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Controlling chaotic orbits in mechanical systems with impacts

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Abstract

We stabilize desired unstable periodic orbits, embedded in the chaotic invariant sets of mechanical systems with impacts, by applying a small and precise perturbation on an available control parameter. To obtain such perturbation numerically, we introduce a transcendental map (impact map) for the dynamical variables computed just after the impacts. To show how to implement the method, we apply it to an impact oscillator and to an impact-pair system.

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1. Introduction

Mechanical systems exhibiting impacts occur in many branches of technology. Such systems are called vibro-impact systems and have been recently the subject of growing interest in the literature [1,2]. Besides, the study of the vibro-impact systems shows that chaotic behavior occurs widely [3–6]. Thus, controlling chaos can improve technological applications [7–12].

In this work, we implement the Ott–Grebogi–Yorke (OGY) method of controlling chaos [13] for the the vibro-impact systems. As examples, we consider an impact oscillator [14,15] and an impact-pair system [16]. The temporal evolution of dynamical variables of these two systems is a combination of smooth motion governed by a linear differential equation interrupted by a series of non-smooth impacts. Using the analytical solution of the differential equation and the impact rule, we can determine a transcendental map [16,17], whose dynamical variables are computed at the impact instants. Furthermore, for these systems the trajectories are discontinues in phase space due to impacts.

The OGY method consists on stabilizing a desired unstable periodic orbit embedded in a chaotic attractor by using only a small perturbation on an available control parameter. For the vibro-impact systems due to discontinuities, the hard part of the control process is how to determine the value of parameter perturbation. For that, we will use the transcendental map, that is two-dimensional in this case. Thus, we calculate the necessary parameter perturbation in the similar way of classical two-dimensional maps, as Hénon map [18]. In addition, it is important to say that the motion of the systems can not be obtained from the transcendental map. This map here is used only to implement the control method.

This paper is organized as follows: In Section 2 we present the mathematical models of the vibro-impact systems and introduce the transcendental maps. In Section 3 we describe the procedure to implement the OGY method for these systems. In Section 4 we show numerical results of this implementation. Our conclusions are presented in Section 5.

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2. Mathematical description

2.1. Impact oscillator

Fig. 1 depicts the model of the impact oscillator. This system is composed by a periodically forced oscillator whose oscillation is limited by an amplitude constraint.

The differential equation of motion of the system between impacts is

$$\ddot{x} + x = \alpha \cos(\omega t), \quad x < x_c \tag{1}$$

where x_c , α and ω are the amplitude constraint, the forcing amplitude, and the forcing frequency, respectively.

Integrating Eq. (1) and invoking initial condition $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$, the displacement x and the velocity \dot{x} between impacts are

$$x = \left[x_0 - \frac{\alpha}{(1 - \omega^2)} \cos(\omega t_0) \right] \cos(t - t_0) + \left[\dot{x}_0 + \frac{\alpha\omega}{(1 - \omega^2)} \sin(\omega t_0) \right] \sin(t - t_0) + \frac{\alpha}{(1 - \omega^2)} \cos(\omega t) \tag{2}$$

$$\dot{x} = \left[-x_0 + \frac{\alpha}{(1 - \omega^2)} \cos(\omega t_0) \right] \sin(t - t_0) + \left[\dot{x}_0 + \frac{\alpha\omega}{(1 - \omega^2)} \sin(\omega t_0) \right] \cos(t - t_0) - \frac{\alpha\omega}{(1 - \omega^2)} \sin(\omega t) \tag{3}$$

An impact occurs wherever $x = x_c$ (amplitude constraint). After each impact, we apply into Eqs. (2) and (3) the Newton law of impact

$$t_0 = t, \quad x_0 = x, \quad \dot{x}_0 = -r\dot{x} \tag{4}$$

where r is a constant coefficient of restitution.

Thus, the evolution of the impact oscillator is given by Eqs. (2)–(4). Consequently, the system depends on the control parameters x_c , α , ω , and r .

In order to determine the transcendental map needed for the control application, we consider the variables x_n , \dot{x}_n , and t_n computed just before the impact n . The variables x_{n+1} , \dot{x}_{n+1} and t_{n+1} are obtained from Eqs. (2) and (3) for the initial conditions:

$$t_0 = t_n, \quad x_0 = x_n, \quad \dot{x}_0 = -r\dot{x}_n \tag{5}$$

Thus, we can introduce the transcendental map (obtained from Eqs. (2), (3) and (5)):

$$\begin{aligned} x_{n+1} &= \left[x_n - \frac{\alpha}{(1 - \omega^2)} \cos(\omega t_n) \right] \cos(t_{n+1} - t_n) + \left[-r\dot{x}_n + \frac{\alpha\omega}{(1 - \omega^2)} \sin(\omega t_n) \right] \sin(t_{n+1} - t_n) \\ &\quad + \frac{\alpha}{(1 - \omega^2)} \cos(\omega t_{n+1}) \\ \dot{x}_{n+1} &= \left[-x_n + \frac{\alpha}{(1 - \omega^2)} \cos(\omega t_n) \right] \sin(t_{n+1} - t_n) + \left[-r\dot{x}_n + \frac{\alpha\omega}{(1 - \omega^2)} \sin(\omega t_n) \right] \cos(t_{n+1} - t_n) \\ &\quad - \frac{\alpha\omega}{(1 - \omega^2)} \sin(\omega t_{n+1}) \end{aligned} \tag{6}$$

where $x_{n+1} = x_n = x_c$.

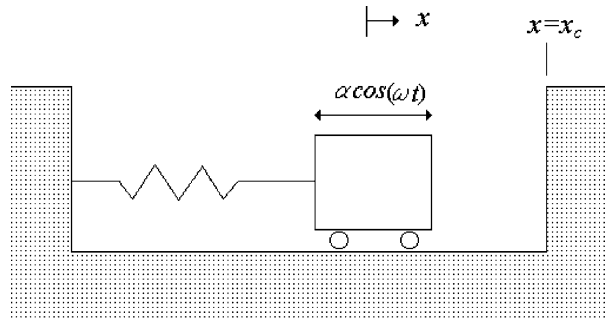


Fig. 1. Model of an impact oscillator.

Here, we use the transcendental map (6) only to specify the parameter perturbation needed to implement the OGY method. In order to study the temporal evolution of dynamical variables we use Eqs. (2)–(4), as mentioned earlier.

2.2. Impact-pair system

The impact-pair system is shown schematically in Fig. 2. This system is composed of a point mass m (whose displacement is denoted by x) and a box with a gap of length v . The mass m is free to move inside the gap and the motion of the box is represented by a periodic function ($A \sin(\omega t)$).

Equation of motion of the point mass m in the absolute coordinate is

$$\ddot{x} = 0 \tag{7}$$

Denoting the relative displacement of the mass m by y , we have

$$x = y + \alpha \sin(\omega t) \tag{8}$$

Substituting Eq. (8) into Eq. (7), equation of motion in relative coordinate is

$$\ddot{y} = \alpha \omega^2 \sin(\omega t), \quad -v/2 < y < v/2 \tag{9}$$

Integrating Eq. (9) and invoking initial conditions $y(t_0) = y_0$ and $\dot{y}(t_0) = \dot{y}_0$, the displacement y and the velocity \dot{y} , between impacts, are

$$y(t) = y_0 + \alpha \sin(\omega t_0) - \alpha \sin(\omega t) + [\dot{y}_0 + \alpha \omega \cos(\omega t_0)](t - t_0) \tag{10}$$

$$\dot{y}(t) = \dot{y}_0 + \alpha \omega \cos(\omega t_0) - \alpha \omega \cos(\omega t) \tag{11}$$

An impact occurs whenever $y = v/2$ or $-v/2$. After each impact, we apply into Eqs. (10) and (11) the new set of initial conditions (the Newton law of impact)

$$t_0 = t, \quad y_0 = y, \quad \dot{y}_0 = -r\dot{y} \tag{12}$$

where r is a coefficient of restitution.

Therefore, the temporal evolution of the dynamical variables of the impact-pair system is given by Eqs. (10)–(12). Thus, the system depends on control parameters v , r , α , and ω .

To obtain the transcendental map, we use the analytical solution (Eqs. (10) and (11)) and the Newton law of impact. Thus, we have

$$\begin{aligned} y_{n+1} &= y_n + \alpha \sin(\omega t_n) - \alpha \sin(\omega t_{n+1}) + [-r\dot{y}_n + \alpha \omega \cos(\omega t_n)](t_{n+1} - t_n) \\ \dot{y}_{n+1} &= -r\dot{y}_n + \alpha \omega \cos(\omega t_n) - \alpha \omega \cos(\omega t_{n+1}) \end{aligned} \tag{13}$$

where $y_n = v/2$ or $-v/2$.

Here, we use the transcendental map (13) only to specify the parameter perturbation needed to implement the OGY method. In order to study the temporal evolution of dynamical variables we use Eqs. (10)–(12), as mentioned earlier.

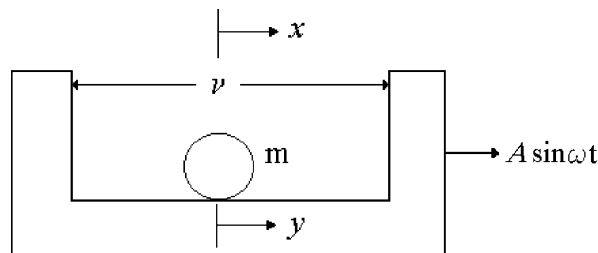


Fig. 2. Model of an impact-pair system.

3. Stabilizing periodic orbits

In this section, we describe the procedure to apply the OGY method of controlling chaos for the vibro-impact systems.

The OGY method consists on stabilizing a desired unstable periodic orbit embedded in the chaotic attractor by using small perturbations on a control parameter. We apply the perturbations when the chaotic orbit is in a region around a periodic orbit. In addition, it is interesting to emphasize that a chaotic attractor has embedded within it a large number of unstable periodic orbits. Besides, due to ergodicity of chaotic systems, a chaotic orbit visits the neighborhood of each one of these periodic orbits.

To implement the OGY method, we first determine a chaotic attractor and choose an unstable periodic orbit. Then, we define a small region (a neighborhood) around this orbit. We can consider this region as a circle, whose radius is here denoted by ϵ . Finally, we tailor the small parameter perturbations.

In our numerical investigations, we use the amplitude of excitation, α , as control parameter to be perturbed. As mentioned earlier, the parameter perturbations, $\Delta\alpha$, are determined from a transcendental map, that is two-dimensional in this case. Thus, we can determine the perturbation in the similar way of classical two-dimensional maps, as Hénon map.

In order to specify the perturbations, first we have to linearize a transcendental map about the parameter value α_0 (the attractor is chaotic for $\alpha = \alpha_0$) and the unstable periodic orbit (\dot{z}_i, t_i) . Thus, we have

$$Z_{n+1} = AZ_n + B(\alpha - \alpha_0) \quad (14)$$

where

$$Z_n = \begin{pmatrix} \dot{z}_n - \dot{z}_i \\ t_n - t_i \end{pmatrix} \quad (15)$$

For the period-1 orbits ($i = 1$), the matrices A and B are given by

$$A(\dot{z}_{n+1}, \dot{z}_n, t_{n+1}, t_n) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial t_{n+1}}{\partial t_n} & \frac{\partial t_{n+1}}{\partial \dot{z}_n} \\ \frac{\partial \dot{z}_{n+1}}{\partial t_n} & \frac{\partial \dot{z}_{n+1}}{\partial \dot{z}_n} \end{pmatrix} \quad (16)$$

$$B(\dot{z}_{n+1}, \dot{z}_n, t_{n+1}, t_n) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial t_{n+1}}{\partial \alpha} \\ \frac{\partial \dot{z}_{n+1}}{\partial \alpha} \end{pmatrix} \quad (17)$$

For the period-2 orbits ($i = 1, 2$), the matrices A and B are given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial t_{n+2}}{\partial t_{n+1}} & \frac{\partial t_{n+2}}{\partial \dot{z}_{n+1}} \\ \frac{\partial \dot{z}_{n+2}}{\partial t_{n+1}} & \frac{\partial \dot{z}_{n+2}}{\partial \dot{z}_{n+1}} \end{pmatrix} \begin{pmatrix} \frac{\partial t_{n+1}}{\partial t_n} & \frac{\partial t_{n+1}}{\partial \dot{z}_n} \\ \frac{\partial \dot{z}_{n+1}}{\partial t_n} & \frac{\partial \dot{z}_{n+1}}{\partial \dot{z}_n} \end{pmatrix} \quad (18)$$

$$B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial t_{n+2}}{\partial t_{n+1}} & \frac{\partial t_{n+2}}{\partial \dot{z}_{n+1}} \\ \frac{\partial \dot{z}_{n+2}}{\partial t_{n+1}} & \frac{\partial \dot{z}_{n+2}}{\partial \dot{z}_{n+1}} \end{pmatrix} \begin{pmatrix} \frac{\partial t_{n+1}}{\partial \alpha} \\ \frac{\partial \dot{z}_{n+1}}{\partial \alpha} \end{pmatrix} \quad (19)$$

The components of these matrices are evaluated at the periodic orbit (\dot{z}_i, t_i) and at α_0 . Besides, the components depend on \dot{z}_n, t_n and \dot{z}_{n+1}, t_{n+1} . In contrast of classical maps, whose components only depend on n th iteration of the dynamical variables.

Second, we consider

$$[Z_{n+1}] \cdot \vec{f}_{u,i+1} = 0 \quad (20)$$

where \vec{f}_u is a contravariant vector and $[Z_{n+1}]$ is a vector whose elements are components of matrix Z_{n+1} .

The contravariant vectors, \vec{f}_u and \vec{f}_s , are obtained from the relations $\vec{f}_s \cdot \vec{e}_s = \vec{f}_s \cdot \vec{e}_u = 1, \vec{f}_u \cdot \vec{e}_s = \vec{f}_s \cdot \vec{e}_u = 0$. The vectors \vec{e}_s and \vec{e}_u are the stable and unstable unit eigenvectors at (\dot{z}_i, t_i) , respectively, and correspond the stable eigenvalue ($|\lambda_s| < 1$) and unstable eigenvalue ($|\lambda_u| > 1$) of the matrix A .

In this case, the eigenvectors are given by

$$\vec{e}_s = \frac{1}{\sqrt{p_1^2 + 1}}(p_1\hat{x} + \hat{y}), \quad \vec{e}_u = \frac{1}{\sqrt{p_2^2 + 1}}(p_2\hat{x} + \hat{y}) \tag{21}$$

where

$$p_1 = (\lambda_s - a_{22})/a_{21}, \quad p_2 = (\lambda_u - a_{22})/a_{21} \tag{22}$$

The contravariant vectors are

$$\vec{f}_s = \frac{\sqrt{p_1^2 + 1}}{(p_1 - p_2)}(\hat{x} - p_2\hat{y}), \quad \vec{f}_u = \frac{\sqrt{p_2^2 + 1}}{(p_2 - p_1)}(\hat{x} - p_1\hat{y}) \tag{23}$$

Finally, from Eqs. (14) and (20), we obtain the parameter perturbation ($\Delta\alpha = \alpha_n - \alpha_0$):

$$\alpha_n = \alpha_0 - \frac{[A_i Z_n] \cdot \vec{f}_{u,i+1}}{[B_i] \cdot \vec{f}_{u,i+1}} \tag{24}$$

Therefore, when the control is applied, we have the new control parameter α_n for the n th iteration (impact). As we use the transcendental maps (impact maps) to obtain the parameter perturbations, the parameter is determined at a given moment of the impact and is evaluated in the next impact. Therefore, between impacts the parameter determined, α_n , does not change.

4. Numerical results

4.1. Impact oscillator

In this section, as numerical examples of the control method implementation, we stabilize both a period-1 and a period-2 orbits for the impact oscillator. For this system, the periodicity of an unstable orbit is equal to the number of the impacts. (This does not happen for the impact-pair system, as we will see in Section 4.2.)

To apply the control method, we compute new parameter α_n from expression (24) when the chaotic orbit falls in the ϵ -neighborhood of the unstable orbit. The components of matrix A are

$$\begin{aligned} \frac{\partial t_{n+1}}{\partial t_n} &= \frac{1}{\dot{x}_{n+1}} [-x_n + \alpha \cos(\omega t_n)] \sin(t_{n+1} - t_n) - r\dot{x}_n \cos(t_{n+1} - t_n) \\ \frac{\partial t_{n+1}}{\partial \dot{x}_n} &= \frac{1}{\dot{x}_{n+1}} [r \sin(t_{n+1} - t_n)] \\ \frac{\partial \dot{x}_{n+1}}{\partial t_n} &= \frac{\partial t_{n+1}}{\partial t_n} [-x_{n+1} + \alpha \cos(\omega t_{n+1})] + [x_n - \alpha \cos(\omega t_n)] \cos(t_{n+1} - t_n) - r\dot{x}_n \sin(t_{n+1} - t_n) \\ \frac{\partial \dot{x}_{n+1}}{\partial \dot{x}_n} &= \frac{\partial t_{n+1}}{\partial \dot{x}_n} [-x_{n+1} + \alpha \cos(\omega t_{n+1})] - r \cos(t_{n+1} - t_n) \end{aligned} \tag{25}$$

and the components of matrix B are

$$\begin{aligned} \frac{\partial t_{n+1}}{\partial \alpha} &= \frac{1}{\dot{x}_{n+1}(1 - \omega^2)} [\cos(\omega t_n) \cos(t_{n+1} - t_n) - \omega \sin(\omega t_n) \sin(t_{n+1} - t_n) - \cos(\omega t_{n+1})] \\ \frac{\partial \dot{x}_{n+1}}{\partial \alpha} &= \frac{\partial t_{n+1}}{\partial \alpha} \left\{ \left[-x_c + \frac{\alpha}{1 - \omega^2} \right] \cos(t_{n+1} - t_n) + \left[r\dot{x}_n - \frac{\alpha\omega}{1 - \omega^2} \sin(t_{n+1} - t_n) \right] \sin(t_{n+1} - t_n) \right\} \\ &\quad + \frac{1}{1 - \omega^2} [\cos(\omega t_n) \sin(t_{n+1} - t_n) + \omega \sin(\omega t_n) \cos(t_{n+1} - t_n) - \omega \cos(\omega t_{n+1})] \end{aligned} \tag{26}$$

Fig. 3(a) shows the two unstable periodic orbits embedded in the chaotic attractor ($\alpha = \alpha_0 = 1.0$) for the dynamical variables computed just before the impacts ($x = x_0 = 0$). Fig. 3(b) depicts a Stroboscopic map for these periodic orbits and this chaotic attractor. Furthermore, comparing these figures we can note that the numbers of the impacts (Fig. 3(a)) and the periodicity of the unstable orbits (Fig. 3(b)) are equal, as mentioned earlier.

For the numerical application we consider the neighborhood radius $\epsilon = 0.01$. The point of period-1 orbit is $(t_1, \dot{x}_1) \approx (1.092635, 0.336177)$ and the points of the period-2 orbits are $(t_1, \dot{x}_1) \approx (1.3978, 0.4447)$ and $(t_2, \dot{x}_2) \approx (0.7352, 0.4669)$.

Fig. 4(a) shows the stabilization of both period-1 orbit and period-2 orbit for the different times. In Fig. 4(b) we depict the indicative of variation of the parameter perturbations, $\delta\alpha$, needed to implement the control, where we can see

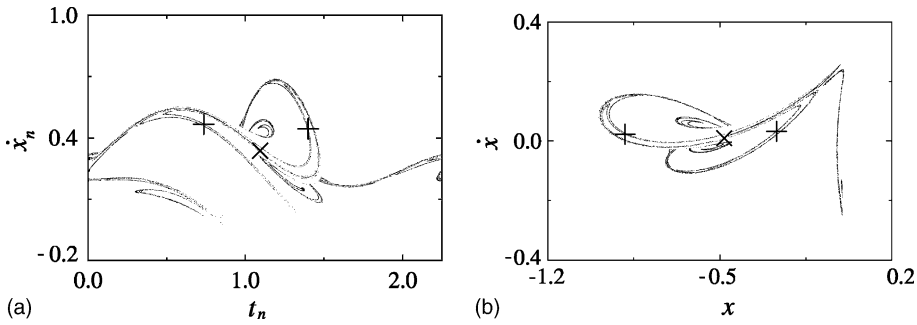


Fig. 3. Chaotic attractor and unstable periodic orbits of period 1 (x) and 2 (+) of the impact oscillator for the control parameters $\alpha = \alpha_0 = 1.0$, $\omega = 2.8$, $r = 0.8$, and $x_c = 0$: (a) Poincaré map just before impact at $x = x_c = 0$ and (b) Stroboscopic map ($t = 2\pi/\omega$).

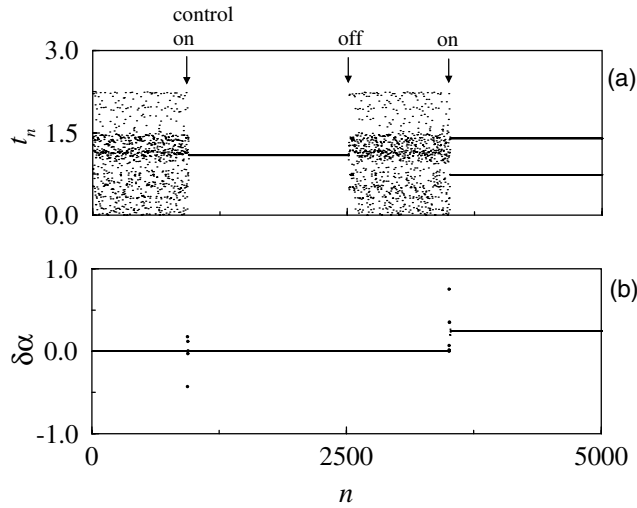


Fig. 4. Stabilization of unstable periodic orbits indicated in Fig. 3: (a) dynamical variable t_n (impact-time) as a function of the impact number n and (b) indicative of variation of the control parameter α ($\delta\alpha = 100(\alpha - \alpha_0)/\alpha_0$).

that $\delta\alpha$ for the period-1 orbit is around zero. However, $\delta\alpha$ for the period-2 orbit is not around zero because the point of this orbit was not obtained with a good precision due to some numerical problems.

Furthermore, we can note in Fig. 4(b) that variation of the parameter perturbation is less than 1%.

4.2. Impact-pair system

Here, we stabilize a period-1 orbit for the impact-pair system. For this orbit, there are two impacts for a time interval $[0, 2\pi/\omega]$. Besides, the parameter perturbation is specified at the moment of impacts. Consequently, to apply the control method for this orbit we have to treat it like a period-2 orbit.

To apply the control method, we compute new parameter α_n from expression (24) when the chaotic orbit falls in the ϵ -neighborhood of the unstable orbit. The components of matrix A are

$$\begin{aligned} \frac{\partial t_{n+1}}{\partial t_n} &= -\frac{1}{\dot{y}_{n+1}} [r\dot{y}_n + \ddot{e}(t_n)(t_{n+1} - t_n)] \\ \frac{\partial t_{n+1}}{\partial \dot{y}_n} &= \frac{r}{\dot{y}_{n+1}} (t_{n+1} - t_n) \\ \frac{\partial \dot{y}_{n+1}}{\partial t_n} &= -\frac{\partial t_{n+1}}{\partial t_n} \ddot{e}(t_{n+1}) + \ddot{e}(t_n) \\ \frac{\partial \dot{y}_{n+1}}{\partial \dot{y}_n} &= -\frac{\partial t_{n+1}}{\partial \dot{y}_n} \ddot{e}(t_{n+1}) - r \end{aligned}$$

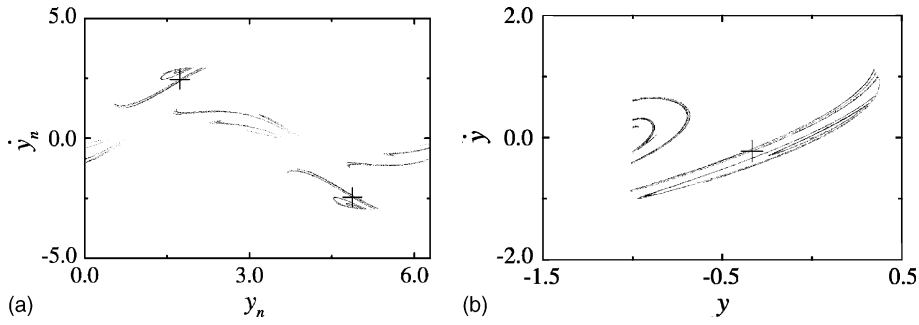


Fig. 5. Chaotic attractor and unstable periodic orbits of period 1 (+) of the impact-pair system for the control parameters $\alpha = \alpha_0 = 2.3$, $\nu = 2.0$, $r = 0.7$, and $\omega = 1.0$: (a) Poincaré map just before impact at $y = |v/2| = 1.0$ and (b) Stroboscopic map ($t = 2\pi/\omega$).

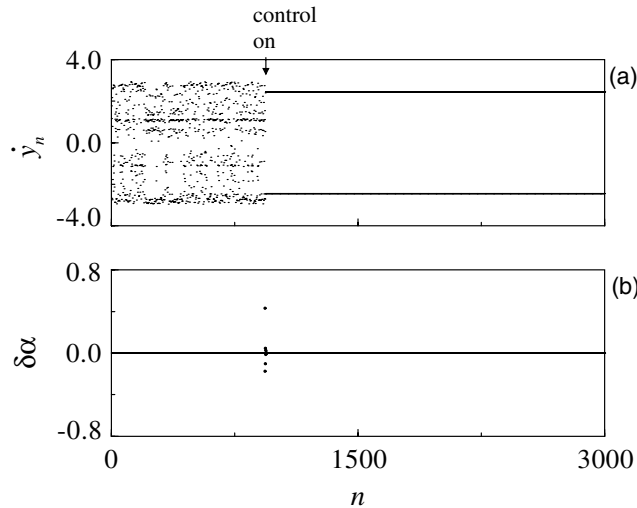


Fig. 6. Stabilization of unstable periodic orbit indicated in Fig. 5: (a) evolution of relative velocity \dot{y}_n as a function of the impact number n and (b) indicative of variation of the control parameter α ($\delta\alpha = 100(\alpha - \alpha_0)/\alpha_0$).

and components of matrix B are

$$\begin{aligned} \frac{\partial t_{n+1}}{\partial \alpha} &= \frac{1}{\dot{y}_{n+1}} [\sin(\omega t_{n+1}) - \sin(\omega t_n) - \omega \cos(\omega t_n)(t_{n+1} - t_n)] \\ \frac{\partial \dot{y}_{n+1}}{\partial \alpha} &= \frac{\partial t_{n+1}}{\partial \alpha} \alpha \omega^2 \sin(\omega t_n) + \omega \cos(\omega t_n) - \omega \cos(\omega t_{n+1}) \end{aligned} \tag{27}$$

Fig. 5(a) shows an unstable period-1 orbit, with two impacts per cycle (time interval $[0, 2\pi/\omega]$), embedded in a chaotic attractor determined for $\alpha = \alpha_0 = 2.3$. In Fig. 5(b) we present the Stroboscopic map for the period-1 orbit and the chaotic attractor shown in Fig. 5(a).

In this case, we consider the neighborhood radius $\epsilon = 0.05$. The points of period-1 orbit are $(t_1, \dot{y}_1) \approx (1.7312311, 2.4494606)$ and $(t_2, \dot{y}_2) \approx (4.8728238, -2.4494606)$.

Fig. 6(a) shows the stabilization of period-1 orbit. In Fig. 6(b) we depict the indicative of variation of the parameter perturbations needed to implement the control, where we can see that variation of the parameter perturbation is less than 0.5%.

5. Conclusion

In conclusion, we present a new procedure to implement the OGY method in mechanical systems with impacts. For that, we use a transcendental map to determine the required small changes on the control parameter. As examples of

this procedure, we apply the OGY control method to an impact oscillator and an impact-pair system. Thus, we stabilize unstable periodic orbits embedding in chaotic attractors commonly observed in the considered systems.

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