São Paulo 13 January 2015

Numerical Methods for the Analysis of Piecewise-smooth Systems

Petri T Piiroinen

National University of Ireland, Galway - Ireland

petri.piiroinen@nuigalway.ie

Piecewise-smooth Systems

What do we mean with by PWS systems?

Complementarity

Δ-function

Grazing

Non-differentiable maps

Friction

Discontinuity induced bifurcations

Saltation matrix

Relay

Impact oscillators

Border crossing

Chattering

Impacts

Filippov

Converter

OÉ Gaillimh NUI Galway

Numerical Analysis

São Paulo 13 January 2015

Local analysis

- Simulation
  - Brute force bifurcation diagrams
- Location and continuation
  - Equilibria and limit cycles
  - Stability analysis
  - Bifurcation points
  - Other

Global analysis

- Simulation
  - Domains of attraction
- Location and continuation
  - Stable and unstable manifolds
  - Boundary crises
  - Other

Simulation

Time stepping vs. Event driven

Existence and Uniqueness vs. Bifurcation Analysis

ODEs vs. Hybrid (ODEs and maps)

Large systems vs. Small systems

SICONOS vs. Researchers' own codes

[INRIA Grenoble] [PP & Kutznetsov 2008, Nordmark & PP 2009]

Types of PWS systems

Discontinuity in the state: impacting systems

$$x^+ = g(x^-), \quad x^-, x^+ \in \Sigma_{12}$$

Discontinuity in the vector field: systems with dry friction

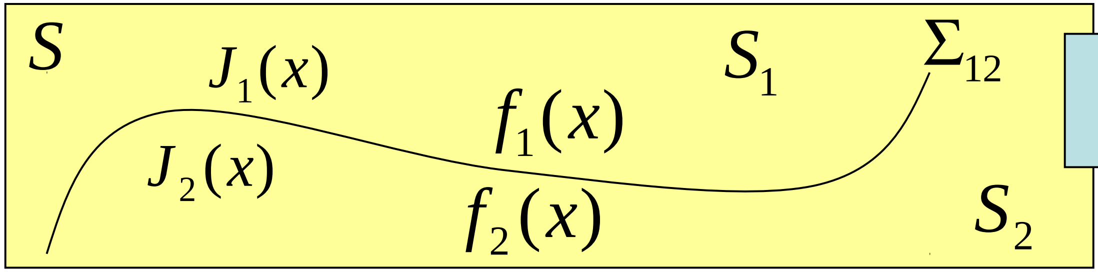
$$f_1(x) \neq f_2(x), \quad x \in \Sigma_{12}$$

Higher order discontinuities: systems with elastic support

$$J_1(x) \neq J_2(x), \quad x \in \Sigma_{12}$$

$$\dot{x} = \begin{cases} f_1(x), & x \in S_1 \\ f_2(x), & x \in S_2 \end{cases}$$

$$S_1 := \{x : h(x) > 0\}$$
$$S_2 := \{x : h(x) < 0\}$$
$$\Sigma_{12} := \{x : h(x) = 0\}$$



$$S = S_1 \cup S_2 \cup \Sigma_{12}$$

PWS systems

$$\dot{x} = f_i(x), \quad x \in S_i$$

$$\Sigma_{ij} := \{ x : h_{ij}(x) = 0 \}$$

$\longrightarrow$

$$S_i := \{ x : h_{ij}(x) > 0 \}$$

$$S_j := \{ x : h_{ij}(x) < 0 \}$$

Impacts law

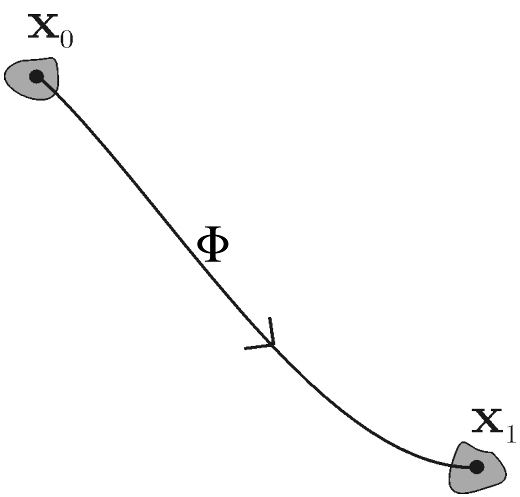
$$x^{+} = g_{ij}(x^{-})$$

Nonsmooth law

$$f^{+}(x^{+}) \leftarrow G_{ij}\left(f^{-}(x^{-}), f_i(x^{-}), f_j(x^{-}) \right)$$

Simulation

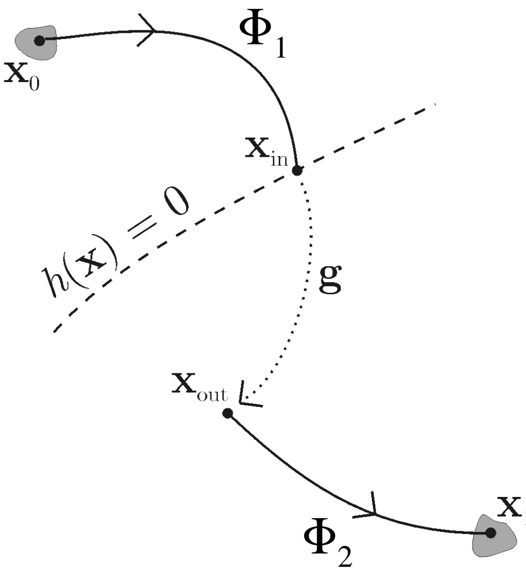
São Paulo 13 January 2015



$$\dot{x} = f(x), \; x(0) = x_0$$
$$x_0 = x_1 = x^*$$

$$\dot{\Phi}(x_0, t) = f(\Phi(x, t))$$
$$\Phi(x, 0) = x_0$$

$$\Phi(x_0, t) = x(t)$$
$$\Phi(x^*, T) = x^*$$



Smooth

$$\Phi(x^*, T) = x^*$$

Piecewise Smooth

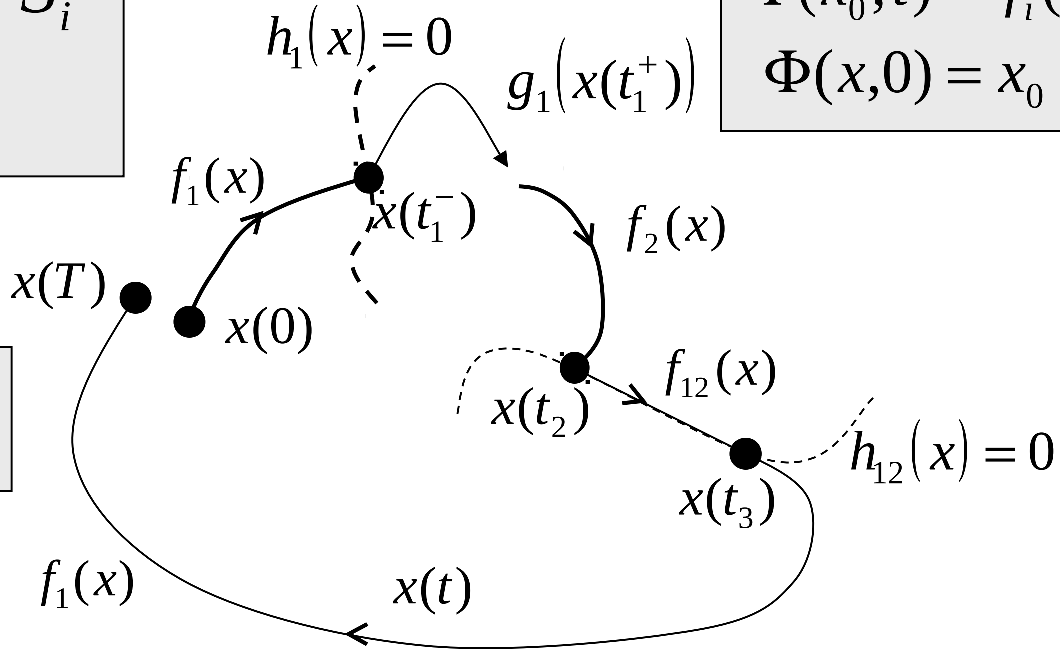
$$\Phi(x^*, T) = \Phi_2(g(\Phi_1(x^*, t_d)), T - t_d) = x^*$$

Simulation

$$\dot{x}(t) = f_i(x), \quad x \in S_i$$
$$x(0) = x_0$$

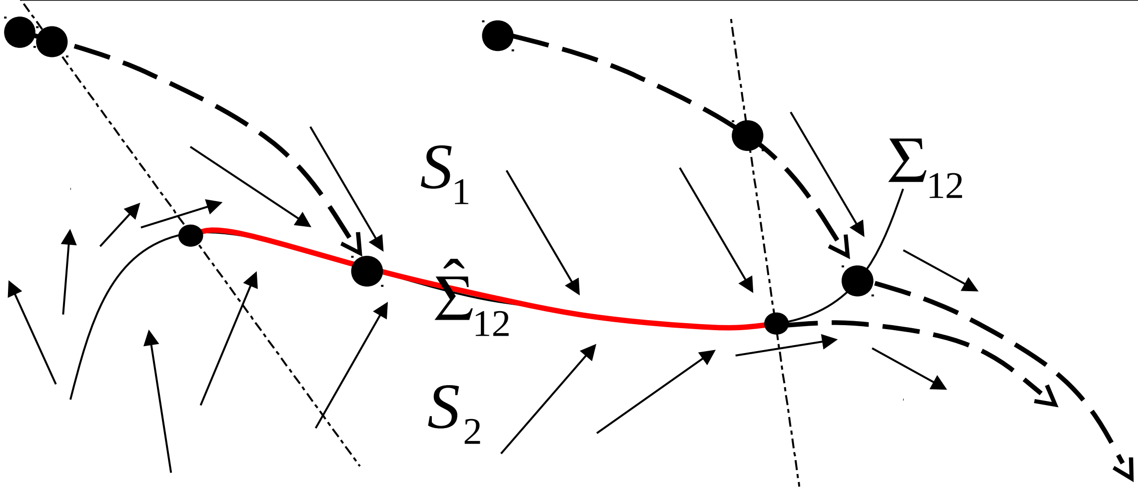
$$x(t^+) = g_i(x(t^-))$$

$$\dot{\Phi}(x_0, t) = f_i(\Phi(x,t))$$
$$\Phi(x,0) = x_0$$



$$f_{12}(x) = \frac{f_1(x) + f_2(x)}{2} + \frac{f_2(x) - f_1(x)}{2}\mu(x) - Ch_{12}(x)h_{12,x}(x)^T$$

Simulation: Filippov systems



$$\dot{x} = \begin{cases} f_1(x), & x \in S_1 \\ f_2(x), & x \in S_2 \end{cases}$$

$$\dot{x} = f_{12}(x), \quad x \in \hat{\Sigma}_{12}$$

$$\langle \nabla h_{12}(x), f_{12}(x) \rangle = 0$$

$$\hat{\Sigma}_{12} \subset \Sigma_{12}$$

Filippov’s convex method

$$f_{12}(x) = f_1(x) + \mu(x)\left(f_2(x) - f_1(x) \right) \boxed{-C(\nabla h_{12})^T h_{12}}$$

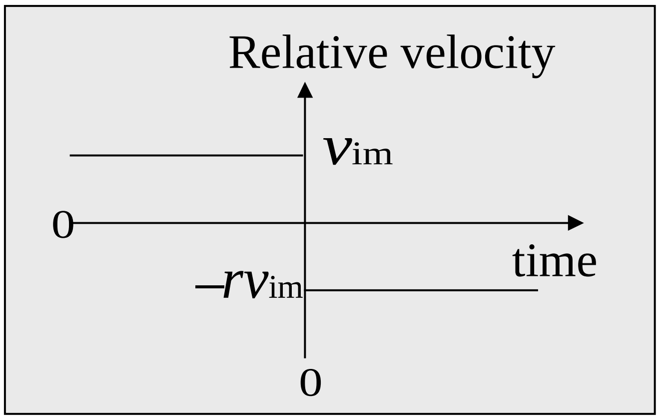
Utkin’s equivalent control

[PP & Kuznetsov, 2008]

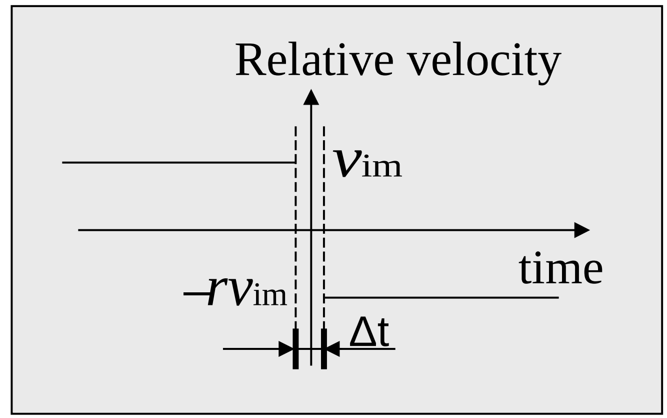
$$f_{12}(x) = \frac{f_1(x) + f_2(x)}{2} + \frac{f_2(x) - f_1(x)}{2}\mu(x) \boxed{-C(\nabla h_{12})^T h_{12}}$$

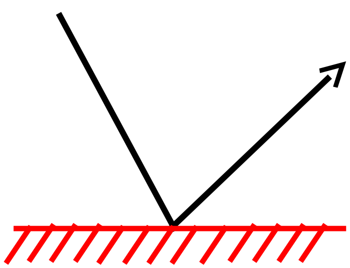
Simulation: Impacts

“Hard impact”

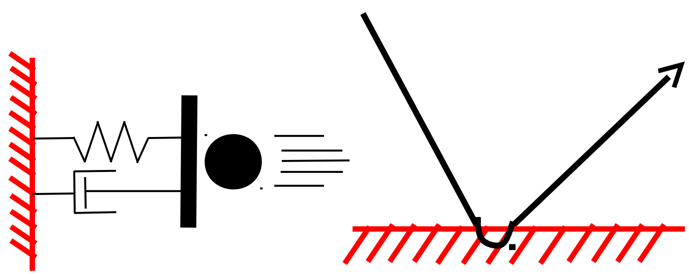


“Soft impact”





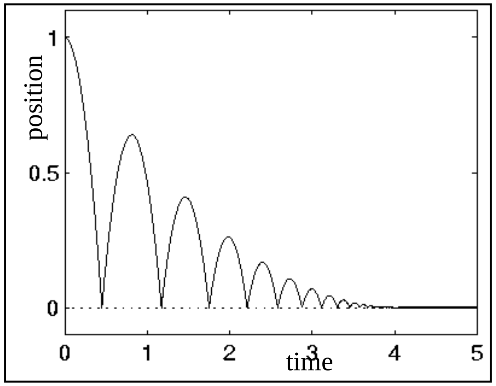
$$x^{+} = g(x^{-})$$

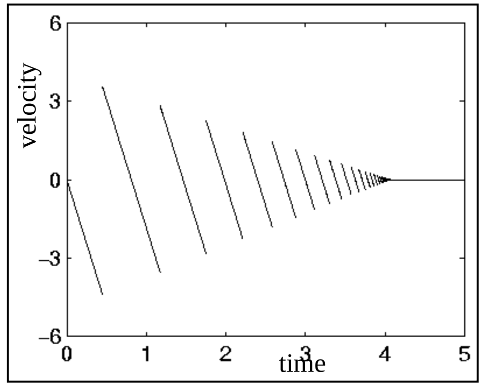


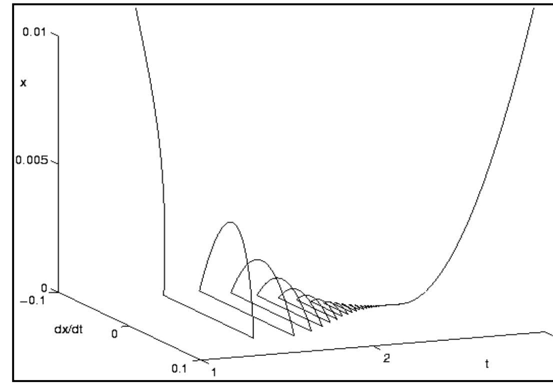
?

$$f_i(x^{-}) \rightarrow f_j(x^{+})$$

Simulation: Chattering







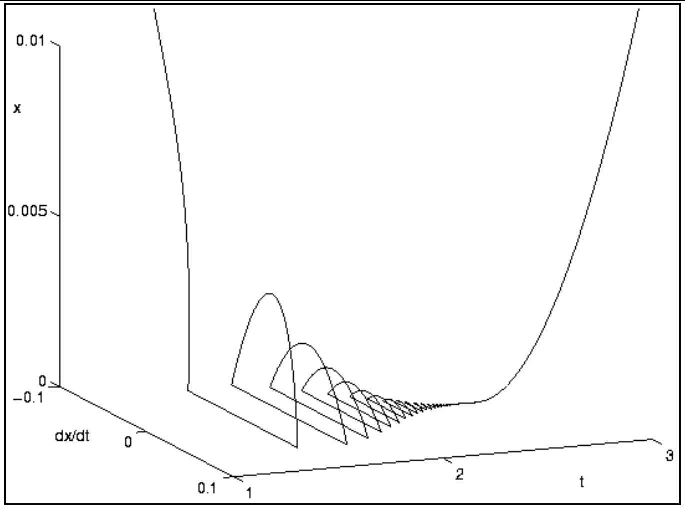
$$\dot{x} = f_1(x), \quad x \in S_1$$

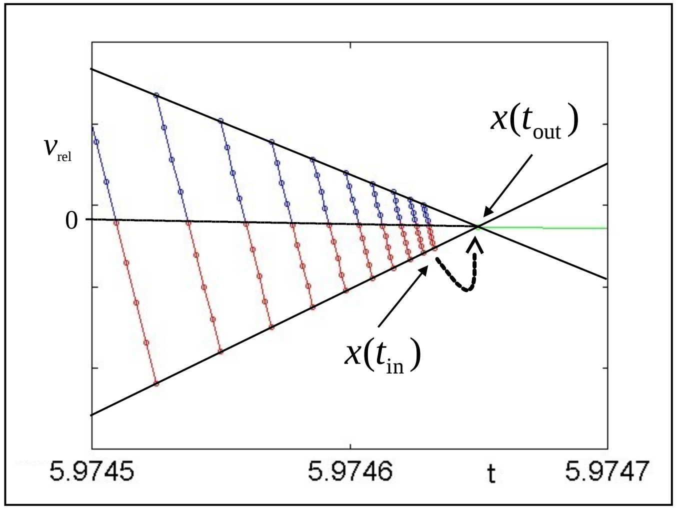
$$x^+ = x^- + W(x^-) h_x(x^-) f_1(x^-)$$

$$\dot{x} = f_1(x) - \lambda(x) W(x) = f_1(x) - \frac{\left(h_x(x) f_1(x)\right)_x f_1(x)}{\left(h_x(x) f_1(x)\right)_x W(x)} W(x)$$

[Nordmark & PP, 2009]

Simulation: Chattering





Local mapping at complete chattering

$$x_{\mathrm{out}} = Q(x_{\mathrm{in}}) = x_{\mathrm{in}} + \frac{1}{1-r}\left(\frac{2 f_{\mathrm{in}} r}{(h_x f_{\mathrm{in}})_x f_{\mathrm{in}}} + W_{\mathrm{in}}\right) h_x f_{\mathrm{in}}$$

$$t_{\mathrm{out}} = q(x_{\mathrm{in}}) = t_{\mathrm{in}} + \frac{2r}{1-r}\frac{h_x f_{\mathrm{in}}}{(h_x f_{\mathrm{in}})_x f_{\mathrm{in}}}$$

$$x = (x_1, \cdots, x_n)^T$$

$$x_{\mathrm{in}} = x(t_{\mathrm{in}})$$

$$x_{\mathrm{out}} = x(t_{\mathrm{out}})$$

$$f_{\mathrm{in}} = f(x_{\mathrm{in}})$$

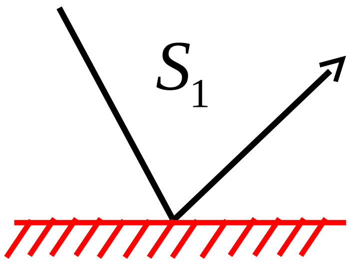
[Nordmark & PP, 2009]

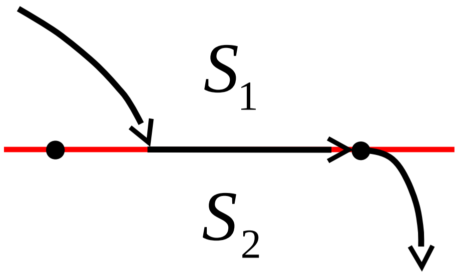
Simulation: Impacts with friction

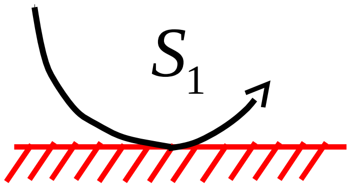
- Recently [Nordmark, Dankowicz & Champneys 08, 11], [Burns & PP 14]
- Impact law: Energetic [Stronge 08, 11]
- Grazing, sliding, chatter and sticking
- 10 possible outcomes of an impact
- Painlevé paradox and reverse chatter
- Brute-force bifurcation analysis and simulation is a challenge

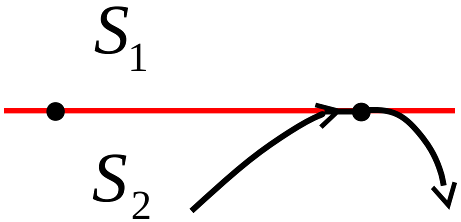
Simulation: Grazing

Problem!!





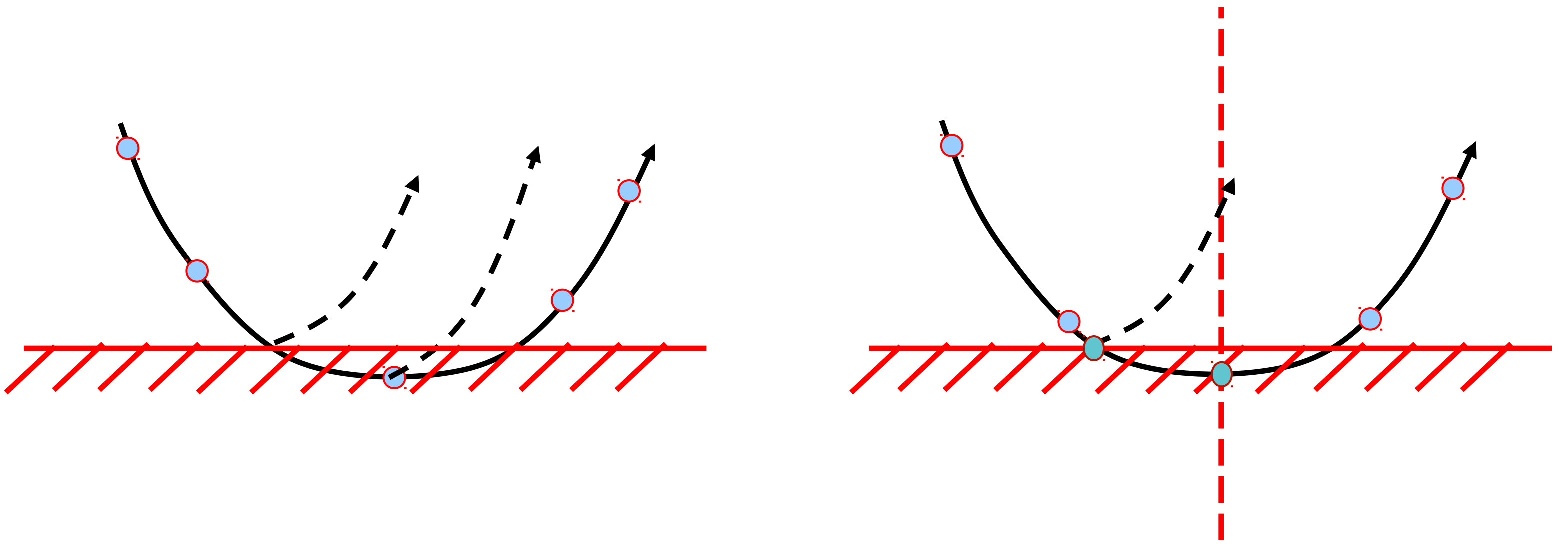




Simulation: Grazing

Problem!!

Time-stepping or event-driven methods for near-grazing trajectories?



Discontinuity-induced bifurcations

Impacting Systems

- **Grazing**
- **Chattering**

Filippov Systems

- **Sliding**

Other

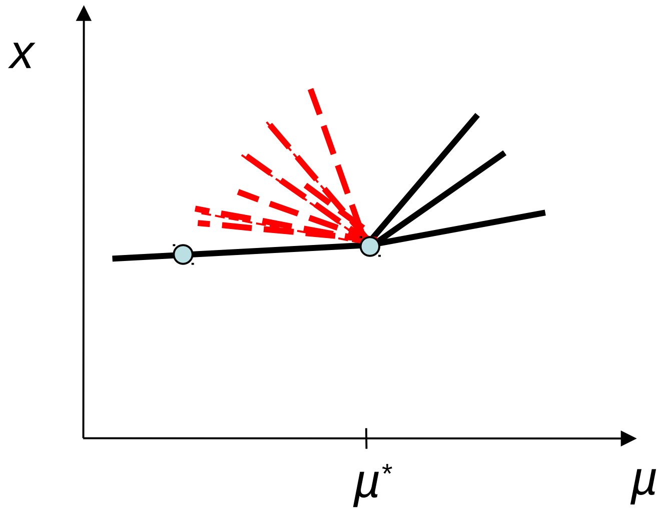
- **Event order**
- **Event number**

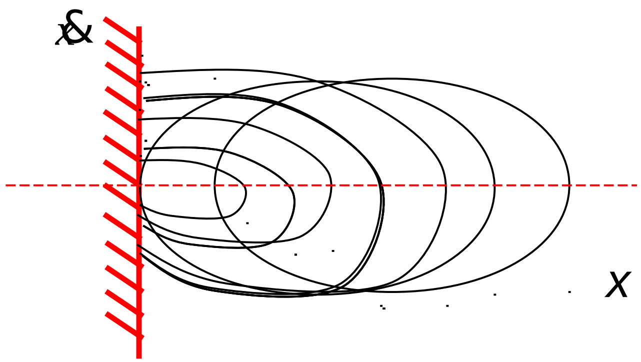
Maps

- **Border crossing**

Grazing bifurcation

Impacting systems



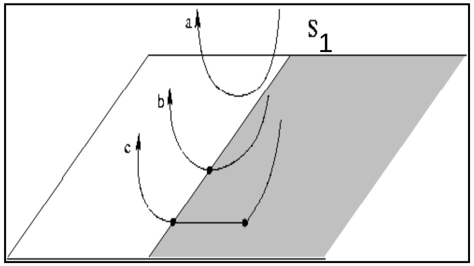


São Paulo 13 January 2015

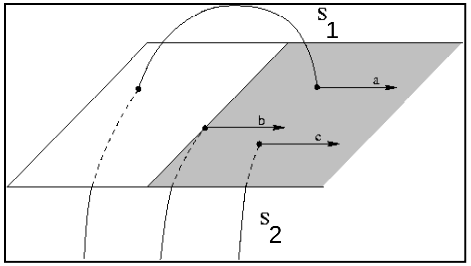
Sliding bifurcations

Filippov systems

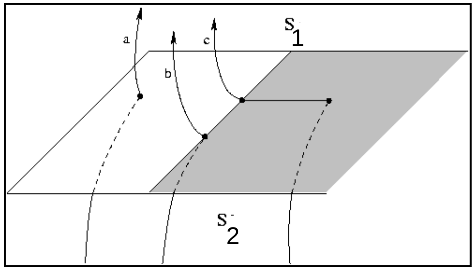
grazing-sliding



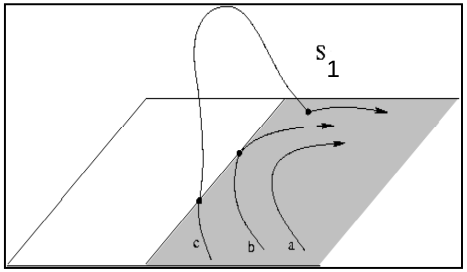
switching-sliding



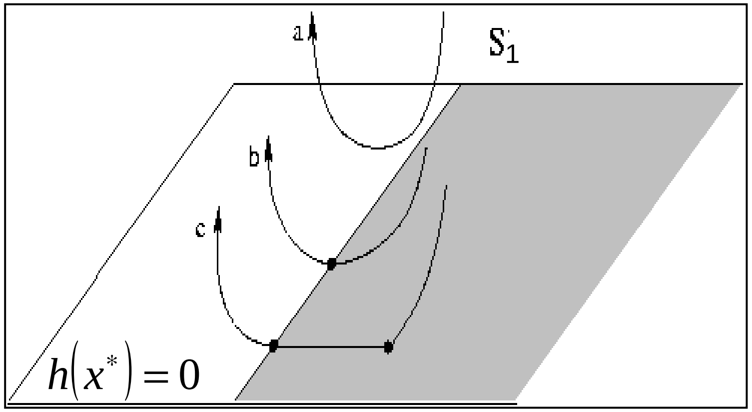
crossing-sliding



adding-sliding



Sliding bifurcations – grazing-sliding



$$\Phi(x^*, T) = x^*$$
$$h(x^*) = 0$$
$$\langle h_x(x^*), f_1(x^*) \rangle = 0$$

switching-sliding

$$\Phi(x^*, T) = x^*$$
$$h(x^*) = 0$$
$$\langle h_x(x^*), f_1(x^*) \rangle = 0$$

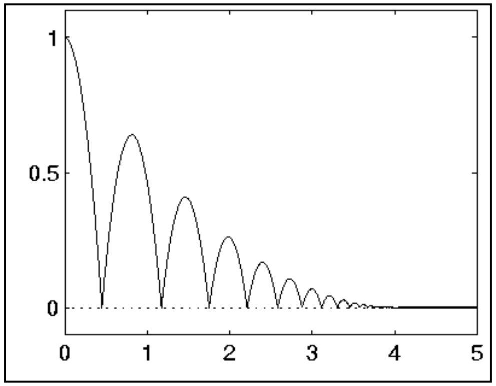
crossing-sliding

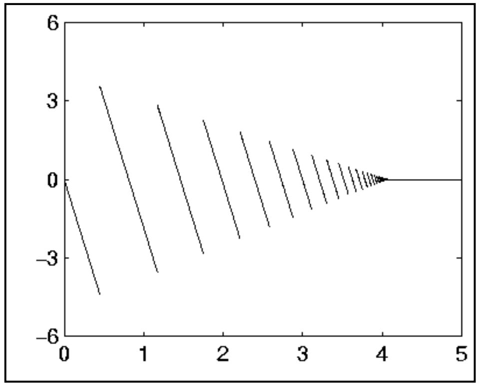
$$\Phi(x^*, T) = x^*$$
$$h(x^*) = 0$$
$$\langle h_x(x^*), f_1(x^*) \rangle = 0$$

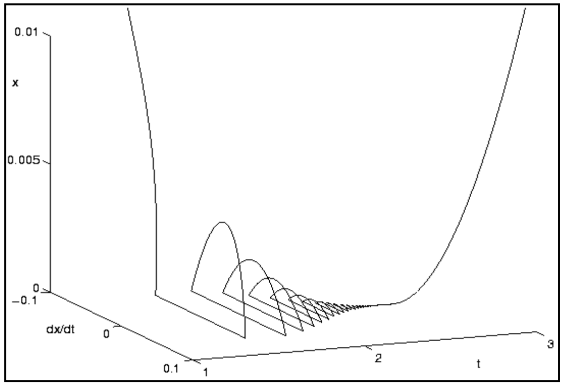
adding-sliding

$$\Phi(x^*, T) = x^*$$
$$\langle h_x(x^*), f_1(x^*) \rangle = 0$$
$$\langle \langle h_x(x^*), f_1(x^*) \rangle_x, f_1(x^*) \rangle = 0$$

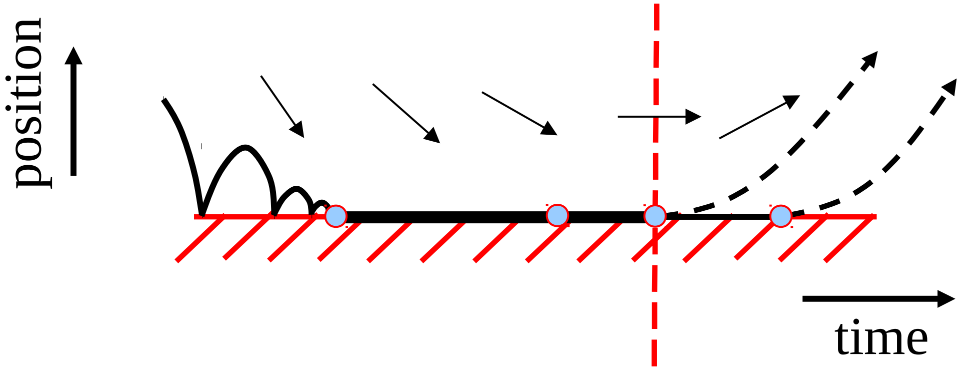
Simulation: Chattering







Problem!!



Stability analysis

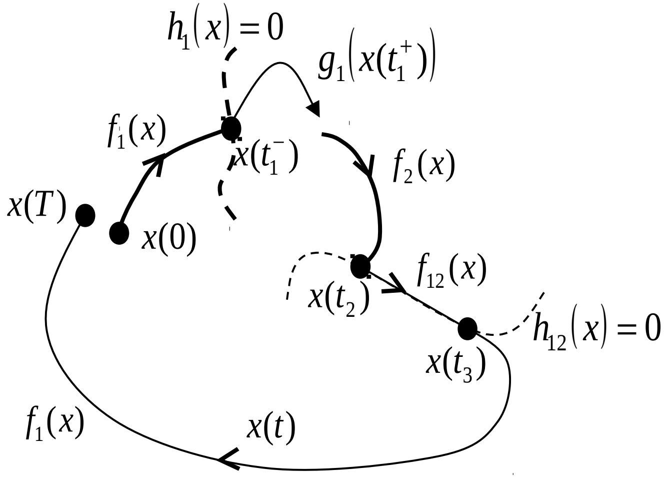
$$\Phi(x_0,T)=\Phi_1(\Phi_{12}(\Phi_2(g(\Phi_1(x_0,t_1)),t_2-t_1),t_3-t_2),T-t_3)=x(T)$$

$$\Phi(x,T)-\Phi(x_0,T)=\Phi_x(x_0,T)(x-x_0)+\text{h.o.t.}$$

$$\Phi_x(x^*,T)=\Phi_{1,x}(x_{\text{out}},T-t_3)G_3(x_3)\Phi_{12,x}(x_2,t_3-t_2)\cdot$$
$$\cdot G_2(x_2)\Phi_{2,x}(x_{\text{out}},t_2-t_1)G_1(x_1)\Phi_{1,x}(x^*,t_1)$$

$$G(x_{\text{in}})=g_{\mathbf{x}}+\frac{(f_{\text{out}}-g_x f_{\text{in}})h_x}{h_x f_{\text{in}}}$$

$$g_x=g_x(x_{\text{in}}),\ h_x=h_x(x_{\text{in}})$$
$$f_{\text{in}}=f(x_{\text{in}}),\ f_{\text{out}}=f(x_{\text{out}})$$



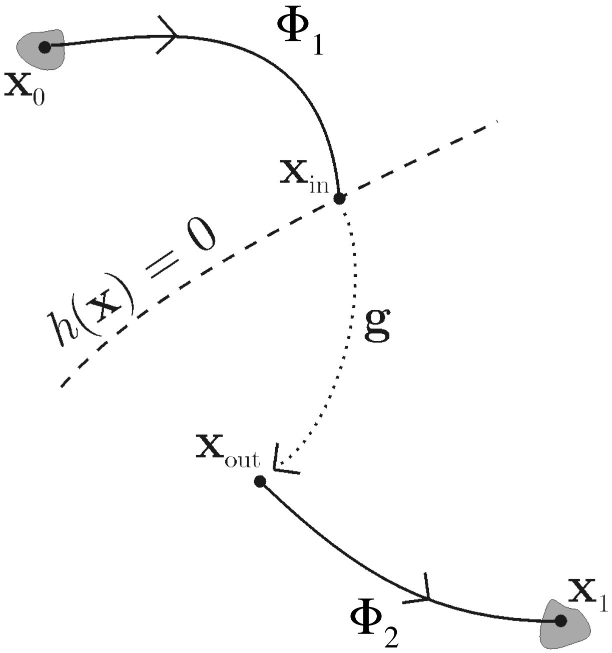
[Adolfsson *et al.*, 2001]

Stability analysis

Saltation matrix

$$G(x_{\text{in}}) = g_{\mathbf{x}} + \frac{(f_{\text{out}} - g_x f_{\text{in}}) h_x}{h_x f_{\text{in}}}$$

$$g_x = g_x(x_{\text{in}}),\ h_x = h_x(x_{\text{in}})$$
$$f_{\text{in}} = f(x_{\text{in}}),\ f_{\text{out}} = f(x_{\text{out}})$$

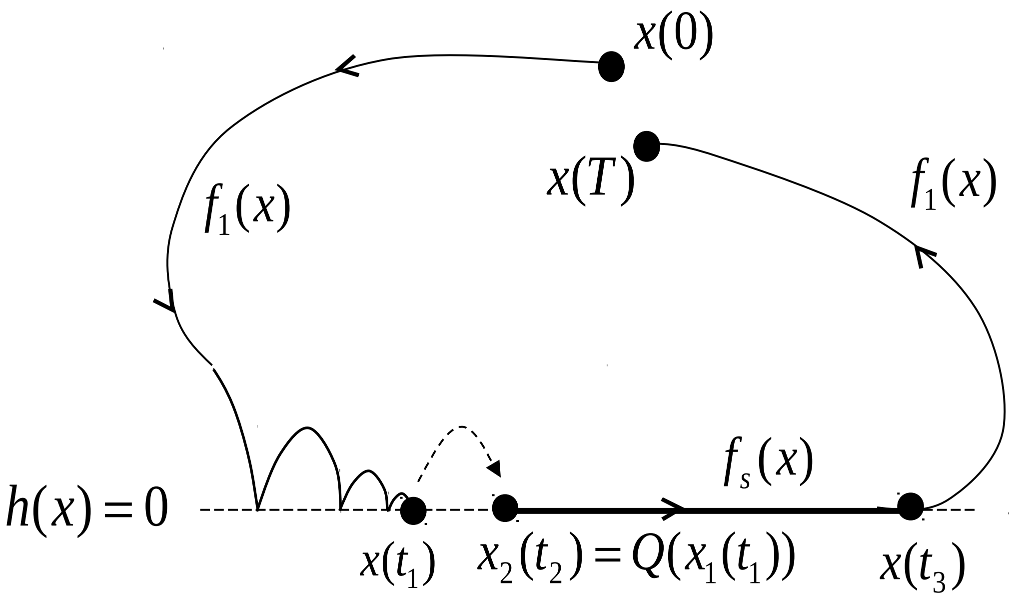


Stability analysis of trajectories with chattering

$$\Phi(x_0,T)=\Phi_1(g_2(\Phi_s(g_1(\Phi_I(x_0,t_1)),t_3-t_2)),T-t_3)=x(T)$$

$$x_i = x(t_i)$$

$$\Phi_x(x^*,T)=\Phi_{1,x}(x_3,T-t_3)G_2(x_3)\Phi_{s,x}(x_2,t_3-t_2)G_1(x_1)\Phi_{I,x}(x_0,t_1)$$



$$x_j = Q(x_i)$$
$$t_j = q(t_i, x_i)$$

[Nordmark & PP, 2009]

Saltation matrices

<u>Impact & Filippov</u>

$$G_i = g_{i,x} + \frac{\left(f_{\text{out}} - g_{i,x} f_{\text{in}}\right) h_{i,x}}{h_{i,x} f_{\text{in}}}$$

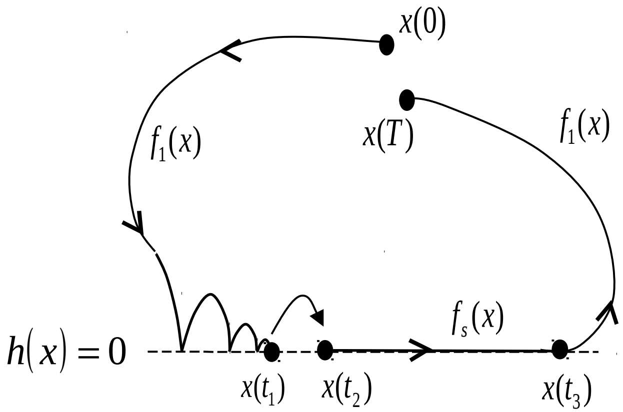
$$h_x = h_x(x_i), q_x = q_x(x_i)$$
$$g_i = g_i(x_i), g_{i,x} = g_{i,x}(x_i)$$
$$f_{in} = f(x_i), f_{out} = f(g_i(x_i))$$

[Nordmark & PP, 2009]

<u>Chattering</u>

$$G_i = C_i + \frac{\left(f_{\text{out}} - C_i f_{\text{in}}\right) h_{i,x}}{h_{i,x} f_{\text{in}}}$$

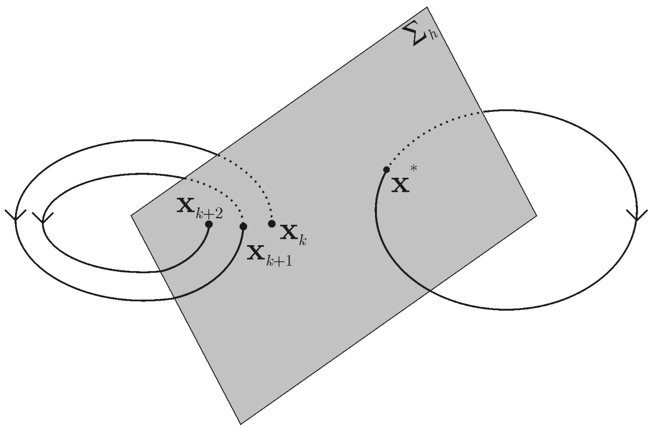
$$C_i = Q_x - f_{\text{out}} q_x$$

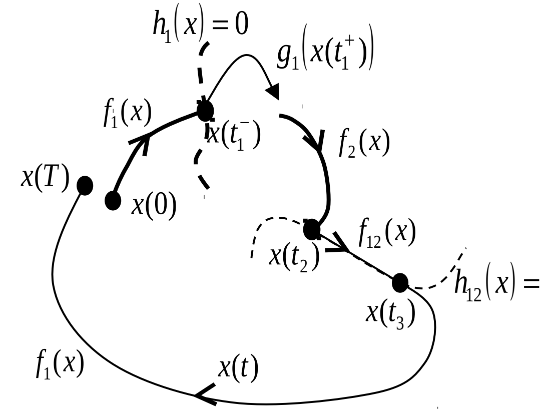


Locating periodic orbits

Sâo Paulo 13 January 2015

$$\Phi(x^*,T) = x^*$$
$$h(x^*) = 0$$



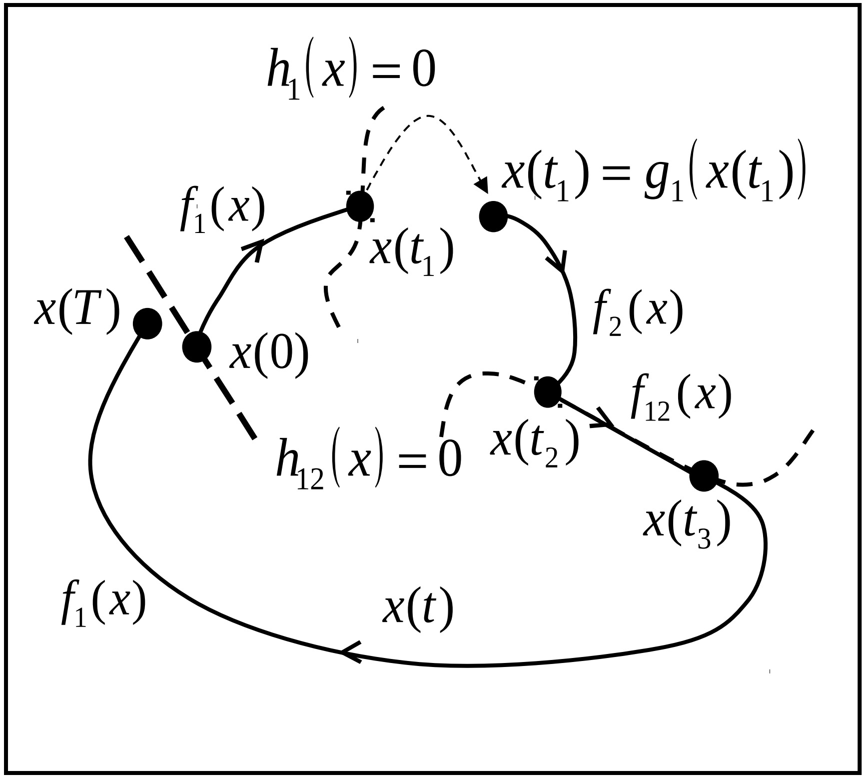


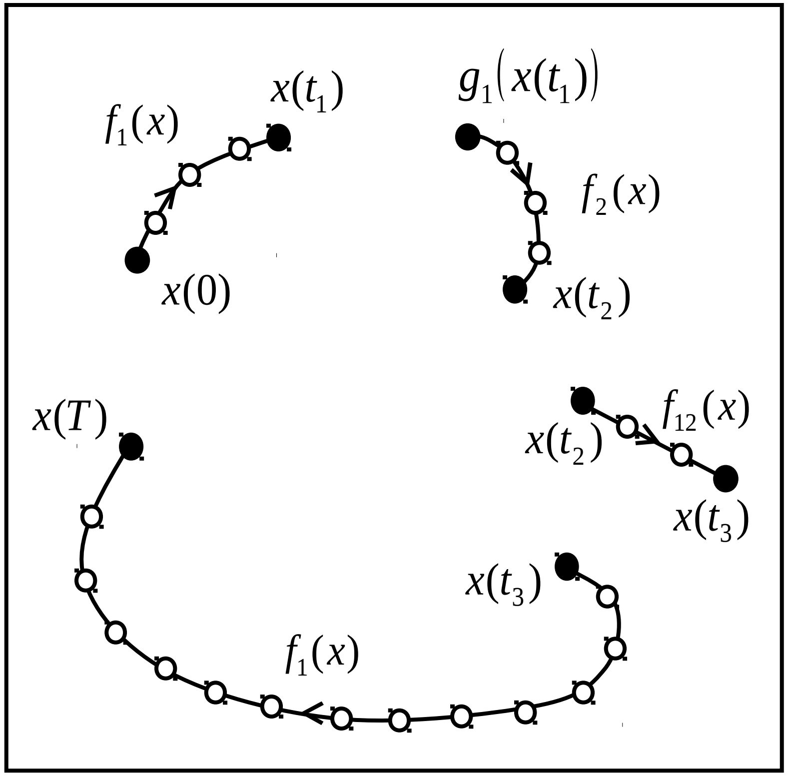
Newton's method

Flow: $$\begin{pmatrix} x_{k+1} \\ T_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ T_k \end{pmatrix} - \begin{pmatrix} \Phi_x(x_k,T_k) - Id & f_i(x_k) \\ h_x(x_k) & 0 \end{pmatrix}^{-1} \begin{pmatrix} \Phi(x_k,T_k) - x_k \\ h(x_k) \end{pmatrix}$$

$\Longrightarrow$ Pseudo-arclength continuation

Shooting vs. Collocation





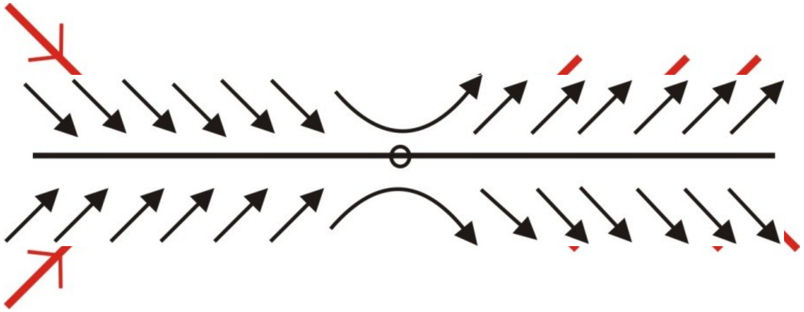
Auto, Content, SlideCont,

TC-Hat, COCO

Simulation: Other issues

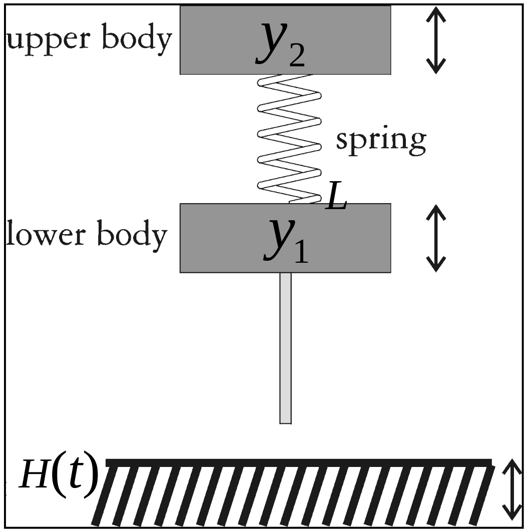
Problem!!

- The number of events changes
- The event order changes
- Nondeterminacy



One-legged jumper

$$m\ddot{y}_1 - k\left(y_2 - y_1 - d_0\right) = -mg$$
$$m\ddot{y}_2 + k\left(y_2 - y_1 - d_0\right) = -mg$$
$$H(t) = \sin(\omega t) + \cos(\omega t)$$



[Dankowicz & PP, 2002]

$$x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}^T$$
$$= \begin{pmatrix} y_1 & \dot{y}_1 & y_2 & \dot{y}_2 & \omega t \end{pmatrix}^T$$

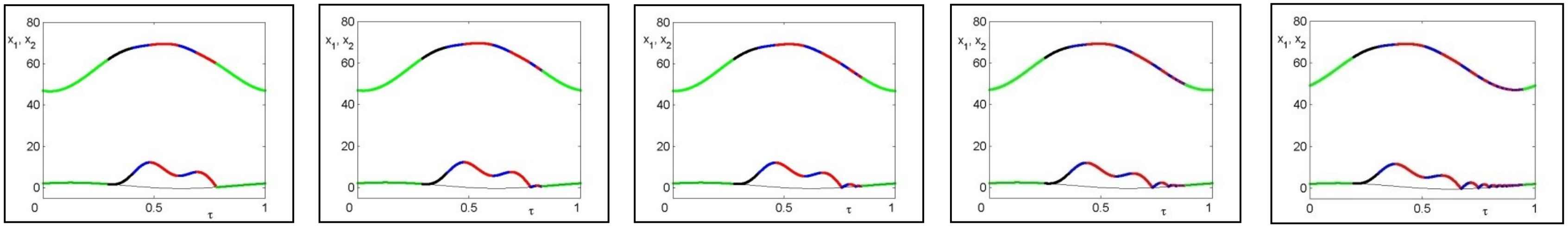
$$h(x) = x_1 - L - \sin(x_5) - \cos(x_5)$$

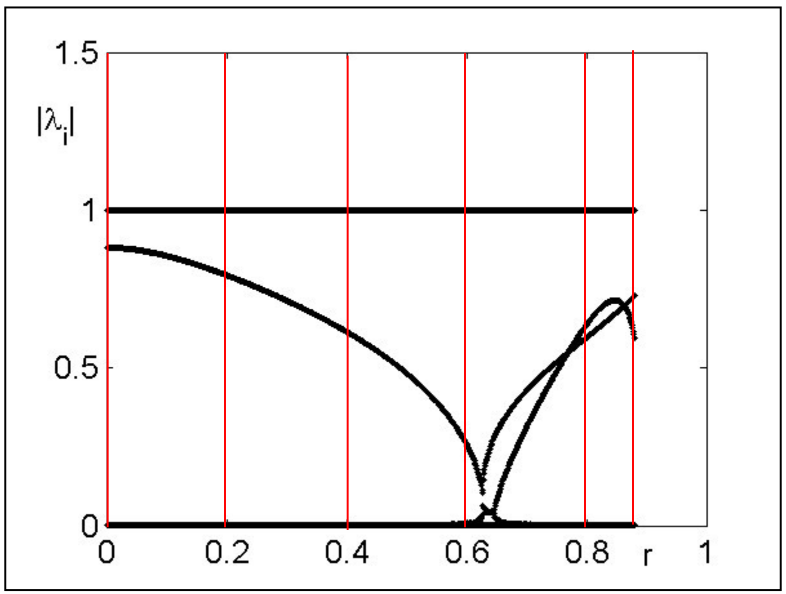
$$x_2^+ = -r x_2^- - (1+r)\omega(\sin(x_5^-) - \cos(x_5^-))$$

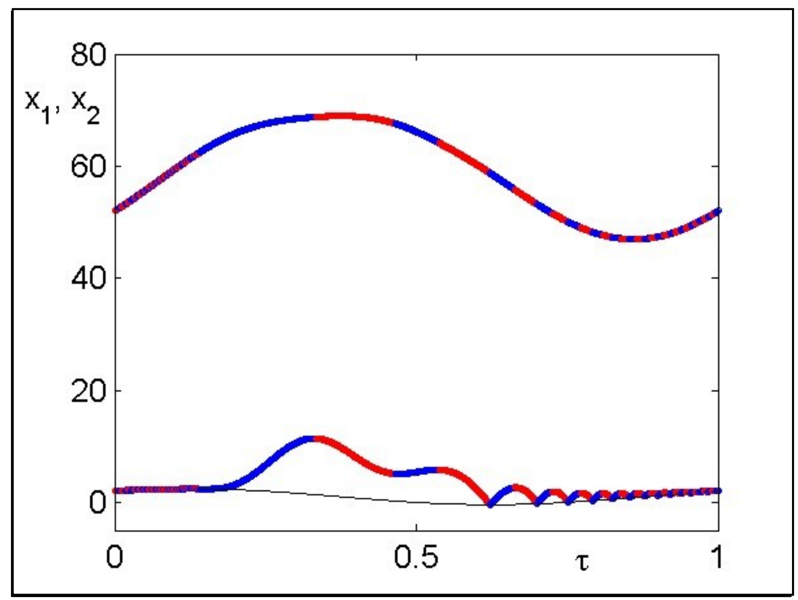
OÉ Gaillimh
NUI Galway

One-legged jumper

São Paulo 13 January 2015







OÉ Gaillimh
NUI Galway

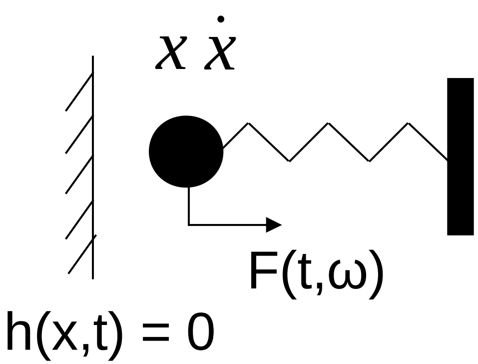
Sâo Paulo 13 January 2015

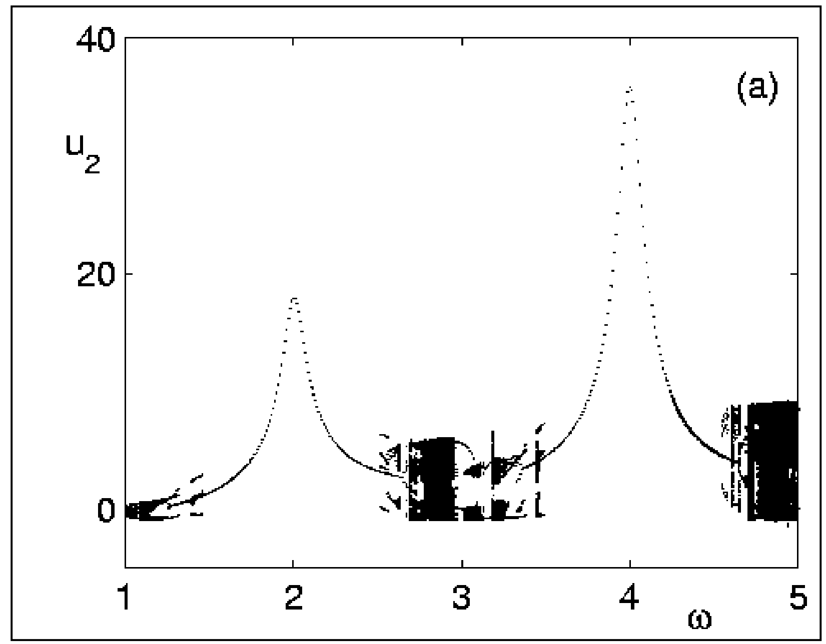
An impact oscillator

$$\ddot{x} + \varepsilon\dot{x} + x = \sin(\omega t)$$

$$\dot{x}(t^{+}) = -r\dot{x}(t^{-}),\ \ x - \sigma = 0$$

[Budd & Dux, 1994]





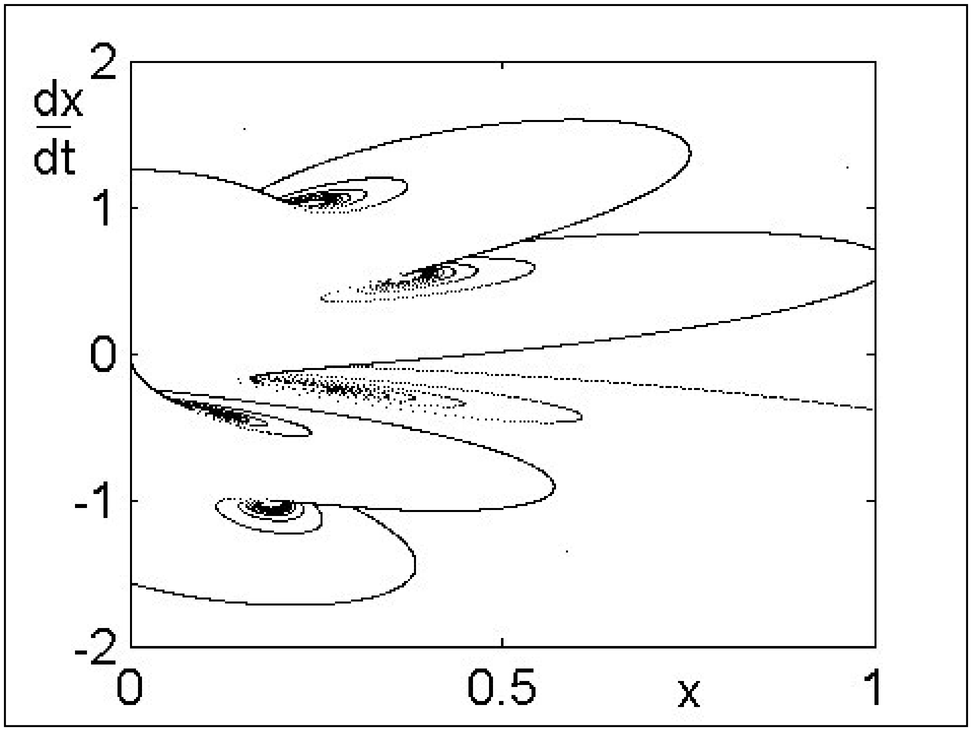
OÉ Gaillimh
NUI Galway

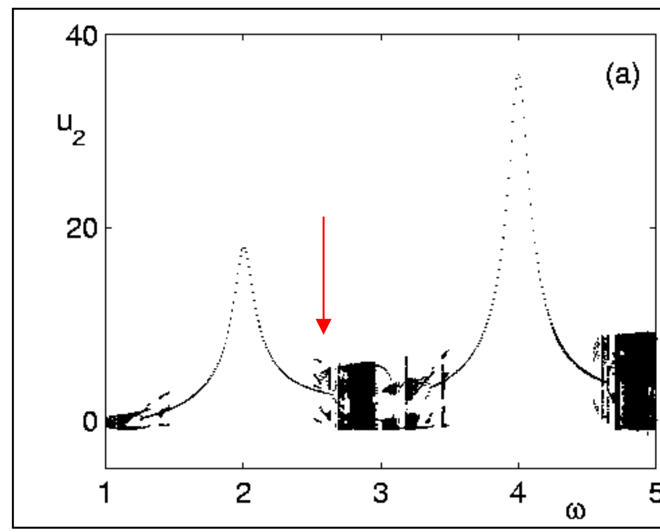
An impact oscillator

Sâo Paulo 13 January 2015

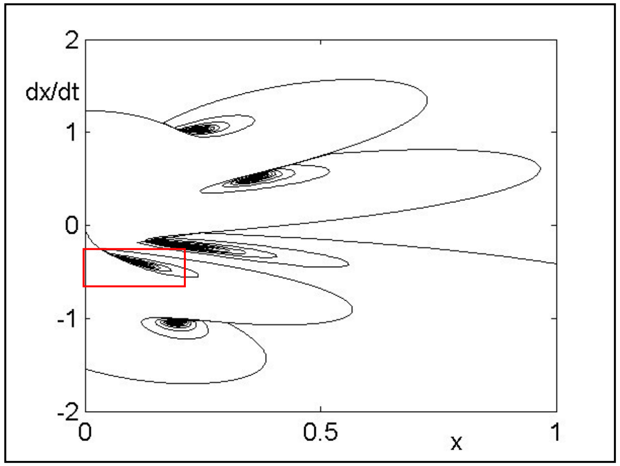
$$\ddot{x} + \varepsilon\dot{x} + x = \sin(\omega t)$$

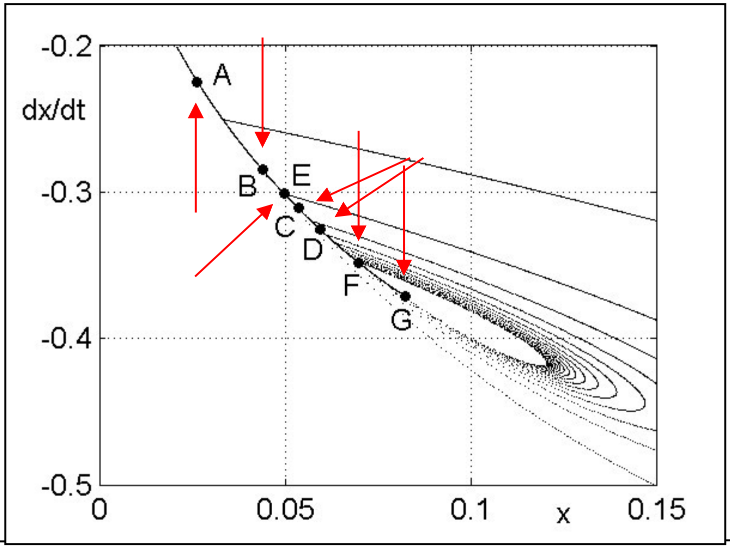
$$\dot{x}(t^{+}) = -r\dot{x}(t^{-})$$

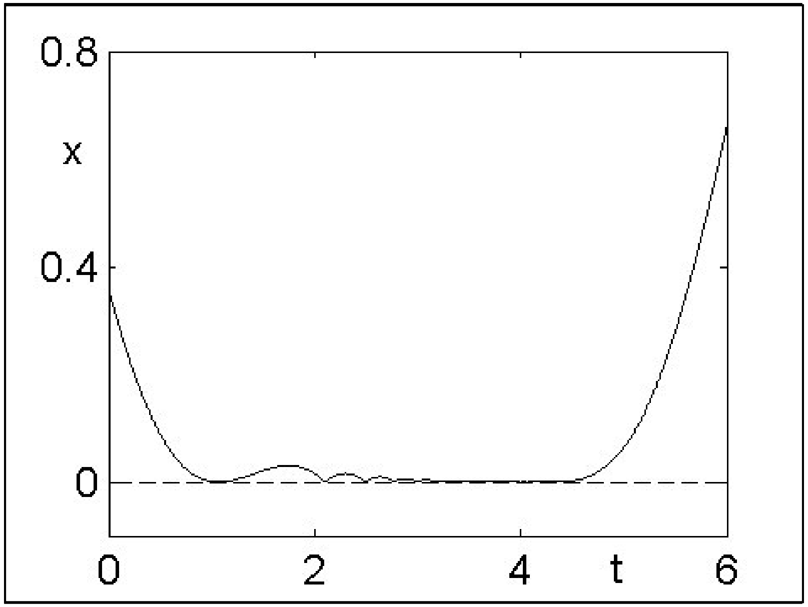




An impact oscillator



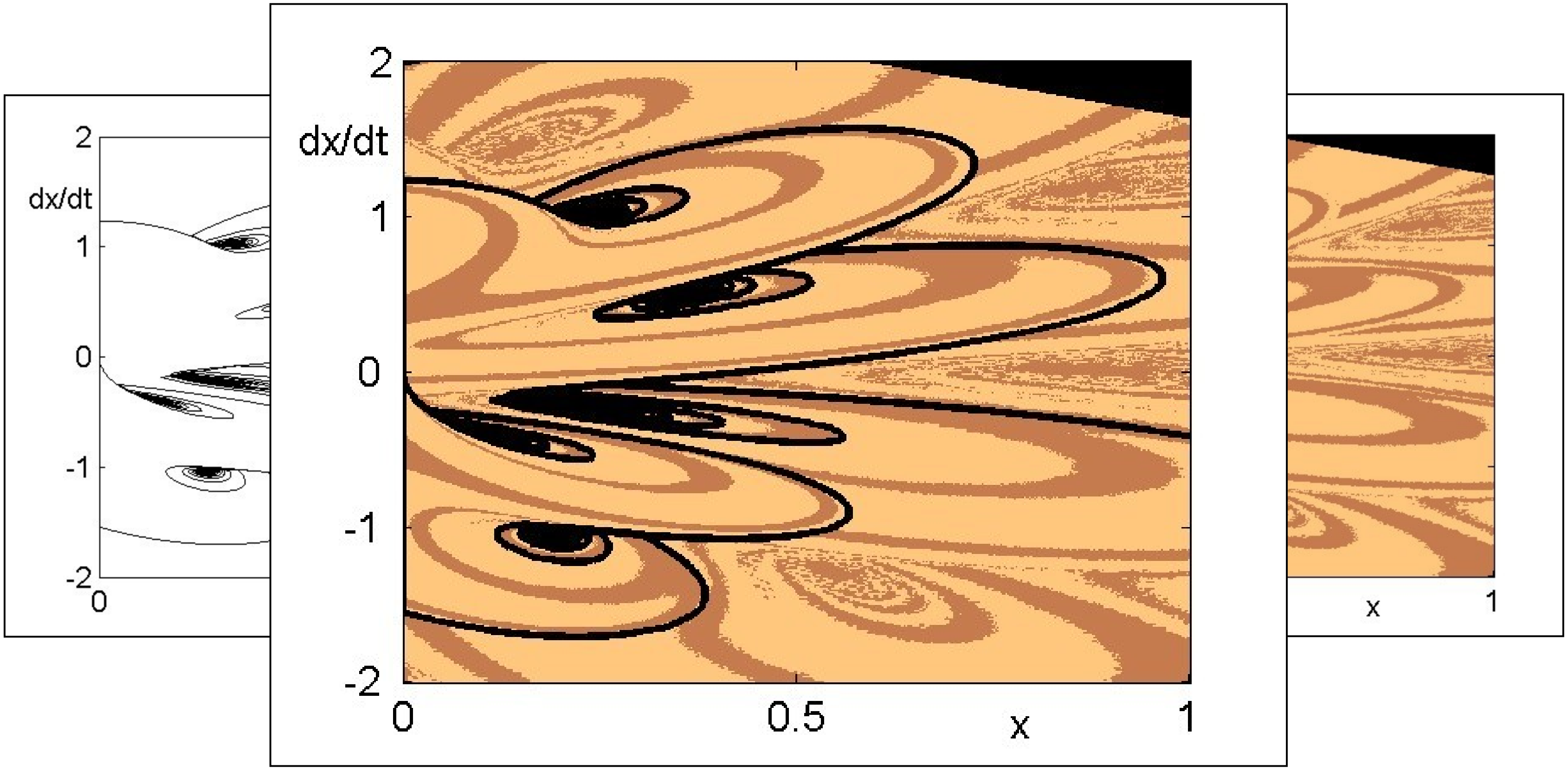




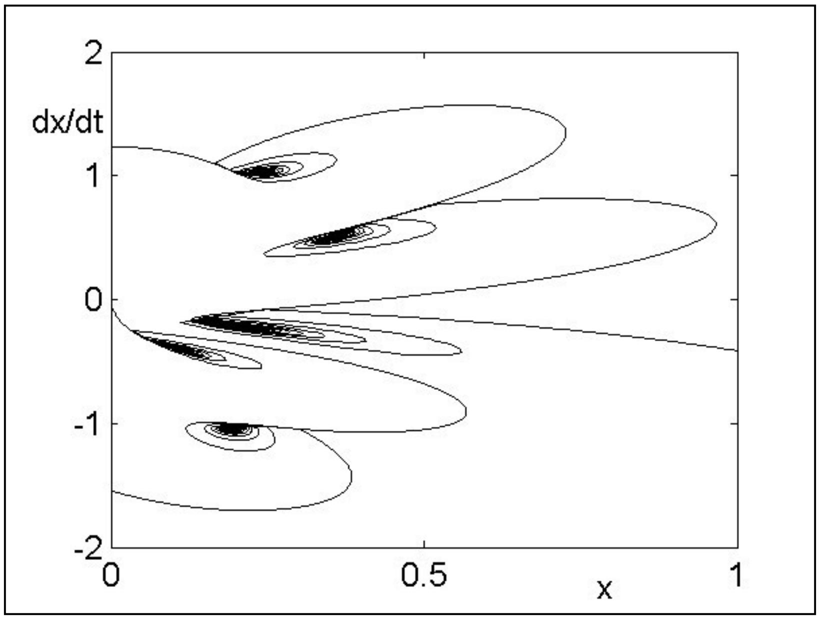
OÉ Gaillimh
NUI Galway

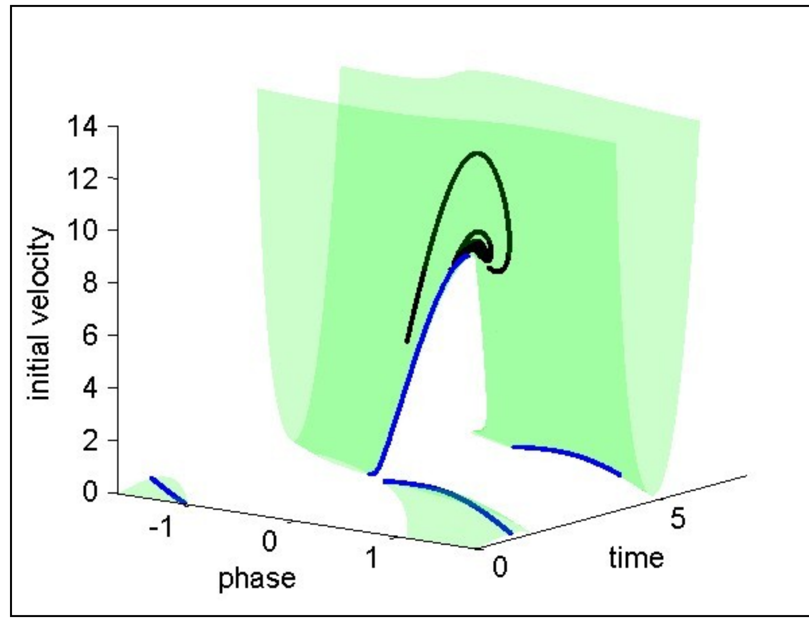
São Paulo 13 January 2015

An impact oscillator



An impact oscillator





For more on discontinuity geometry see

[Chillingworth 02,10], [Humphries & PP 11]

Summary & Outlook

Local analysis

- Simulation of Filippov and impacting systems
- Brute-force bifurcation and domain-of-attraction diagrams
- Stability analysis and continuation
- Analysis of PWS system combines a variety of analytic and numerical methods
- Create as general methods as possible

Summary & Outlook

There are still many mathematical/numerical problems to tackle in this area:

- Manifolds
- Discontinuous boundaries, crossing of two or more boundaries
- Impacts with friction
- Topological analysis of nonsmooth systems,
- …and many more.

Thank you!