

VII- E

Alguns Mapas Bidimensionais

1 - Novas Características Dinâmicas

- Além de pontos fixos, há pontos de selas.
- Ponto de sela: contração em uma direção e expansão na outra.
- Mapa de Poincaré bidimensional de uma órbita tridimensional.

2 - Mapa de Hénon

Hénon (Comm. Math. 50, 69, 1976) introduziu o mapa

$$(x_{n+1}, y_{n+1}) = f(x_n, y_n) = (a - x_n^2 + b y_n, x_n)$$

a, b: parâmetros de controle

Para $a = 1,28$ e $b = -0,3$; $(x_0, y_0) = (0, 0)$,

trajetória converge para órbita com período 2

Os pontos iniciais convergem para essa órbita ou para $x \rightarrow \infty$

Bacias de atração variam com a, b

Para $a = 1,28$ e $b = -0,3$; fronteira das bacias é contínua

Para $a = 1,40$ e $b = -0,3$; fronteira das bacias é fractal

Mapa de Hénon

$b = -0.3$

$a = 1.28$

$a = 1.4$

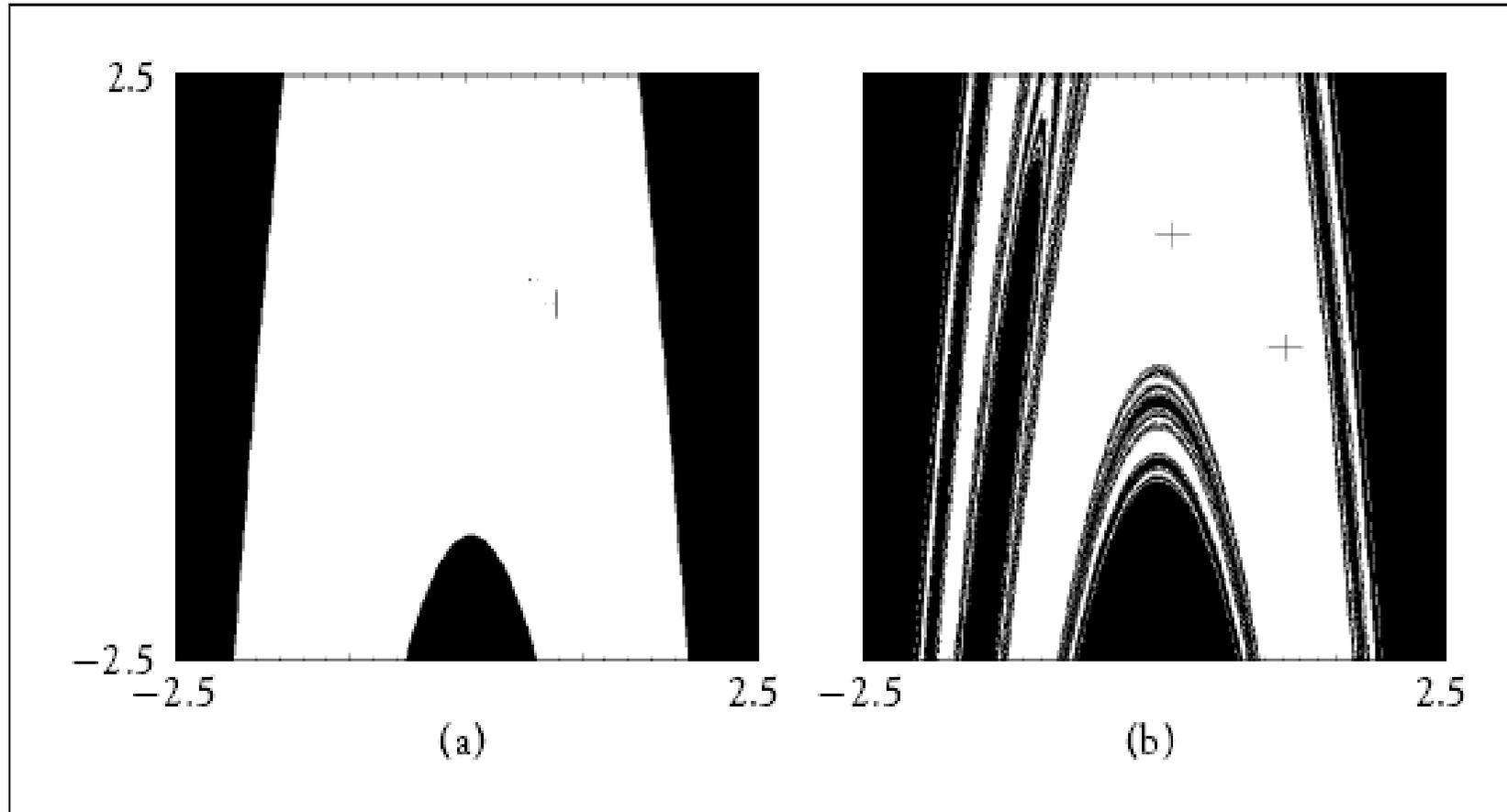


Figure 2.3 A square of initial conditions for the Hénon map with $b = -0.3$.

Initial values whose trajectories diverge to infinity upon repeated iteration are colored black. The crosses show the location of a period-two sink, which attracts the white initial values. (a) For $a = 1.28$, the boundary of the basin is a smooth curve. (b) For $a = 1.4$, the boundary is “fractal” .

(Alligood et al.
Chaos...)

Atratores, Repulsores e Pontos de Sela

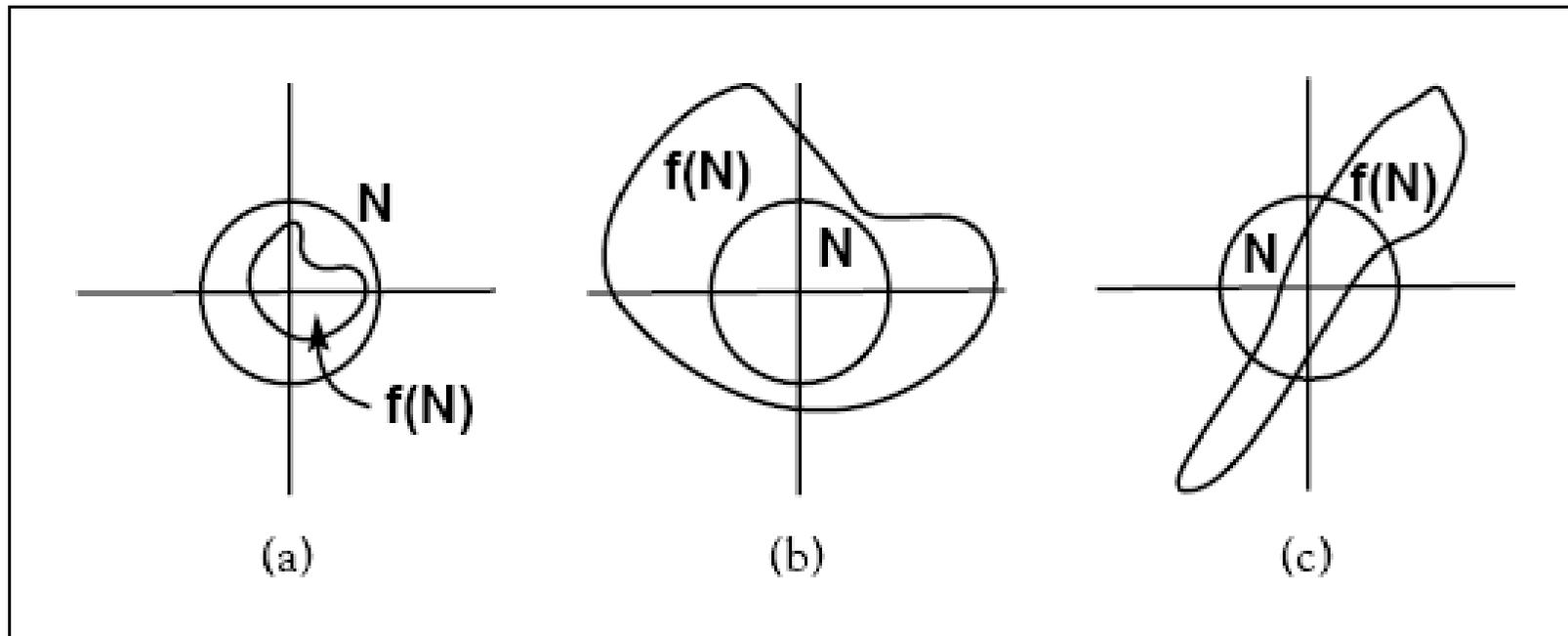
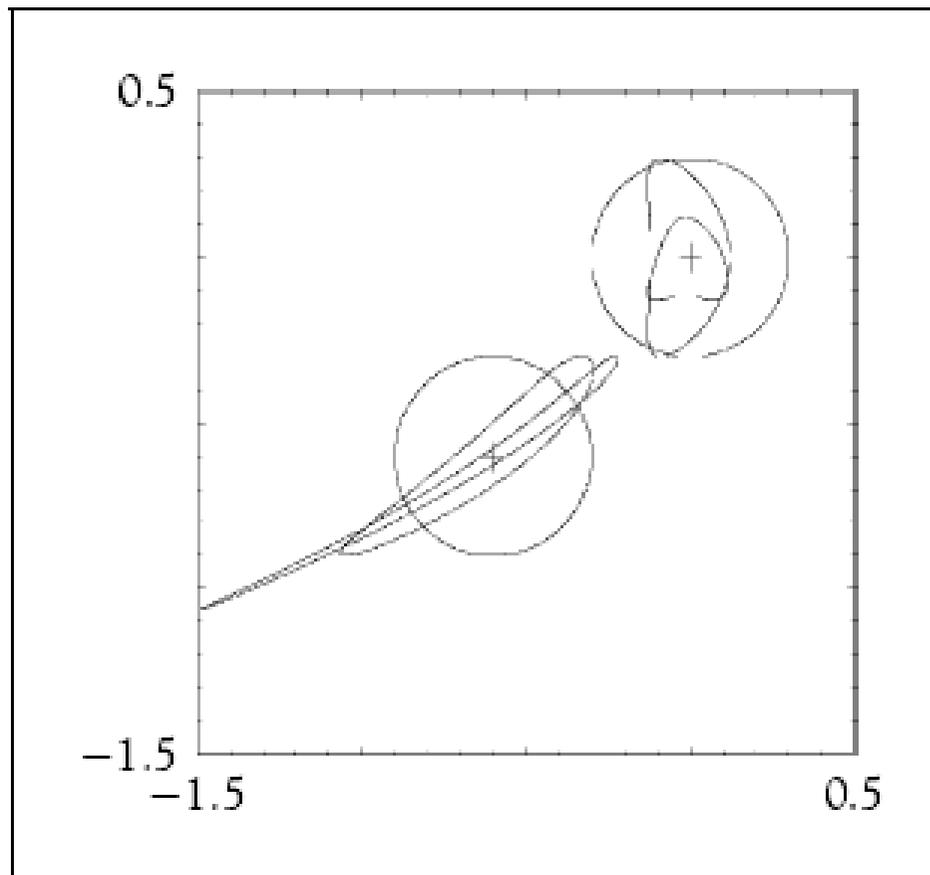


Figure 2.8 Local dynamics near a fixed point.

The origin is (a) a sink, (b) a source, and (c) a saddle. Shown is a disk N and its iterate under the map f .

(Alligood et al.
Chaos...)

Atrator e Ponto de Sela no Mapa de Hénon



$$a = 0$$
$$b = 0.4$$

Figure 2.9 Local dynamics near fixed points of the Hénon map.

The crosses mark two fixed points of the Hénon map f with $a = 0$, $b = 0.4$, in the square $[-1.5, 0.5] \times [-1.5, 0.5]$. Around each fixed point a circle is drawn along with its two forward images under f . On the left is a saddle: the images of the disk are becoming increasingly long and thin. On the right the images are shrinking, signifying a sink.

4 - Definições de Atratores e Repulsores

Definição: O comprimento (euclidiano) de um vetor

$$\vec{u} = (x, y) \text{ em } \mathbb{R}^2 \text{ é } |\vec{u}| = \sqrt{x^2 + y^2}.$$

A vizinhança é $N_\varepsilon(\vec{p}) : \{\vec{v} \in \mathbb{R}^2 : |\vec{v} - \vec{p}| < \varepsilon\}$

Seja \vec{f} um mapa em \mathbb{R}^2 , \vec{p} um ponto fixo, $\vec{f}(\vec{p}) = \vec{p}$

Definições:

\vec{p} é um atrator se $\exists \varepsilon > 0$ tal que $\vec{v} \in N_\varepsilon(\vec{p}) \Rightarrow \lim_{k \rightarrow \infty} \vec{f}^k(\vec{v}) = \vec{p}$

\vec{p} é um repulsor se $\exists \varepsilon > 0$ tal que $\vec{v} \in N_\varepsilon(\vec{p}) \Rightarrow \lim_{k \rightarrow \infty} \vec{f}^k(\vec{v}) \notin N_\varepsilon(\vec{p})$

Bacias de atração para o Mapa de Hénon

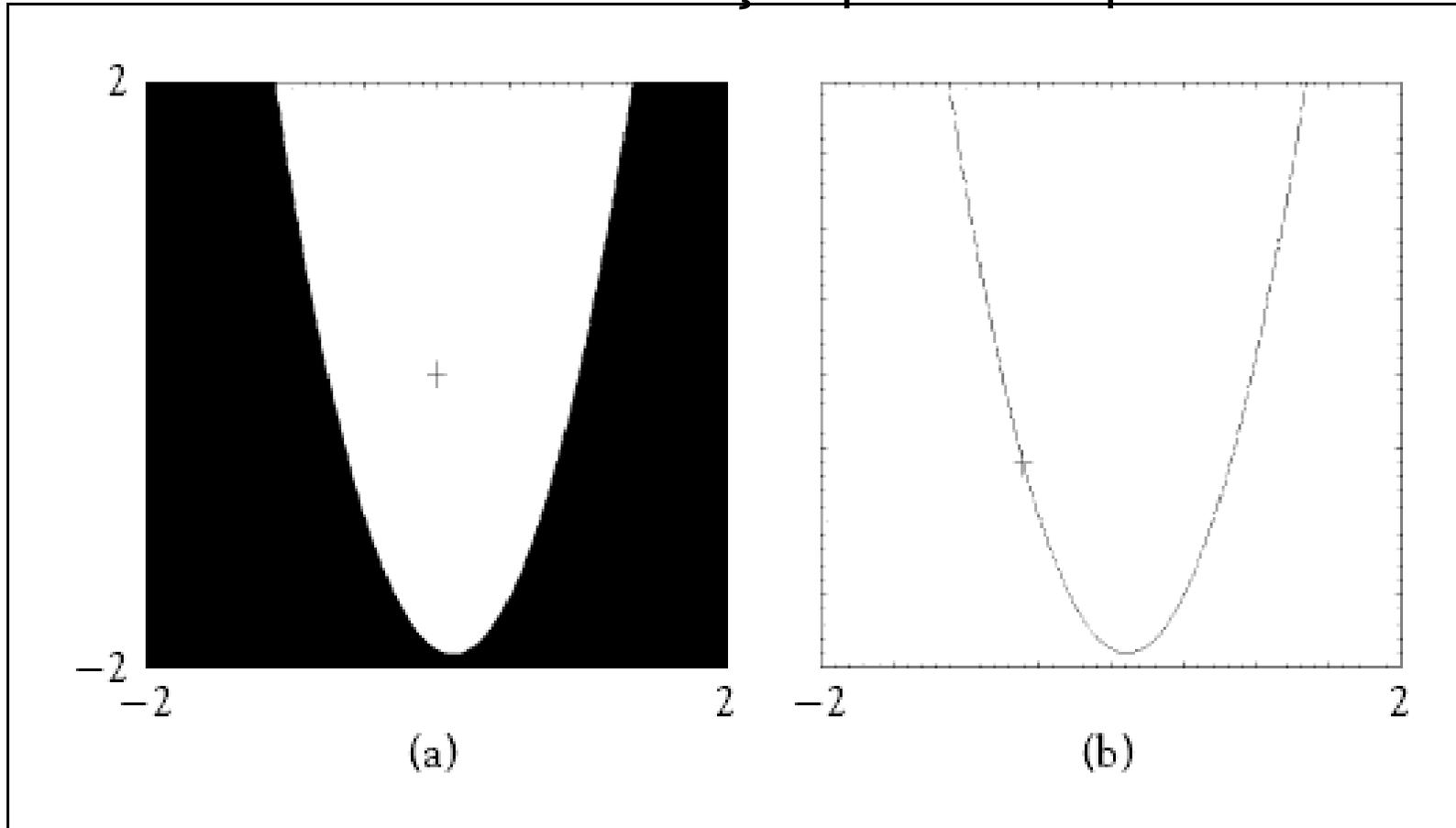


Figure 2.10 Basins of attraction for the Hénon map with $a = 0, b = 0.4$.

(a) The cross marks the fixed point $(0, 0)$. The basin of the fixed point $(0, 0)$ is shown in white; the points in black diverge to infinity. (b) The initial conditions that are on the boundary between the white and black don't converge to $(0, 0)$ or infinity; instead they converge to the saddle $(-0.6, -0.6)$, marked with a cross. This set of boundary points is the stable manifold of the saddle (to be discussed in Section 2.6).

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Trator Caótico para o Mapa de Hénon

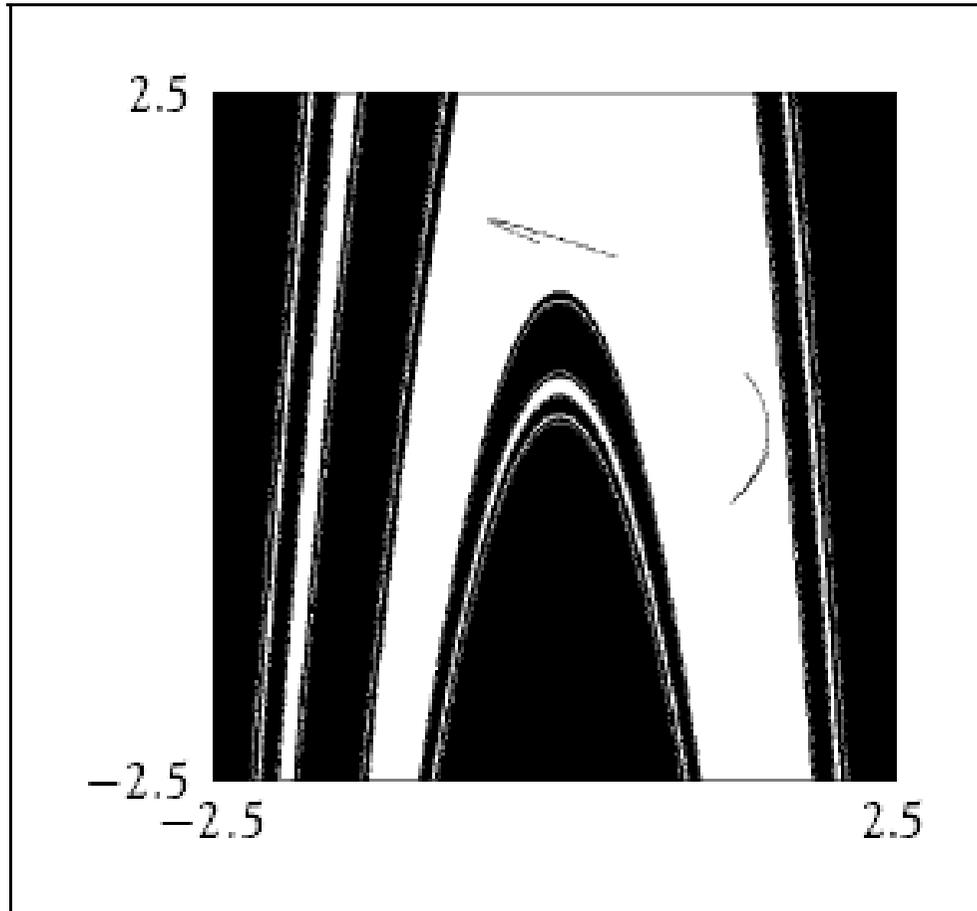


Figure 2.11 Attractors for the Hénon map with $a = 2$, $b = -0.3$.

Initial values in the white region are attracted to the hairpin attractor inside the white region. On each iteration, the points on one piece of the attractor map to the other piece. Orbits from initial values in the black region diverge to infinity.

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5 - Mapas Lineares

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix} \quad \vec{V} \equiv \begin{pmatrix} x \\ y \end{pmatrix}$$

Definição: A é linear se $A(a\vec{v} + b\vec{w}) = aA(\vec{v}) + bA(\vec{w})$

λ é um auto-valor da matriz A se (para $\vec{v} \neq 0$)

$$A\vec{v} = \lambda\vec{v}$$

$$\vec{v}_{n+1} = A\vec{v}_n \Rightarrow \vec{v}_{n+1} = \lambda^{n+1}\vec{v}_0$$

Exemplo

Para a matriz $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a \text{ é auto - valor e } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ auto - vetor}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow b \text{ é auto - valor e } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ auto - vetor}$$

Expansão e Contração no Mapeamento de Um Disco de Pontos Iniciais

Para n iterações $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$$\begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a^n \\ 0 \end{pmatrix} = a^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ b^n \end{pmatrix} = b^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pontos iniciais em um disco de raio unitário são mapeados em uma elipse com semi-eixos $|a^n|$ na direção x e $|b^n|$ em y .

Um disco de raio ε , i. e.,

na vizinhança $N_\varepsilon(0,0)$ do ponto $\vec{P}(0,0)$,

torná-se uma elipse com semi-eixos $\varepsilon|a^n|$ e $\varepsilon|b^n|$

$|a| > 1 \Rightarrow$ repulsão na direção x

$|b| < 1 \Rightarrow$ atração na direção y

Exemplo: $A = \begin{pmatrix} 2 & 0 \\ 0 & 0,5 \end{pmatrix}$ e $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2x_0 \\ 0,5y_0 \end{pmatrix}; A^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 4x_0 \\ 0,25y_0 \end{pmatrix}$$

Em geral, $A(\vec{v}) = |\vec{v}| A\left(\frac{\vec{v}}{|\vec{v}|}\right)$

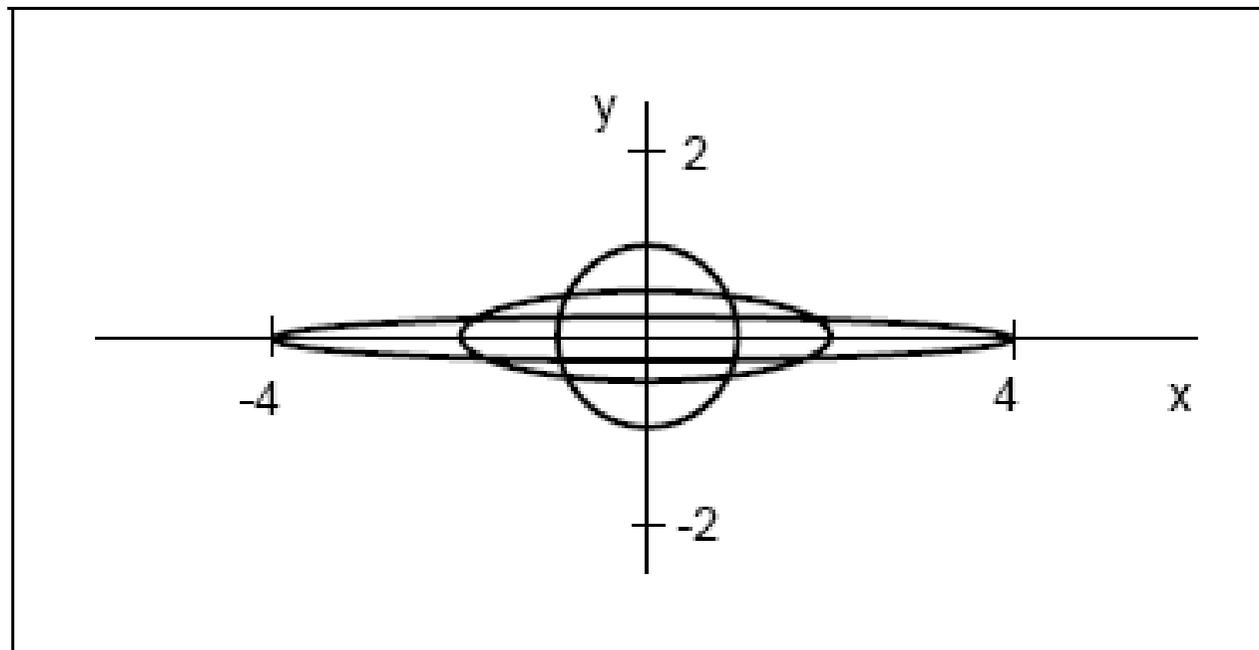


Figure 2.12 The unit disk and two images of the unit disk under a linear map.

The origin is a saddle fixed point.

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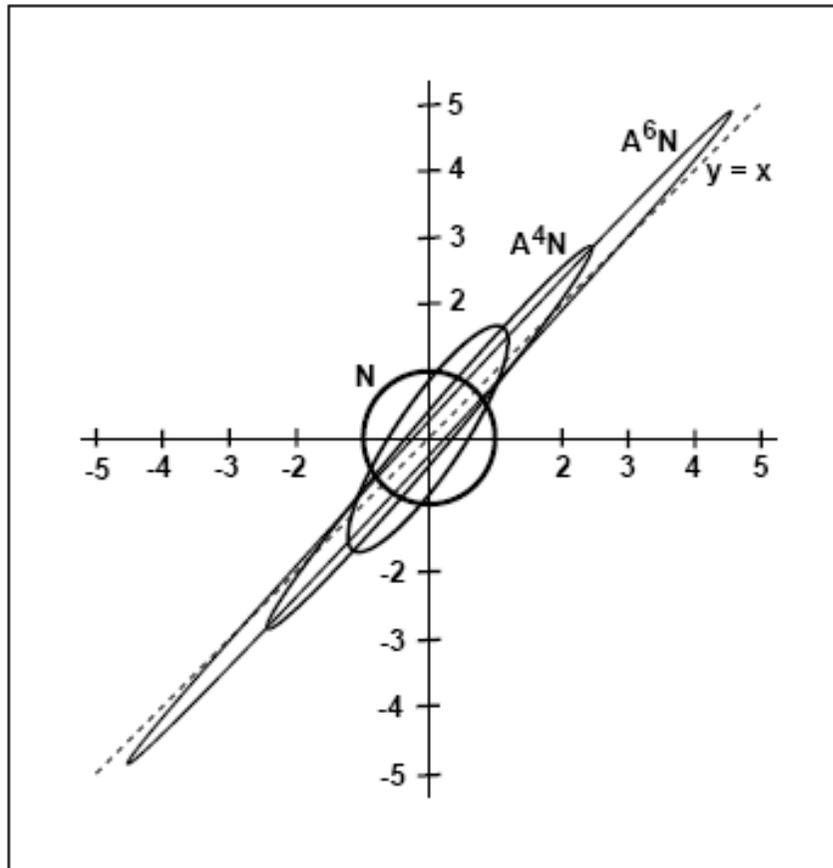


Figure 2.25 Successive images of the unit circle N for a saddle fixed point. The image of a circle under a linear map is an ellipse. Successive images are therefore also ellipses, which in this example line up along the expanding eigenspace.

Exemplo: $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \Rightarrow$ ponto iterado gira de θ e dilata de r

Auto - valores $a \pm ib$ auto - vetores $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$

Transformação $a = r \cos \theta$, $b = r \sin \theta$

$$r^2 = a^2 + b^2, \quad \theta = \arctg \frac{a}{b}$$

$$A = r \begin{pmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{pmatrix} = \sqrt{a^2 + b^2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$A \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = r \begin{pmatrix} x_0 \cos \theta \\ x_0 \sin \theta \end{pmatrix}$$

Rotação de θ e dilatação (contração) de r

7 – Matriz Jacobiana

Em uma dimensão, com um ponto fixo $p = f(p)$,
 $f(p + h) \cong f(p) + h f'(p)$

$|f'(p)| < 1 \Rightarrow p$ é um atrator

$|f'(p)| > 1 \Rightarrow p$ é um repulsor

Em duas dimensões, com um ponto fixo $\vec{p} = \vec{f}(\vec{p})$

$$\vec{f}(\vec{p} + \vec{h}) \cong \vec{f}(\vec{p}) + \vec{D}\vec{f}(\vec{p}) \cdot \vec{h} = \vec{p} + \vec{D}\vec{f}(\vec{p}) \cdot \vec{h}$$

Teorema :

1- Se os módulos dos auto-valores da matriz $\vec{D}\vec{f}(\vec{p})$ forem menores que 1, \vec{p} é um atrator.

2- Se os módulos dos auto-valores da matriz $\vec{D}\vec{f}(\vec{p})$ forem maiores que 1, \vec{p} é um repulsor.

Definição : Se um auto-valor for maior que 1 e o outro menor que 1, \vec{p} é um ponto de sela.

Em uma dimensão, $f^2(p_0)' = f(p_0)' f(p_1)'$

Em duas dimensões, $D\vec{f}^2(\vec{p}_0) = D\vec{f}(\vec{p}_0) D\vec{f}(\vec{p}_1)$

Exemplo: Mapa de Hénon $f(x, y) = (a - x^2 + b y, x)$ $a = 0$ e $b = -0,4$

Ponto fixo $(0, 0)$

$$J = \begin{pmatrix} \partial_x (a - x^2 + b y) & \partial_y (a - x^2 + b y) \\ \partial_x x & \partial_y x \end{pmatrix} = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0,4 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 - \lambda & b \\ 1 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow |\lambda| = |\pm \sqrt{0,4}| < 1 \Rightarrow \text{atrator}$$

Ponto fixo $(-0,6, -0,6)$

$$J = \begin{pmatrix} \partial_x (a - x^2 + b y) & \partial_y (a - x^2 + b y) \\ \partial_x x & \partial_y x \end{pmatrix} = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1,2 & 0,4 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1,2 - \lambda & b \\ 1 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow |\lambda_1| = |1,472| > 1 ; |\lambda_2| = |-0,272| < 1 \Rightarrow \text{ponto de sela}$$

Esqema Dinâmico de um Ponto de Sela

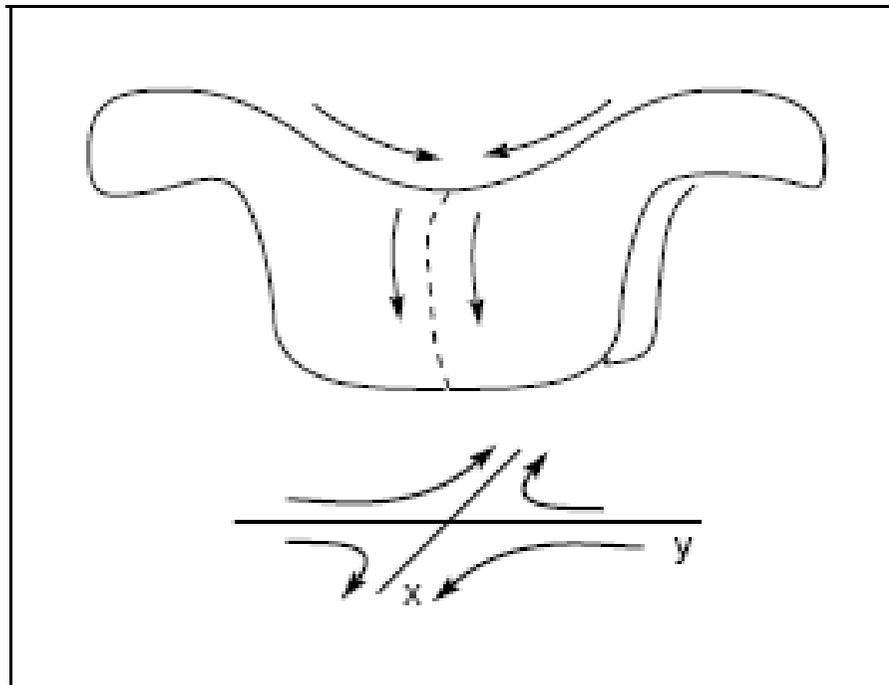


Figure 2.13 Dynamics near a saddle point.

Points in the vicinity of a saddle fixed point (here the origin in the xy -plane) move as if responding to the influence of gravity on a saddle.

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Órbitas Próximas de Um Ponto de Sela

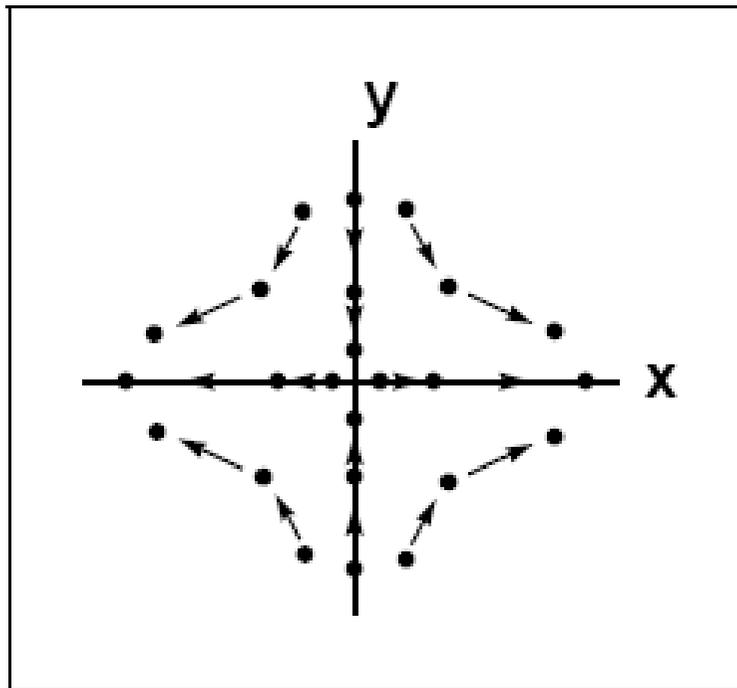


Figure 2.14 Saddle dynamics.

Successive images of points near a saddle fixed point are shown.

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Exemplo: Mapa de Hénon $f(x, y) = (a - x^2 + by, x)$ $a=0,43$ $b=0,4$

Órbita de período 2: $(0.7, -0,1) \Leftrightarrow (-0,1, 0.7)$

$$J^2 = JJ = \begin{pmatrix} -2(-0.1) & 0.4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2(0.7) & 0.4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0.12 & 0.008 \\ -1.4 & 0.4 \end{pmatrix}$$

$$\begin{vmatrix} 0.12 - \lambda & 0.008 \\ -1.4 & 0.4 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0.26 \pm i 0.30 \Rightarrow |\lambda| = 0.40 < 1 \text{ atrator}$$

Mudança de Atratores do Mapa de Hénon

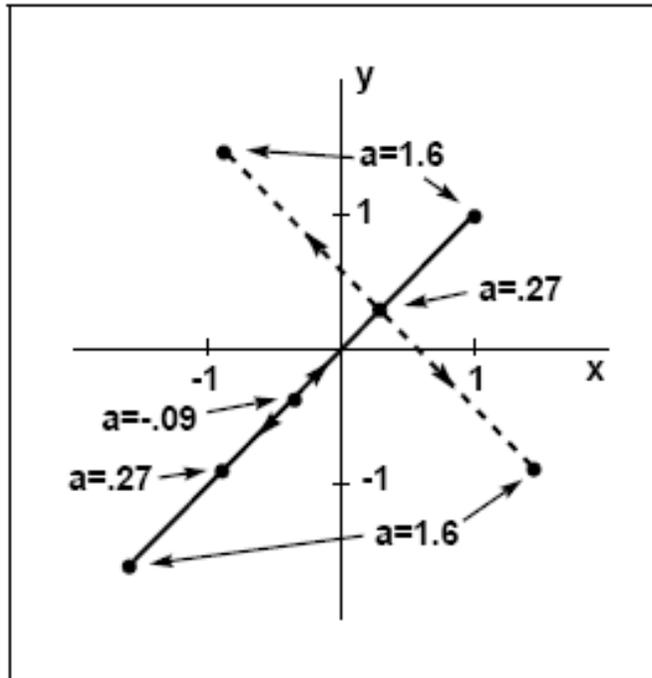


Figure 2.15 Fixed points and period-two points for the Hénon map with b fixed at 0.4.

The solid line denotes the trails of the two fixed points as a moves from -0.09 , where the two fixed points are created together, to 1.6 where they have moved quite far apart. The fixed point that moves diagonally upward is attracting for $-0.09 < a < 0.27$; the other is a saddle. The dashed line follows the period-two orbit from its creation when $a = 0.27$, at the site of the (previously) attracting fixed point, to $a = 1.6$.

Diagrama de Bifurcação do Mapa de Hénon

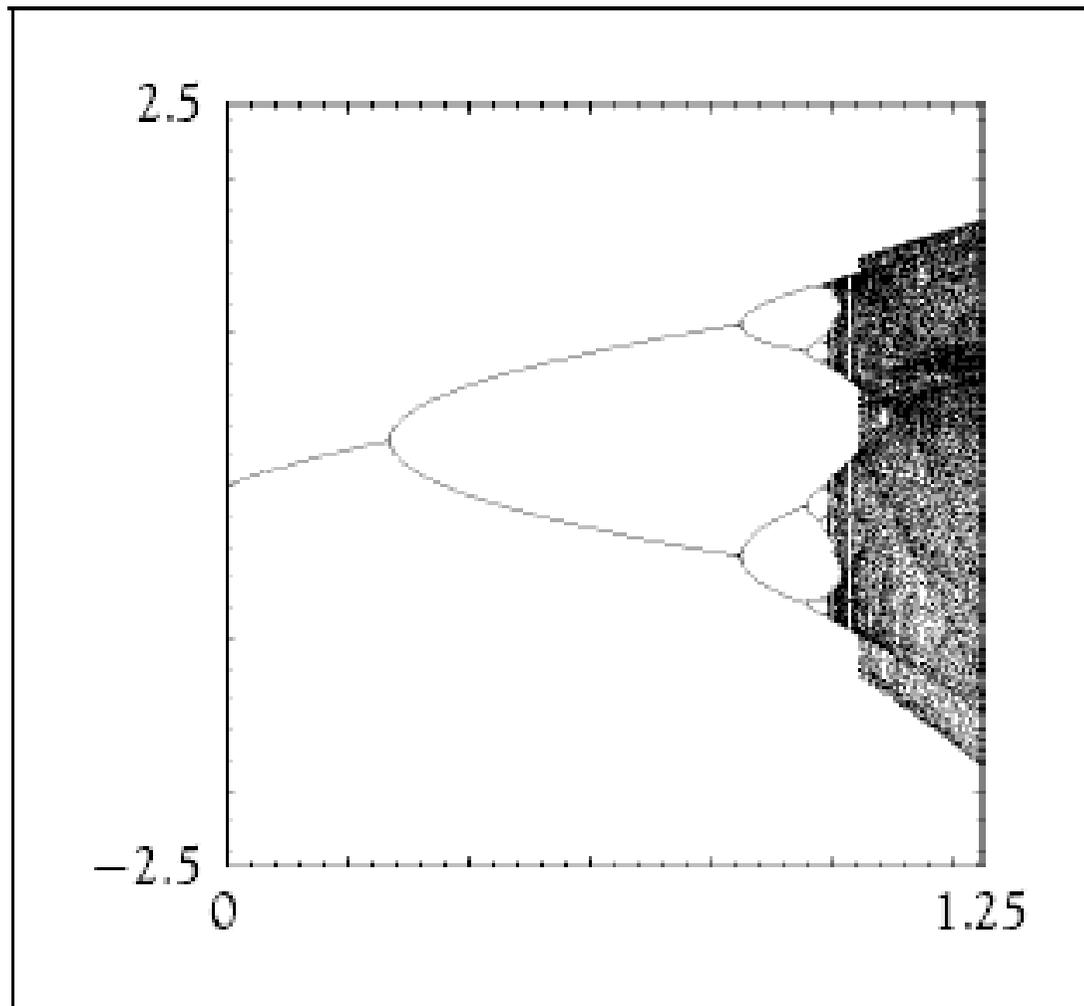
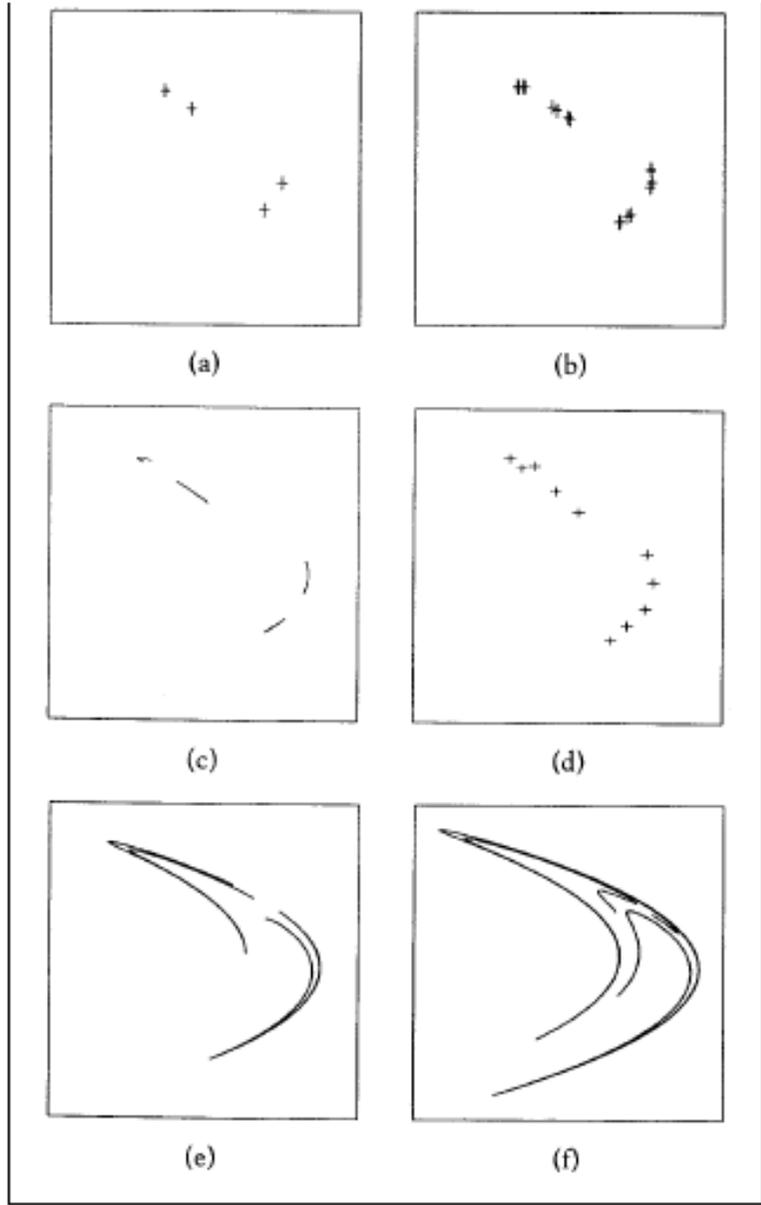


Figure 2.16 Bifurcation diagram for the Hénon map, $b = 0.4$.

Each vertical slice shows the projection onto the x -axis of an attractor for the map for a fixed value of the parameter a .

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Mudança de atratores com a , para $b=0.4$

Figure 2.17 Attractors for the Hénon map with $b = 0.4$. Each panel displays a single attracting orbit for a particular value of the parameter a . (a) $a = 0.9$, period 4 sink. (b) $a = 0.988$, period 16 sink. (c) $a = 1.0$, four-piece attractor. (d) $a = 1.0293$, period-ten sink. (e) $a = 1.045$, two-piece attractor. The points of an orbit alternate between the pieces. (f) $a = 1.2$, two pieces have merged to form one-piece attractor.

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