



Finite-time rotation number: A fast indicator for chaotic dynamical structures



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ARTICLE INFO

Article history:

Received 21 September 2012

Received in revised form 14 December 2012

Accepted 18 December 2012

Available online 21 December 2012

Communicated by C.R. Doering

ABSTRACT

Lagrangian coherent structures are effective barriers, sticky regions, that separate chaotic phase space regions of different dynamical behavior. The usual way to detect such structures is by calculating finite-time Lyapunov exponents. We show that similar results can be obtained for time-periodic systems by calculating finite-time rotation numbers, which are faster to compute. We illustrate our claim by considering examples of continuous- and discrete-time dynamical systems of physical interest.

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It is noteworthy that chaos is observed in physically interesting systems, both in Nature and in mathematical models. Indeed, chaotic dynamics is now commonplace in such diverse disciplines as celestial mechanics [1], atomic physics [2], fluid mechanics [3], plasma physics [4], etc. The main means of determining chaos is with Lyapunov exponents, which can be notoriously difficult to compute in fast and reliable ways, particularly if only experimental data are available and the governing equations are unknown. Hence, an active area of research is the search for *fast indicators* (FIs), i.e., computational diagnostics that characterize chaos quickly [5]. Here we introduce a new FI, the *finite-time rotation number* (FTRN), for determining effective transport barriers or sticky regions within chaotic domains.

Since Lyapunov exponents measure the exponential rate of divergence of nearby trajectories, FIs are usually based on them, examples being the finite-time Lyapunov exponent (FTLE) [6], the smaller (SALI) and generalized (GALI) alignment indices [7], and the mean exponential growth of nearby orbits (MEGNO) [8]. These FIs not only determine the strength of chaos, but can detect invariant tori and other issues related to integrability. Alternatively, there are diagnostics based on frequency decomposition, which are well-suited for weakly chaotic motion with any number of degrees of freedom [9].

A relatively new application of FIs is to quantitatively characterize chaotic transport, an important basic physical process occurring

in many contexts, from fluid dynamics [3,10] to fusion plasma confinement [4]. A now popular method for determining chaotic transport is based on *Lagrangian coherent structures* (LCS) [11]. LCS represent effective barriers that separate regions of different dynamical behavior. This FI has been used to investigate time-dependent flows occurring in many applications of fluid mechanics, such as transport in ocean currents [12], flow over an airfoil [13], and chaotic mixing in forced tanks [14], and it has recently been introduced into plasma physics and used to describe plasmas turbulence [15] and magnetic reconnection [16]. The main purpose of this Letter is to argue that for two-dimensional time-periodic systems FTRN is a suitable alternative to compute LCS.

In practice LCS are unveiled via computation of FTLEs for many points, i.e. the FTLE field. One then finds ridges in this FTLE field that indicate inhibition of transport between different regions. Such ridges, although not *bona fide* invariant sets, provide effective transport barriers. Intuitive arguments based on local stretching of phase space have been proposed along with a very large number of examples to support this claim [11], and a recent theoretical justification for these results has been proposed [17].

The detection of LCS by FTLEs is versatile – it can be applied to periodic, quasiperiodic, or broadband vector fields defining the flow. However, in two dimensions when the vector field is area-preserving and time-periodic, the FTRN proposed in this Letter is a convenient alternative. Although for tractability many early studies of mixing in fluids [3] considered time-periodic vector fields, physical velocity fields contain many frequencies or broadband turbulent spectra. Moreover, area preservation in fluids arises from the solenoidal approximation of the velocity field, a common assumption, of e.g. geophysical interest (e.g. [10,18]), with varying degrees of validity. However, for analyzing snapshots of magnetic

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fields in toroidal plasma devices or, more generally, astrophysical or other magnetic fields in the vicinity of any stable closed field line [19], the time-like coordinate is a toroidal angle [16] and the system is exactly periodic. Also, area preservation is an exact consequence of $\nabla \cdot \mathbf{B} = 0$. Thus for such systems, the FTRN is a natural fast indicator for characterizing chaos and transport [20].

For such time-periodic frequency systems, we show results on LCS using the FTLE can be obtained by the simple and computationally fast method based on the FTRN. As for the FTLE, LCS are ridges of the FTRN field computed from a grid of initial conditions. But, the rotation number does not require the evaluation of spatial derivatives; thus a fine mesh, although desirable, is not essential and this substantially reduces the computational time to obtain LCS with good resolution. For conventional FTLE methods the evaluation of derivatives at each grid point requires the calculation of four nearby orbits. Using such a method, we find for all of our calculations that the FTLE takes between four and five times more CPU time. However, for more efficient FTLE methods (e.g. [21]) this can be reduced, but the amount of this reduction depends on the nature of the vector field. Thus, it is not unreasonable to expect a savings of at least a factor of two in this part of the calculation. A theoretical formulation of our procedure to determine LCS from the FTRN should be further explored in other investigations.

A time- T -periodic dynamical system with annular phase space \mathcal{D} is determined by a Poincaré map, $M : \mathcal{D} \rightarrow \mathcal{D}$, where each period- T is represented by one iteration of M . If $\mathbf{x} \in \mathcal{D}$ lies on an invariant circle S^1 , then M maps S^1 to itself. The rotation number (e.g. [22]) for an orbit starting at \mathbf{x}_0 is

$$\omega = \lim_{n \rightarrow \infty} \frac{1}{n} \Pi \cdot (M^n(\mathbf{x}_0) - \mathbf{x}_0), \tag{1}$$

which is lifted to \mathbb{R} and Π is a suitable angular projection. Under mild conditions on M , this limit exists for every initial condition $\mathbf{x}_0 \in S^1$ and does not depend on \mathbf{x}_0 . Rotation numbers have been well-studied since the time of Poincaré [23] and to date many convergence results exist [24], but our approach is empirical like those for existing FTLE studies of LCS.

Consider a simple example, the rigid rotation $M(\mathbf{x}) = (x + w, y)$, where $\mathbf{x} = (x, y)$ and $x \in S^1$. Here the rotation number $\omega = w$. If w is a rational number p/q , the trajectory is a period- q orbit of the map M , whereas if w is irrational, then the ensuing (quasiperiodic) orbit covers densely the circle S^1 . The FTRN is the time- NT truncation for the flow and corresponds in the preceding definition to N iterations of M . Below we use

$$\omega_N(\mathbf{x}_0) := \frac{1}{n} \Pi \cdot (M^N(\mathbf{x}_0) - \mathbf{x}_0), \tag{2}$$

with only $N \in \mathbb{Z}$ iterations. In general, ω_N , like any truncation, depends on the initial condition. While the infinite-time rotation number is not defined for chaotic orbits, which do not lie on any S^1 , the finite-time counterpart exists for any orbit. Roughly speaking, ω_N measures the average rotation angle swept out by a trajectory over a time interval NT with respect to a nearby elliptic fixed point (according the island neighborhood chosen to be analyzed), and thus conveys information about the local behavior of trajectories, just as the FTLE does (which measures local rates of contraction or expansion). We identify LCS and Lagrangian barriers with ridges of the FTRN. Note, observed ridges of the FTRN amount to finite pieces of invariant tori, which appear (for a finite number of iterations) because the full rotation number has not been calculated.

FTRN can be viewed as dual to FTLE. For completely integrable systems, all points lie on invariant tori and thus there is a complete foliation by transport barriers. Also, for such integrable systems, all periodic orbits are parabolic and positive Lyapunov exponents do not exist for any orbits. Thus one would expect the FTLE field to

be flat and there to be no ridges of use for detecting barriers. Conversely, for integrable systems rotation numbers exist for all points of phase space and generically there are no extrema in this quantity. Thus, neither FTRN nor FTLE are expected to display ridges, and neither is expected to be particularly useful for studying nearly integrable systems. Their power comes into play for understanding structures within chaotic seas, typical of almost-integrable systems, and it is in this context that we make comparisons of the two indicators.

In order to illustrate how the FTRN detects LCS, we consider three examples and compare each with FTLE calculations. Essentially the same results are obtained with FTRN, but with lower computational cost. One example is a time-periodic two-dimensional fluid flow, the second is a discrete-time map of a flow used for passive advection, and the third is a magnetic field-line map. In all cases a uniform grid of 800×800 points is advanced.

Example 1 (Periodic double gyre flow). We consider a two-dimensional fluid flow with a stream function,

$$\psi(x, y) = A \sin[\pi f(x, t)] \sin(\pi y), \tag{3}$$

defined in the domain $\mathcal{D} := \{0 \leq x \leq 2, 0 \leq y \leq 1\}$, where

$$f(x, t) = a(t)x^2 + b(t)x, \tag{4}$$

$$a(t) = \epsilon \sin\left(\frac{2\pi t}{T}\right), \tag{5}$$

$$b(t) = 1 - 2\epsilon \sin\left(\frac{2\pi t}{T}\right), \tag{6}$$

and A is the maximum value of the horizontal velocity u [25].

The velocity field $\mathbf{v} = (u, v)$ is given by

$$u = -\frac{\partial \psi}{\partial y}, \tag{7}$$

$$v = \frac{\partial \psi}{\partial x}. \tag{8}$$

For $\epsilon = 0$ the flow is integrable, with equilibrium points

$$A: (1/2, 1/2), \quad B: (3/2, 1/2), \quad C_i: (x_{ci}, y_{ci}) \\ (i = 1, \dots, 6),$$

where $x_{ci} \in \{0, 1, 2\}$ and $y_{ci} \in \{1, 0\}$. The points A and B are centers, whereas C_i are saddles connected by heteroclinic trajectories. The latter are boundaries of two gyres surrounding A and B , with clockwise and counterclockwise rotations, respectively. The heteroclinic trajectory \mathcal{H} connecting $C_3: (1, 0)$ and $C_4: (1, 1)$ separates two distinct gyres and thus is a natural place to focus attention when looking for LCS.

For $\epsilon \neq 0$ the flow is time-dependent and nonintegrable, yet \mathcal{D} remains invariant. As is well known, the former heteroclinic connections are structurally unstable; upon perturbation an entanglement of stable and unstable manifolds with concomitant horseshoe dynamics appears. The vertical line \mathcal{H} is no longer invariant, but can be thought of as roughly separating two gyres with time-varying amplitudes: \mathcal{H} oscillates in the horizontal direction with amplitude $\approx \epsilon$ (for small ϵ) and frequency $2\pi/T$. The T -periodicity permits to choose a time- $4T$ to compute the rotation number,

$$(\omega_4)_{A,B}(\mathbf{x}) = \frac{1}{4} \sum_{i=1}^4 \theta_{A,B}(\mathbf{x}_i), \tag{9}$$

where the rotation angles around the gyre centers A and B are given, respectively, by

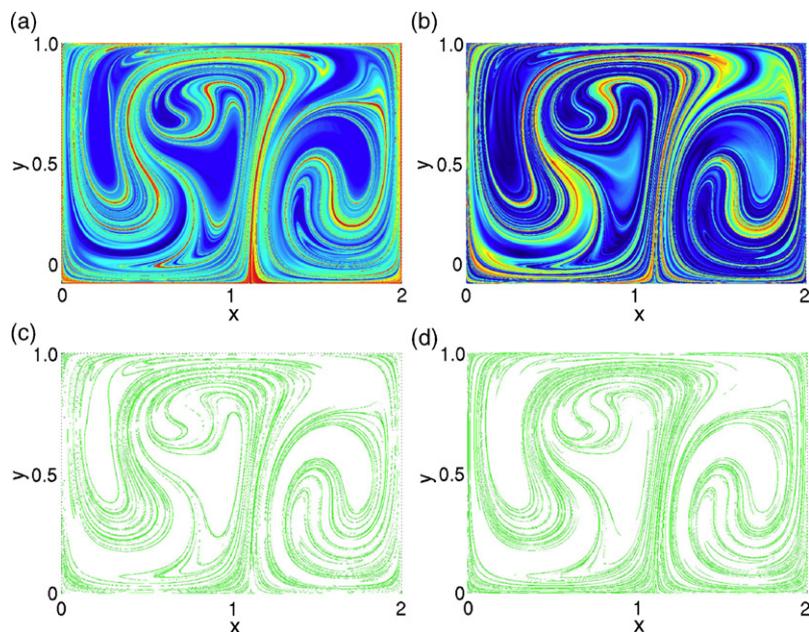


Fig. 1. (Color online.) (a) Time- $4T$ Lyapunov exponent and (b) time- $4T$ rotation number for the double gyre system, with period $T = 10$, amplitude $A = 0.1$ and forcing strength $\epsilon = 0.25$. (c) and (d) depict the Lagrangian coherent structures corresponding to ridges of (a) and (b), respectively.

$$\tan \theta_A(\mathbf{x}) = \frac{x_A - x}{y_A - y}, \quad (10)$$

$$\tan \theta_B(\mathbf{x}) = \frac{x_B - x}{y_B - y}, \quad (11)$$

$$x_{n+1} = x_n + a \sin(2\pi y_n) \pmod{1}, \quad (12)$$

$$y_{n+1} = y_n + a \sin(2\pi x_{n+1}) \pmod{1}. \quad (13)$$

for the time- $4T$ stroboscopic map of this nonintegrable system. This defines the projection Π . Similarly, we use the same corresponding time- $4T$ to compute the Lyapunov exponent, which requires computing five nearby orbits, necessary to evaluate spatial derivatives, instead of only one orbit for the FTRN. For this reason, computation of FTRNs is faster than FTLEs.

The results of the FTLEs and the FTRNs for the double gyre system are depicted in Figs. 1(a) and (b), respectively, where blue (red) depict low (high) values of the corresponding quantity. To obtain Fig. 1(b) we add the FTRN calculated with respect to points A and B . The ridges of both, i.e. the crests of higher values are shown in Figs. 1(c) and (d) for FTLE and FTRN, respectively. The pictures are indeed very similar, notwithstanding the wide difference in the CPU-time necessary to produce them, mainly due to the derivatives computed for the FTLE. Moreover, Figs. 1(c) and (d) reveal the existence of LCS for the nonintegrable system. For small values of y , the LCS approach the oscillating vertical line \mathcal{H} that separates the gyres. In fact, despite the absence of well-defined stable and unstable manifolds of equilibria for time-periodic flows, the ridges displayed by Fig. 1 are quasi-invariant: if a passive scalar (tracer) were put on such ridges, it would be advected by flow and remain in the vicinity of the ridge for a long time (on the order of the experiment duration). In other words, even though the ridge no longer separates the gyres for an arbitrarily long time (there may be a small transverse flux), trajectories starting on the left-hand side (right-hand side) chiefly remain in the left-hand side (right-hand side). In practical terms, however, this suffices to characterize an effective transport barrier.

Example 2 (*Advection of a passive scalar*). In Example 1, investigation of LCS in a time- T -periodic flow required the numerical evaluation of the time- T stroboscopic map M . Sometimes when researchers consider advection of a passive scalar by two-dimensional flows, M is given by replacing the mixing action of a flow by an explicit area-preserving map. An example [26] of this is

This map is also known in the literature as the Harper map [27].

The map (13) corresponds to a velocity field that is the superposition of two sinusoidal shear flows in the x and y directions. In this case, the flow shear reverses sign along some shearless curve, as is the case for zonal flows of geophysical, atmospheric, and plasma physical interest [10,19]. The map (13) is symplectic and represents a Hamiltonian system for any value of a . The fixed points in the torus $\mathcal{D} = [0, 1) \times [0, 1)$ are the centers $P: (0, 1/2)$, $Q: (1/2, 0)$ and the saddles $R: (0, 0)$ and $S: (1/2, 1/2)$.

Since the system is nonintegrable for $a \neq 0$, the stable and unstable manifolds stemming from the saddle points R and S intersect in a heteroclinic tangle and there are chaotic orbits that do not lie on continuous invariant circles. This structure is responsible for the mixing effect of the chaotic advection. Nevertheless, this chaotic layer acts as an effective transport barrier separating the two gyres with invariant tori encircling P and Q .

This is clearly seen after computing the time- $100T$ rotation number for M (with Π projecting onto the x -axis) and then extracting the corresponding ridges with high values of ω_{100} . These ridges trace out quasi-invariant sets that shadow the heteroclinic connections, especially in the vicinity of the saddle points R and S , reinforcing their interpretation as LCS. For both maps and flows one has to decide which projection is relevant to compute FTRN. Generally for maps like Eqs. (12)–(13) one is interested in transport in the momentum (y) direction, and the angle (x) is the natural variable. The computation of FTRNs is particularly fast for maps, so calculation of ω_{100} is not difficult, but the results are essentially identical for ω_{10} .

In Figs. 2(a) and (b) we depict the FTLE and FTRN, respectively, for orbits of the advection map (13) for $a = 0.25$, whose ridges (points with largest relative values) are shown in Figs. 2(c) and (d), respectively. Both diagnostics indicate that the LCS are quasi-invariant sets about a chaotic separatrix layer. This layer acts as a transport barrier that separates quasiperiodic curves encircling the centers P and Q ; the layer being a ridge implies these are LCS of this system.

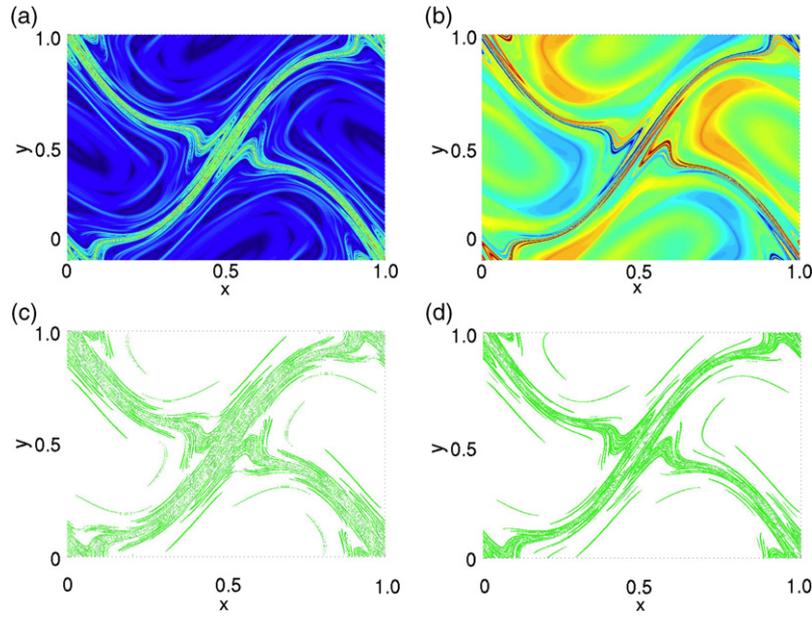


Fig. 2. (Color online.) (a) Time-100T Lyapunov exponent and (b) time-100T rotation number for the advection map (13) with mixing parameter $a = 0.25$. (c) and (d) depict the Lagrangian coherent structures corresponding to ridges of (a) and (b), respectively.

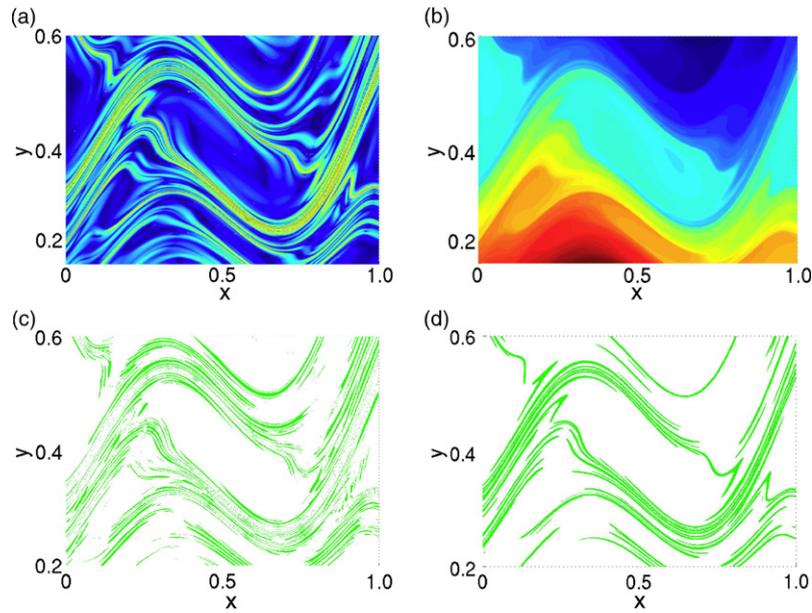


Fig. 3. (Color online.) (a) Time-15T Lyapunov exponent and (b) time-15T rotation number for the tokamak (15) with $L = 4$. (c) and (d) depict the Lagrangian coherent structures corresponding to ridges of (a) and (b), respectively.

Example 3 (Magnetic field line map). Magnetic field lines in the equilibrium states of toroidal magnetic plasma confinement devices, such as tokamaks and stellarators, are orbits of a one degree-of-freedom integrable Hamiltonian system, where a toroidal-like angle plays the role of time (e.g. [19]). In the simplest case, canonically conjugate variables are the spatial coordinates of an annulus and points (x_n, y_n) represent the n -th field line intersection with a surface-of-section at a fixed value of the toroidal angle. A similar situation arises in the vicinity of any stable (elliptic) closed magnetic field line.

Perturbations due to external electric currents or internal MHD instabilities break symmetry and render the system nonintegrable, giving rise to chaotic field lines. Since the magnetic field configuration here is strictly static in time, the term chaos means that two infinitesimally close field lines Lyapunov exponentiate as they wind around the torus.

An example of a nonintegrable system that models such field lines is provided by the so-called *tokamak* [28]:

$$y_{n+1} = y_n - \frac{L}{2\pi} \frac{y_{n+1} \sin(2\pi x_n)}{1 + y_{n+1}}, \tag{14}$$

$$x_{n+1} = x_n + \frac{1}{q(y_{n+1})} - \frac{L}{(2\pi)^2} \frac{\cos(2\pi x_n)}{(1 + y_{n+1})^2}, \tag{15}$$

where L is a parameter measuring nonintegrability (proportional to the perturbation strength) and

$$\frac{1}{q} = \frac{1}{4}(2 - y)(2 - 2y + y^2) \tag{16}$$

is the inverse of the rotational transform of the field lines.

For $L = 4$ the FTLE and FTRN (with \mathcal{I} again the x -projection) of the tokamak are shown in Figs. 3(a) and (b), respectively, with (c)

and (d) depicting the corresponding ridges. In both cases the latter correspond, as in the advection map, to the homoclinic intersection of manifolds of the saddle fixed points of the tokamak (15), defining a thin chaotic layer originating from such intersections. The separatrix layer is again a LCS, which indicates a transport channel along the layer, yet restricting diffusion across the barrier.

In conclusion, here we have proposed the FTRN as a FI, which for physical systems with a single period is superior, being faster and simpler, than the FTLE. The three examples treated demonstrate this point. We note, however, that the speed of calculation of both FTRNs and FTLEs is achieved at the cost of sacrificing the detailed picture of transport provided by an analysis of turnstiles through cantori (e.g. [29]), which requires a search for periodic orbits [22]. In future studies we propose relaxing the single frequency limitation by a more detailed frequency analyses (e.g. [9]), and considering statistical analyses of the FTRN akin to that of FTLE (e.g. [5]).

Acknowledgements

This work was funded by FAPESP, CNPq, CAPES, MCT/CNEN, Fundação Araucária, and the USDOE Contract DE-FG05-80ET-53088.

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