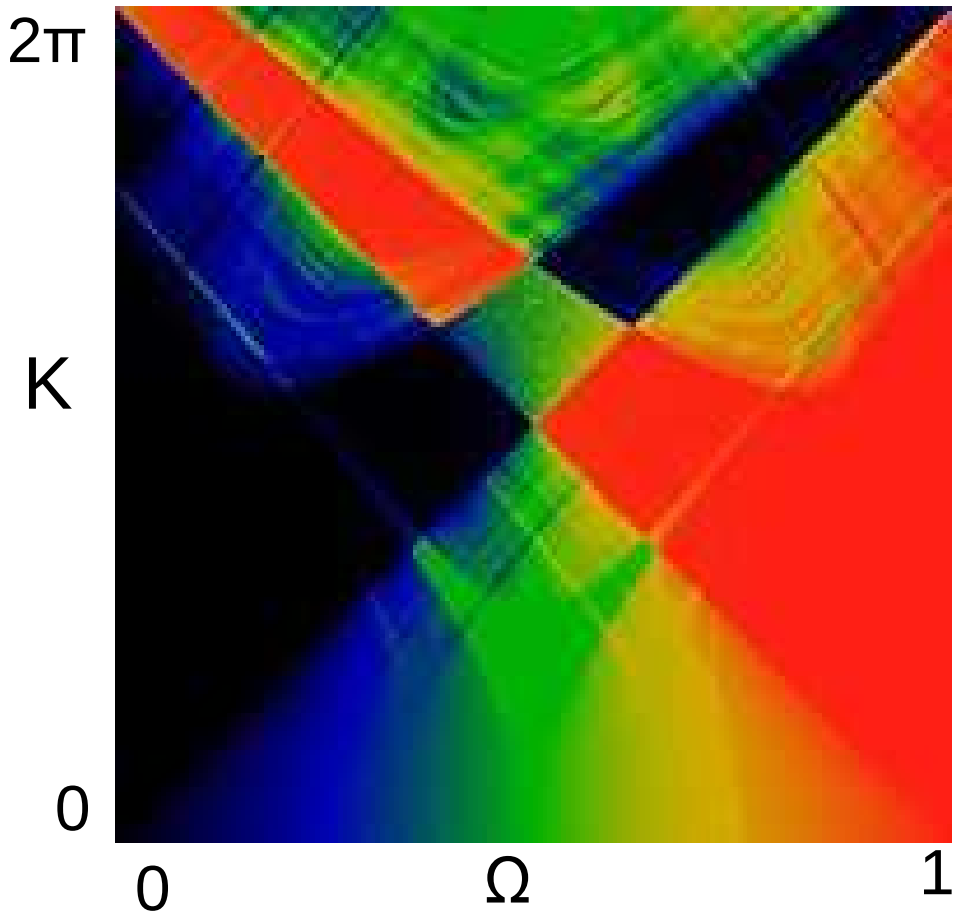
W = m/n, constante, menos no conjunto de Cantor (dimensão $0{,}9 \times 10^{-4}$)




Número de rotação x frequência (K = 1)
(W não depende da condição inicial)

onal number. This is a monotonic increasing "staircase" for

# Cálculo do Número de Rotação ω




Preto: ω = 0
Verde: ω = 0,5
Vermelho: ω = 1

# III- cenário de Ruelle-Takens

# Ruelle-Takens Theorem

D. Ruelle and F. Takens, Comm. Math. Phys. **20**, 167 (1971)

An immensely influential paper in the history of the study of chaos was the work of Ruelle and Takens [1], later extended with Newhouse [2] on the robustness of quasiperiodic dynamics with 3 or more incommensurate frequencies. This addressed the question of whether complexity in dynamics was likely to occur through the accumulation of Hopf-like bifurcations adding additional frequencies (and presumably spatial "modes"), or instead through the onset of low dimensional chaos. The statement of the theorem was quite mathematical, which led initially to a misunderstanding of the strength of the result in the physics community, but nevertheless it remains a powerful result.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# Rota em um Mapa de Poincaré Bidimensional (Fluxo Tridimensional)

C. Grebogi, E. Ott and J.A. Yorke Phys. Rev. Lett. **51**, 339 (1983); Physica **D15**, 354 (1985)

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega_1 + \varepsilon P_1(\theta_n, \phi_n) \mod 1 \\ \phi_{n+1} &= \phi_n + \omega_2 + \varepsilon P_2(\theta_n, \phi_n) \mod 1 \end{aligned}$$

where $P_1$, $P_2$ are nonlinear functions periodic in both $\theta$ and $\phi$ with period $2\pi$. The explicit forms used were sums of sinusoidal functions $A_{rs} \sin[2\pi(r\theta + s\phi + B_{rs})]$ with $A_{rs}$ and $B_{rs}$ chosen randomly for $(r, s)$ taking the values $(0, 1)$, $(1, 0)$, $(1, 1)$, $(1, -1)$. Equation (21.2) takes the form of two, nonlinearly coupled circle maps.

(P), as well as chaos. The types of motion were identified through the values of the two Lyapunov exponents (together with the third value 0 for the corresponding flow):

| Map exponents | Flow Exponents | Dynamics |
|---|---|---|
| 0, 0 | 0, 0, 0 | 3-frequency QP |
| 0, − | 0, 0, − | 2-frequency QP |
| −, − | 0, −, − | 1-frequency (P) |
| +, ? | +, 0, ? | Chaotic |

They found the percentage of the values of leading to each type of motion

| Attractor | $\varepsilon/\varepsilon_c = 0.375$ | $\varepsilon/\varepsilon_c = 0.75$ | $\varepsilon/\varepsilon_c = 1.125$ |
|---|---|---|---|
| 3-frequency QP | 82% | 44% | 0% |
| 2-frequency QP | 16% | 38% | 32% |
| Periodic | 2% | 11% | 31% |
| Chaotic | 0% | 7% | 36% |

where $\varepsilon_c$ is the value of $\varepsilon$ for which the map becomes noninvertible. (When the map is noninvertible it can be shown that there is no 3-frequency periodic motion, c.f. the 1d circle map at $K > 1$.) Thus in this example at least, 3-frequency quasiperiodic

# Evidências Experimentais

Experimentally 3-frequency quasiperiodic motion has been documented by Gollub and Benson [4], and 4-frequency and 5-frequency quasiperiodic motion by Walden et al.[5], both in fluid convection.


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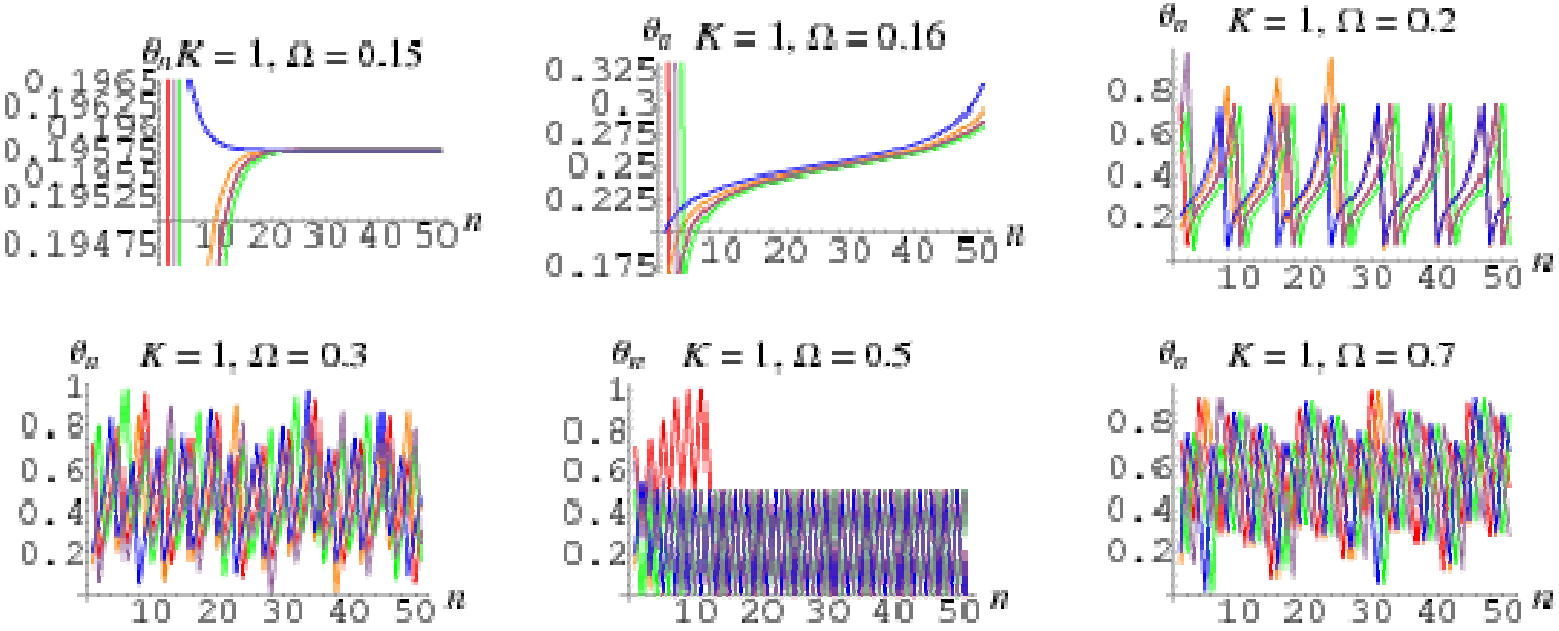

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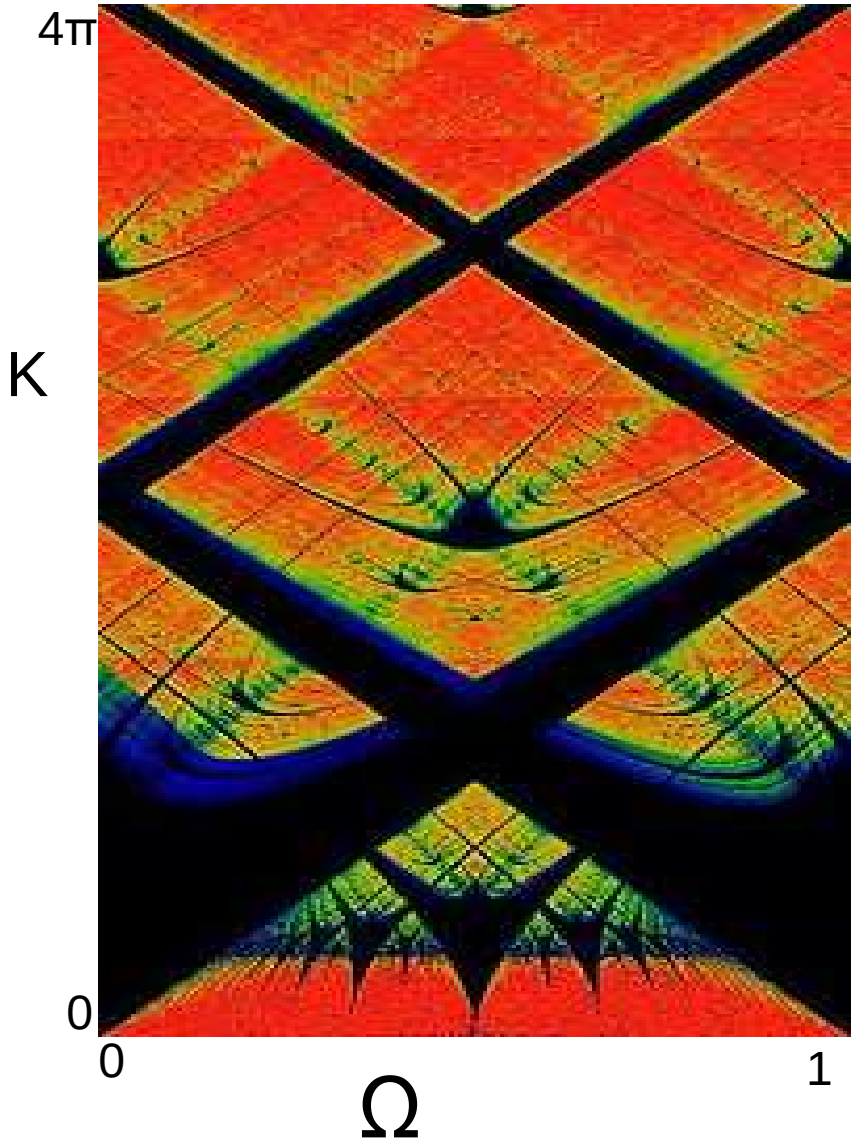


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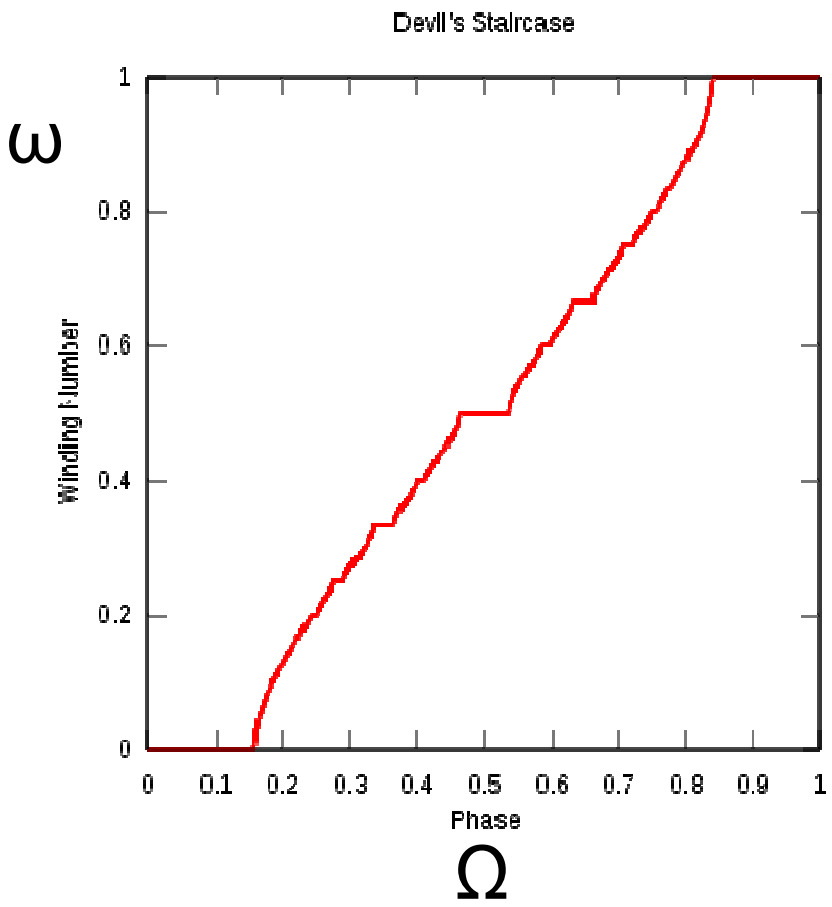
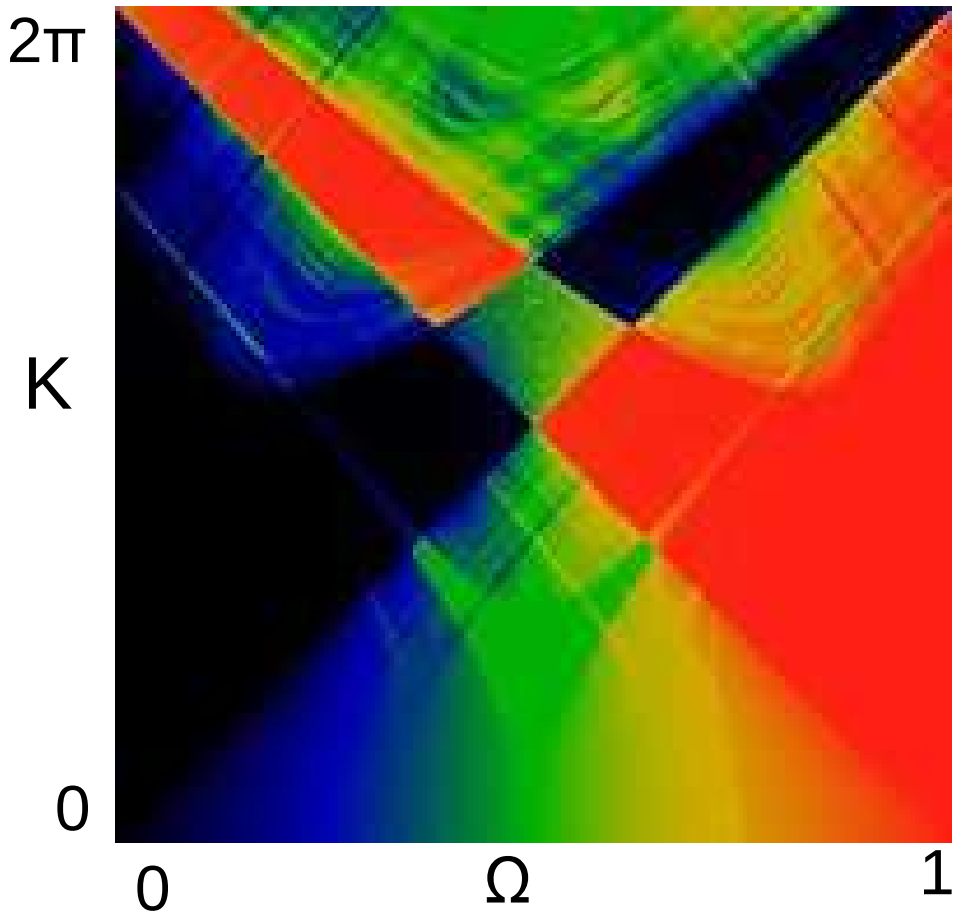



Número de rotação ω x frequência Ω
para K = 1

# Cálculo do Número de Rotação ω

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# IV- Janelas Periódicas:
# Um Cenário Homoclínico

Rene O. Medrano-T., Iberê L. Caldas

IF-USP

Abril / 2009

Motivação:

Espaço dos parâmetros das equações não lineares com soluções caóticas

Propriedades a serem explicadas:

a - Janelas periódicas (para mapas e equações diferenciais)

b - Distribuição dessas janelas

c - Sequência de janelas com períodos progressivamente crescentes

Sistema de Roessler

$$\dot{x} = -y - z$$

$$\dot{y} = x + a\,y$$

$$\dot{z} = b + (x - c)\,z$$

Variáveis: $x, y, z \rightarrow$ espaço de fase tridimensional

Parâmetros de controle: $a,\ b,\ c$

# Atrator Caótico de Roessler

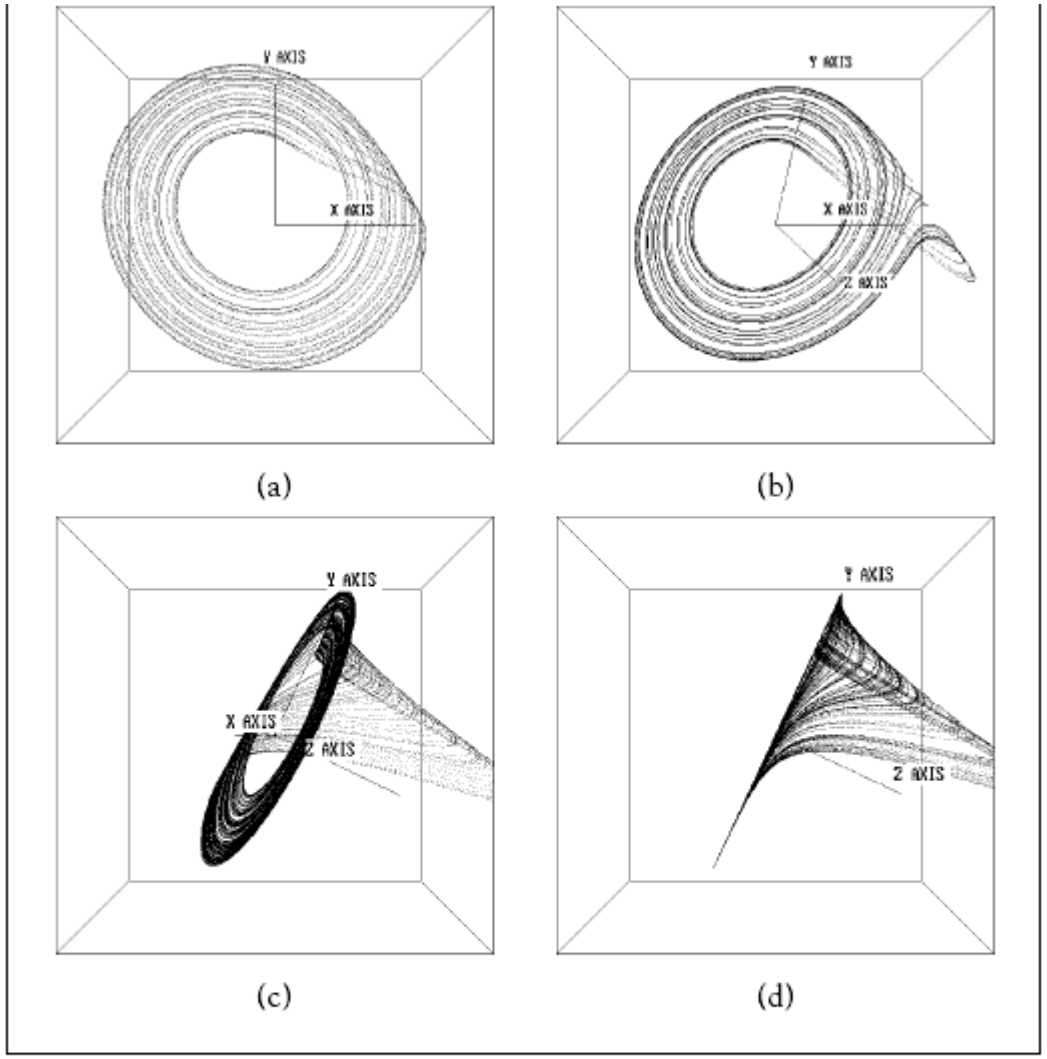



**Figure 9.6 The Rössler attractor.**
Parameters are set at $a = 0.1$, $b = 0.1$, and $c = 14$. Four different views are shown. The dynamics consists of a spiraling out from the inside along the $xy$-plane followed by a large excursion in the $z$-direction, followed by re-insertion to the vicinity of the $xy$-plane. Part (d) shows a side view. The Lyapunov dimension is 2.005—indeed it looks like a surface.

Chaos
Alligood et al.

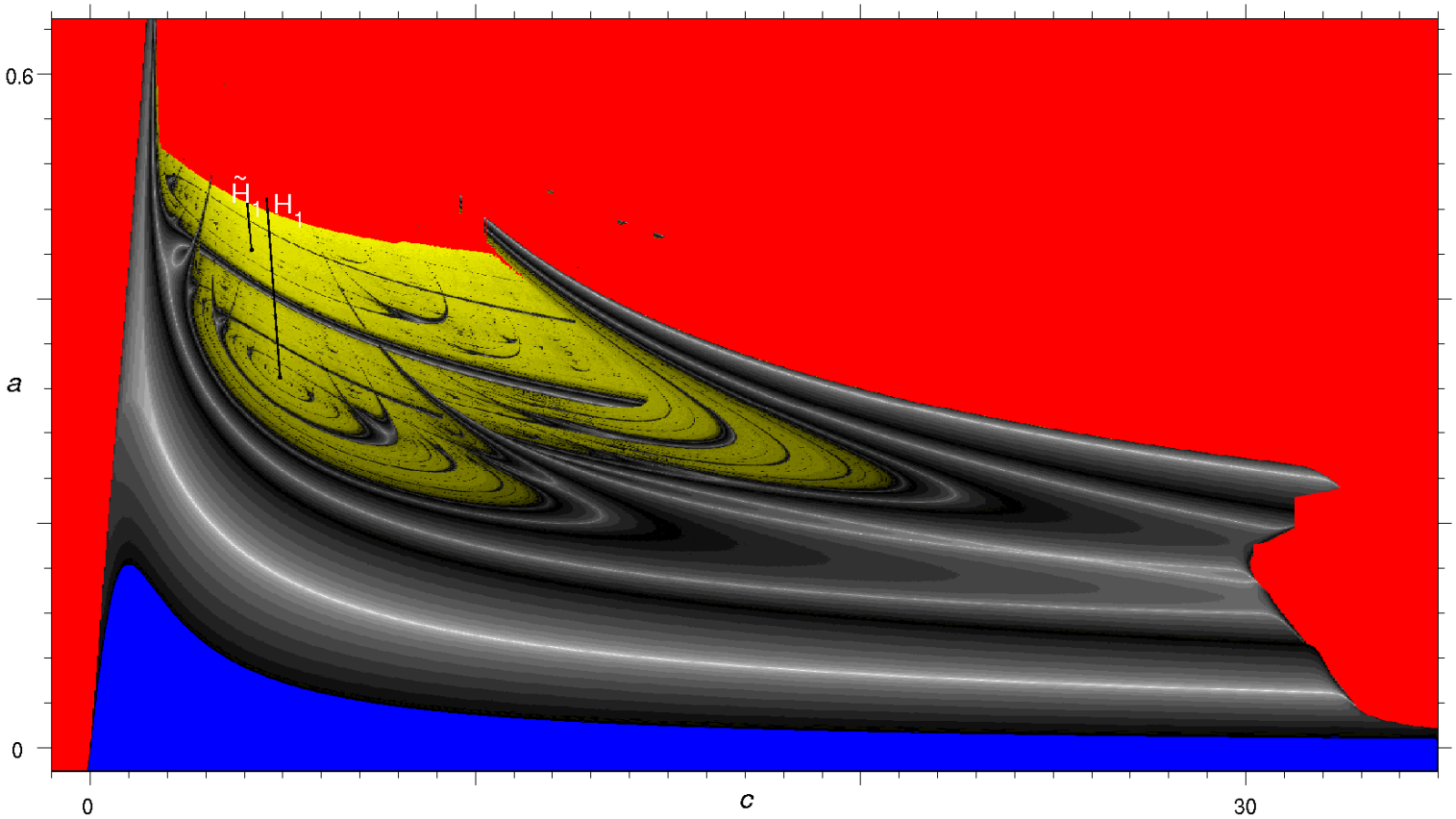
Resultados obtidos para as equações de Roessler

$$\text{Previsto pelo teorema de Shilnikov} \Rightarrow \begin{cases} \text{órbitas homoclínicas} \\ \\ \text{distribuição, no espaço dos parâmetros,} \\ \text{de janelas periódicas em torno} \\ \text{das órbitas homoclínicas} \end{cases}$$

Aplicação do teorema de Shilnikov:

Descoberta da distribuição de órbitas homoclínicas no espaço dos parâmetros $\Rightarrow$ distribuição das janelas periódicas

# Sistema de Roessler
# Atratores no Espaço dos Parâmetros

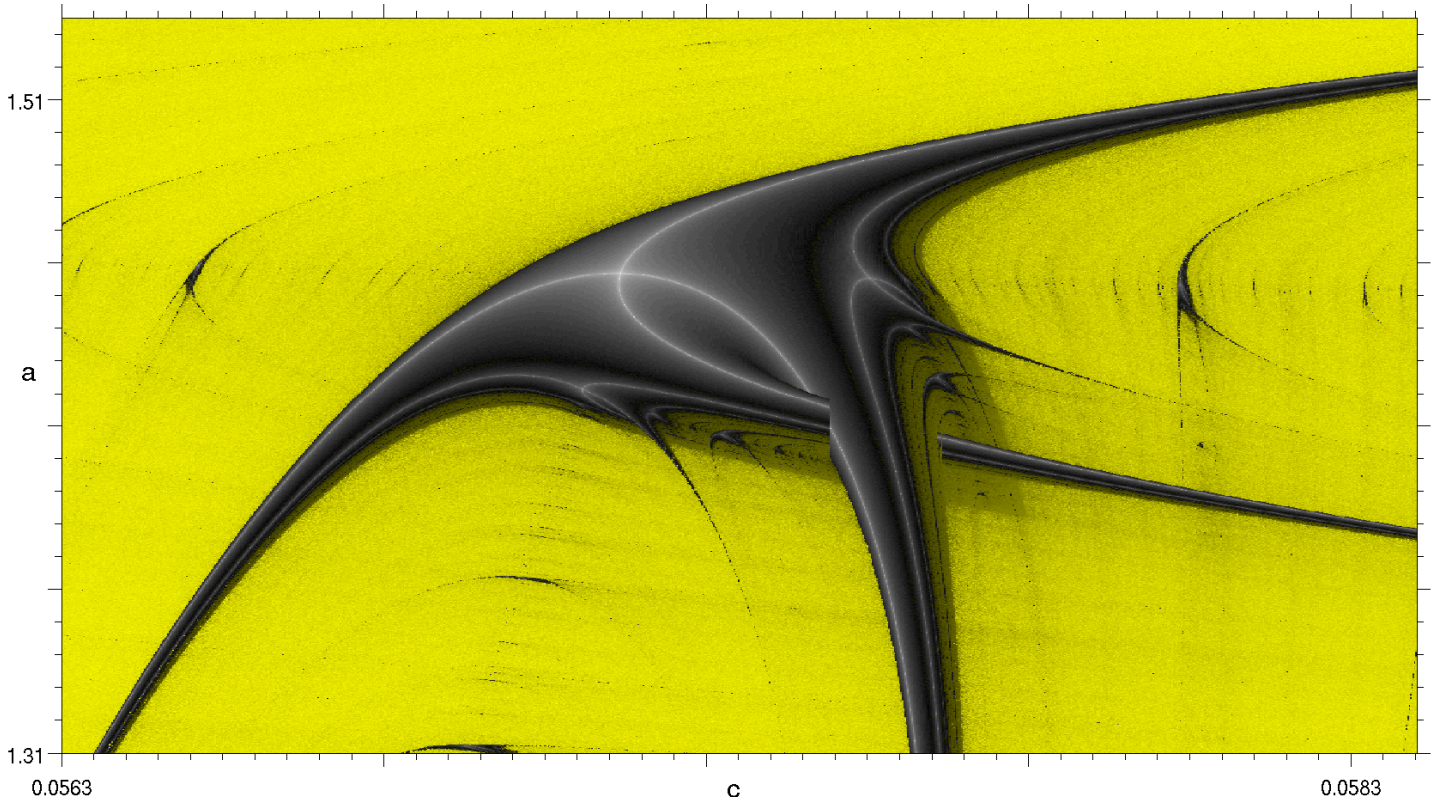



Azul: ponto fixo
Vermelho: infinito
Amarelo: atrator caótico
Preto a branco: atrator periódico

Expoente de Lypunov do atrator periódico
(graduação crescente de branco para preto)
b = 0,3

# Janela Periódica no Espaço dos Parâmetros

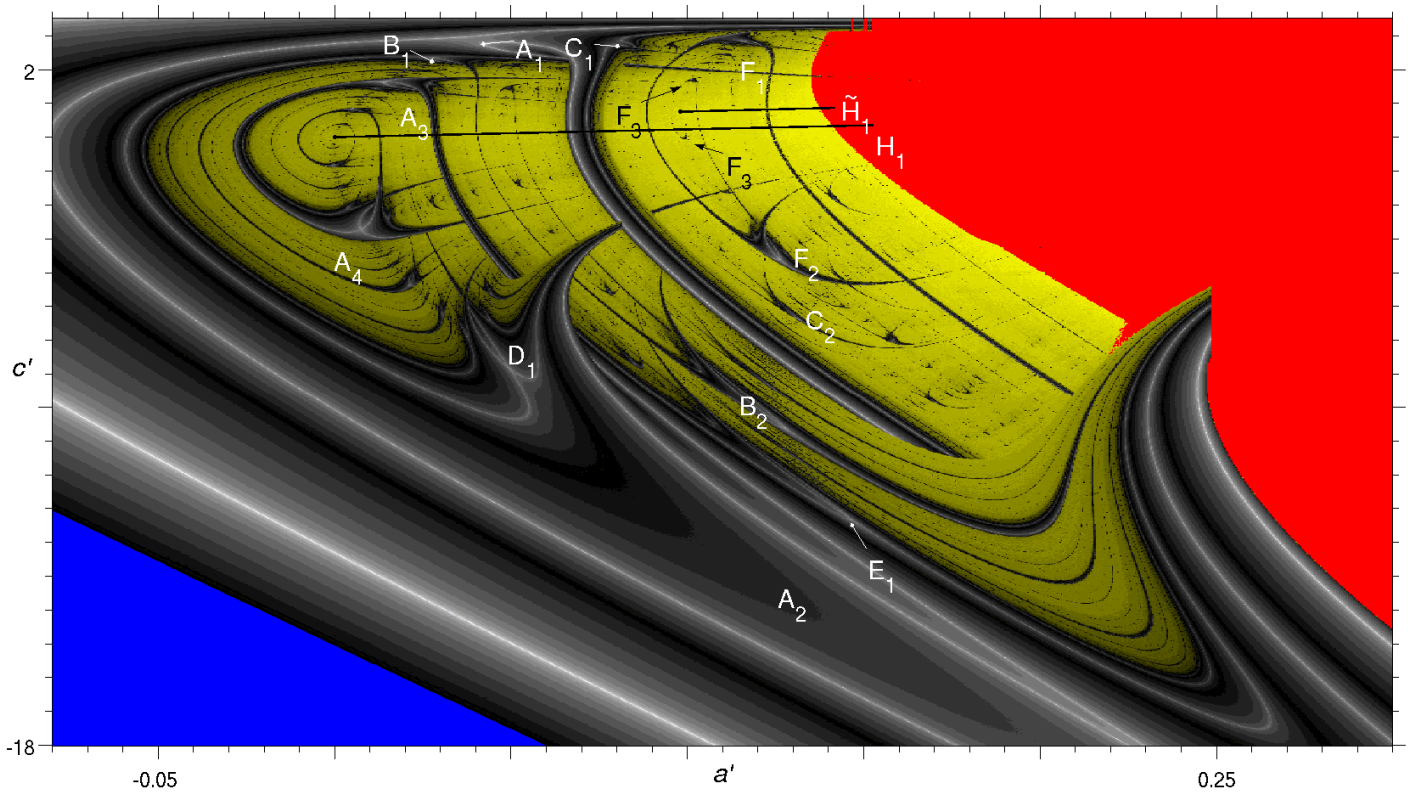
(b = 0.3)




Amarelo: atrator caótico

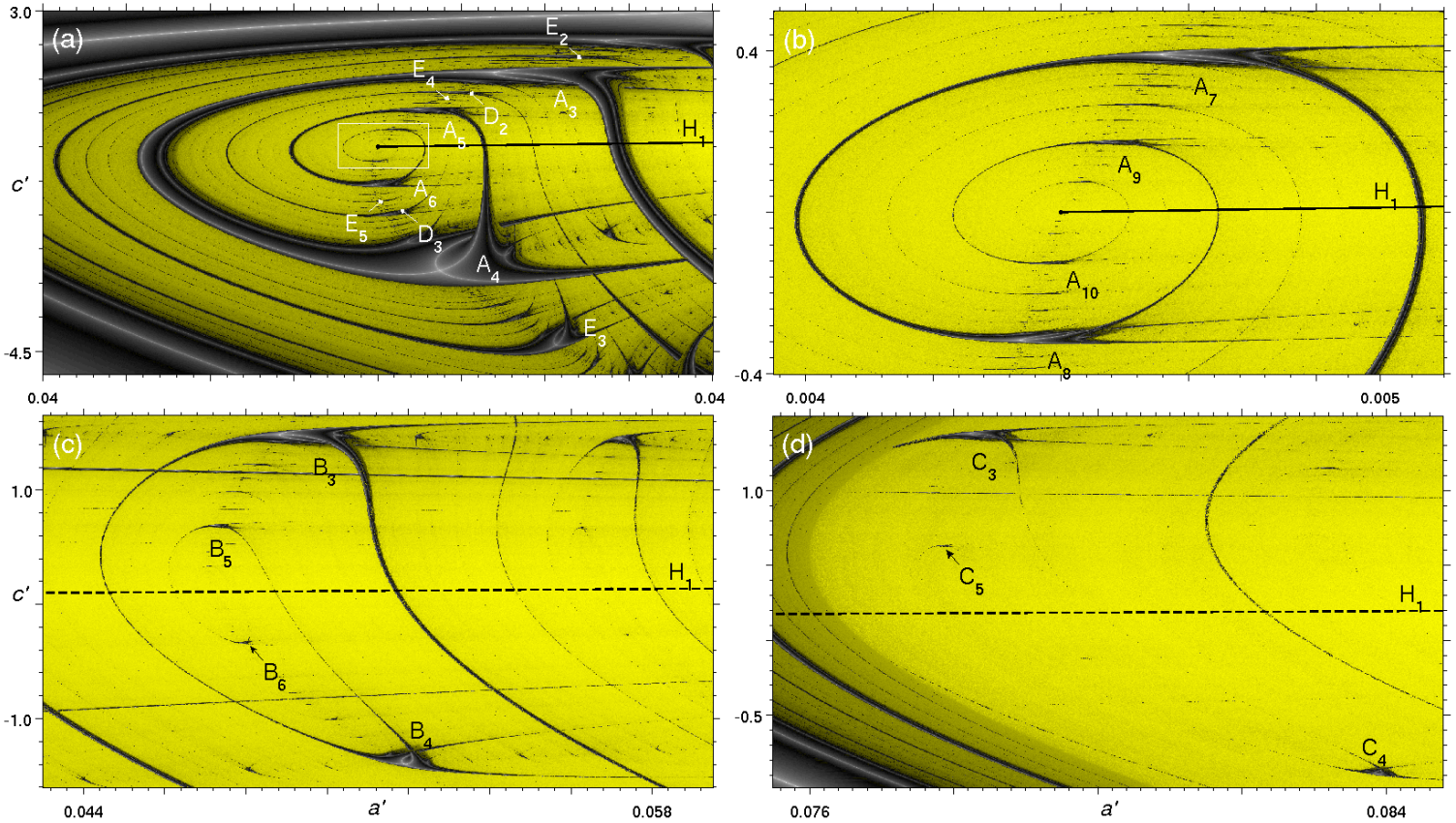
Graduação crescente de branco para preto: expoente de Lypunov do atrator periódico

# Sistema de Roessler
# Órbitas Homoclínicas e Atratores no Espaço dos Parâmetros




Linhas $H_1$ e $\tilde{H}_1$ : parâmetros com órbitas homoclínicas

# Sequências de Janelas Periódicas
# Órbitas Homoclínicas H1



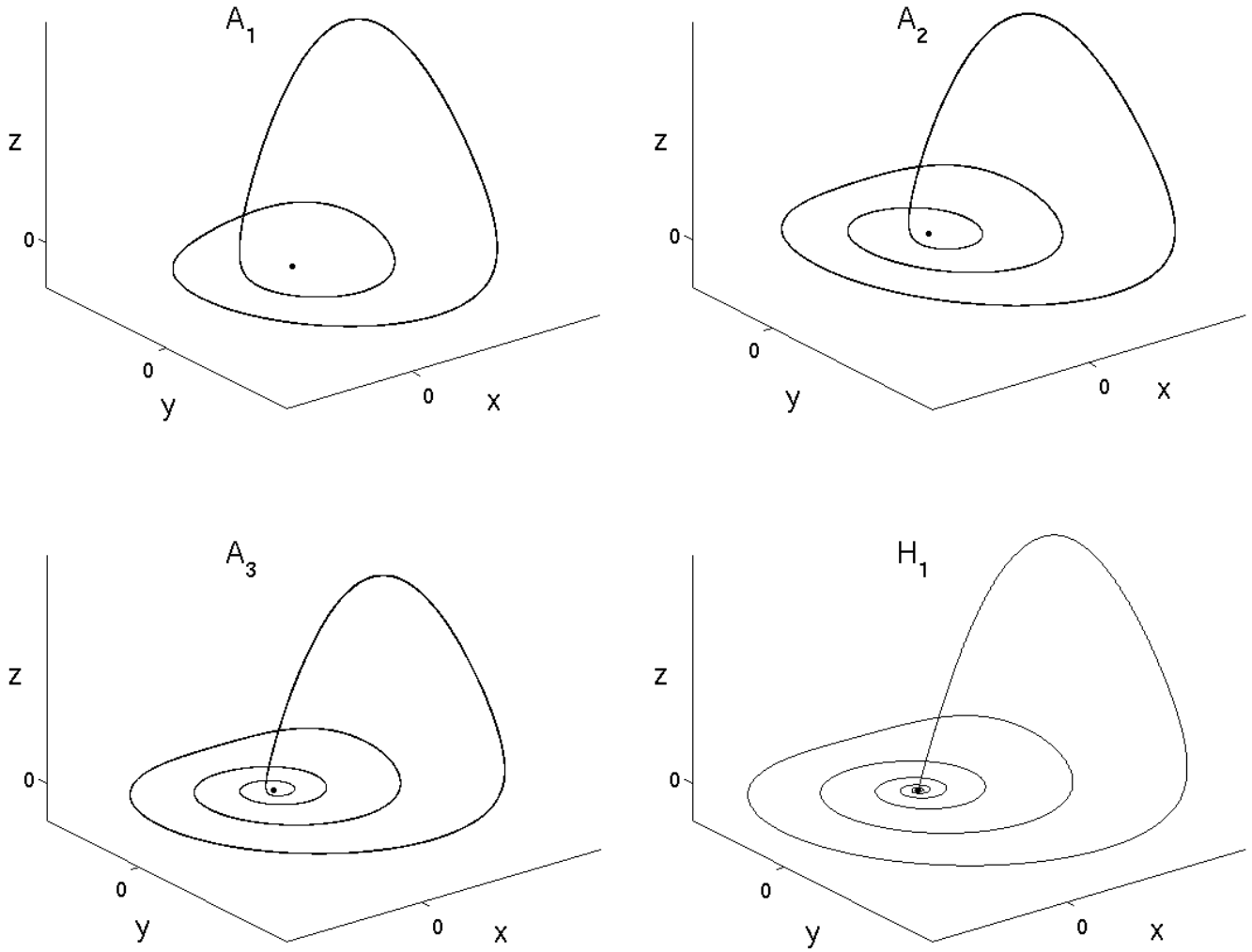

(a) Janelas periódicas $A_3, A_4, A_5, A_6$

(b) Janelas $A_7, A_8, A_9, A_{10}$

(c) Janelas $B_3, B_4, B_5, B_6$

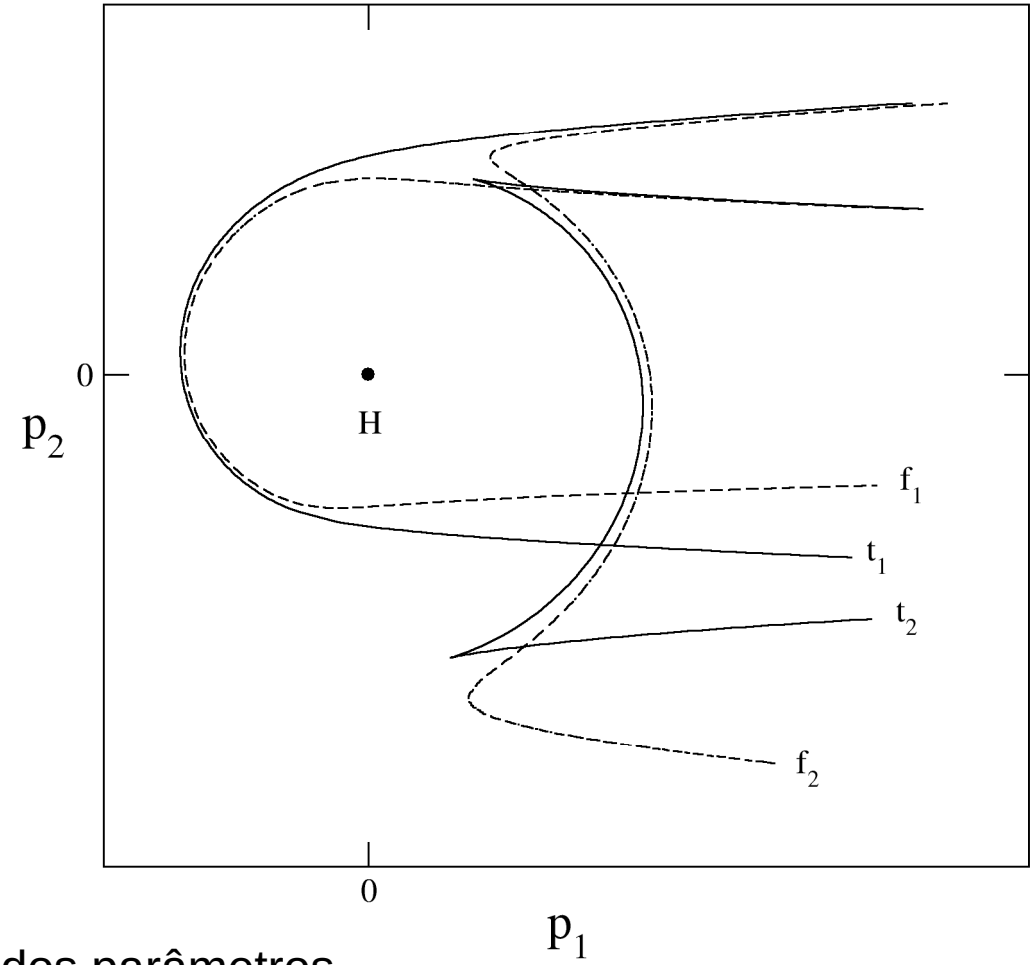
(d) Janelas $C_3, C_4, C_5$

# Órbita Homoclínica e Órbitas Periódicas Associadas



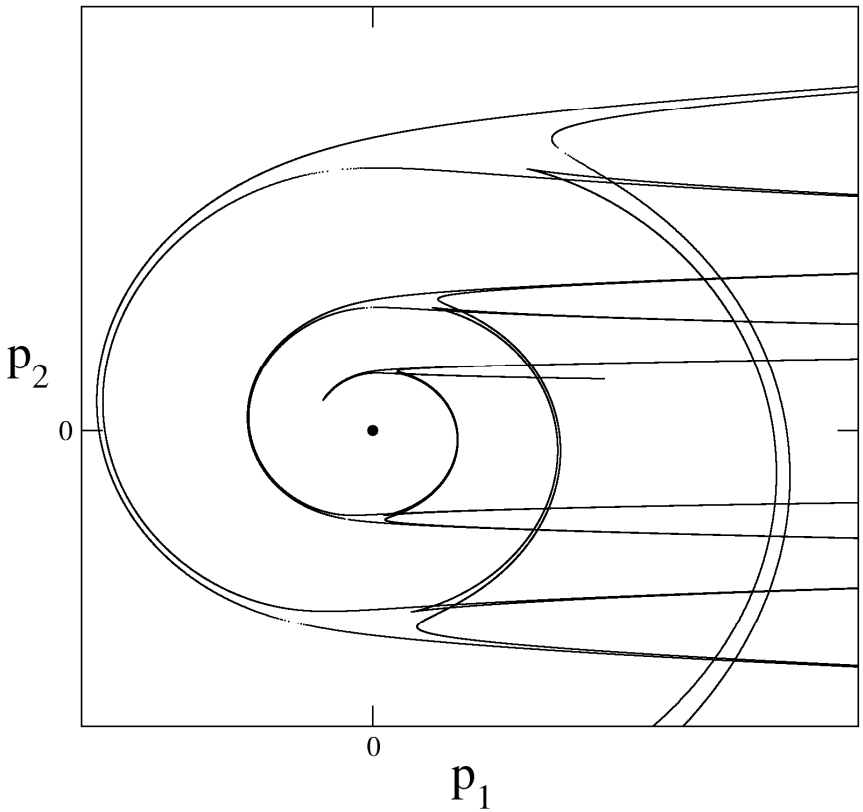

Órbita homoclínica

# Estrutura da Janela Periódica
# Obtida Aplicando Teorema de Shilnikov




Espaço dos parâmetros
H – ponto com órbita homoclínicas
Rotas para o caos (bifurcações tangente e de duplicação de período)

# Janela Periódica




Fronteiras obtidas da linearização em torno da órbita homoclínica