



## Synchronization of coupled neurons in a master-slave configuration

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**Abstract.** This paper presents bursting synchronization for unidirectional coupled Hindmarsh-Rose neurons according to a master-slave configuration. The synchronisation can be associated with some pathological conditions, such as Parkinson's disease and epilepsy. We verify bursting synchronisation for a wide interval of the coupling strength. Moreover, we show results for the Kolmogorov-Sinai entropy. The entropy decreases for strong coupling when the neurons are synchronised.

### 1 Introduction

The neuron is a cell of the neuronal system that processes information through action potentials. The action potentials are variations in the neurons membrane potential generated by external stimuli. Such variations are spikes and bursts followed by a rest period [1]. The studies about potential and spike/bursts patterns of neurons are important to understand the mechanism of neuronal coding [2]. The electrical activity of the action potential is supported and propagated by ionic currents such as Sodium ( $\text{Na}^+$ ), Potassium ( $\text{K}^+$ ), Calcium ( $\text{Ca}^+$ ), and Chlorine ( $\text{Cl}^+$ ), which cross the neurons membrane according to the electrochemical gradients [3].

To reproduce the neuronal dynamics we used the Hindmarsh-Rose model [4]. This model was developed in order to study the synchronisation between spikes of the mollusc *Lymnaea stagnalis* [5]. The model is described by a set of three first-order differential equations in which the state variable

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shows a succession of states (spikes or bursts), rest periods, and interaction between time and space [6].

One of the goals of this work is to study the synchronisation of neurons. Synchronisation is one of the central mechanisms in processing neuronal information within an area of the brain, as well as for communication between different brain areas. Pecora and Carroll [7] found the necessary conditions for two coupled chaotic systems to display stable synchronisation. From this finding, many works on synchronisation of dynamic systems were published. For the proposed system of neurons, we evaluate the regions that show the occurrence of chaotic behaviour according to the current intensity parameter. The conditions for the synchronisation of this system are verified as a function of the coupling term in the equation that represents the slave neuron.

With regarding to synchronisation, the tools for characterising chaos and estimating the change in dynamics of a neuron are the Lyapunov exponents [8], bifurcation diagram [9], and Kolmogorov-Sinai entropy (KS) [9]. The Lyapunov exponents are related to the contraction or expansion ratio of a small volume of the phase space, whereas the Kolmogorov-Sinai entropy is delimited by the sum of the positive Lyapunov exponents and allows to quantify the average rate of production of information in the system or the average increase in uncertainty due to small disturbances.

This work is organised as follows: in Section 2, we present the methodology, Section 3 describes the biological neuron and its behaviour, in Section 4, we show the Hindmarsh-Rose model for the isolated neuron and for coupled neurons, Section 5 exhibits the tools and results. Finally, the conclusions are shown in Section 6.

## 2 Methodology

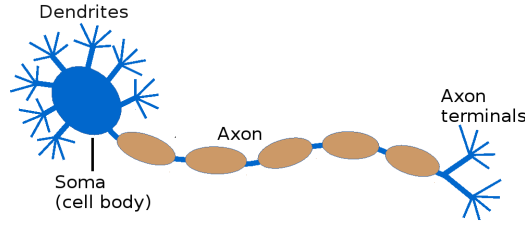
For the development of this work, we use computational tools and parameter values collected from the literature. In our simulations, we use Language C [11]. We choose the Hindmarsh-Rose model [4] to mimic the neuronal dynamics. We perform numerical simulations with the intention of representing the dynamic behaviour of isolated neurons, and later two coupled neurons through the master-slave formalism, an unidirectional type of coupling [7].

We consider different external currents and coupling parameter intensities to establish comparisons with previously published studies, and to identify the efficiency of our routines. In this work, the current intensity  $I$  and the coupling  $\varepsilon$  are the control parameters. We perform analyses of the synchronisation of neuronal spikes [12], spectrum of Lyapunov exponents [8], bifurcation diagram and Kolmogorov-Sinai entropy [10]. By means of graphic platforms<sup>1</sup> we show the generated patterns for the some parameters and discuss the neuronal model and its behaviour.

## 3 Neuron

The neuron, a cell of the nervous system, works with other neurons and cells called glial [13]. The neuron has a plasma membrane that surrounds a cytoplasm containing organelles with different functions [2]. The difference of the neuron in relation to the other cells is its adapted morphology for the information processing [14]. In Fig. 1, we observe the basic structure of a neuron: Soma (cell body), dendrites, axon, and axon terminals [15]. The axon is responsible for the conduction of electric impulses. The dendrites and axon terminals act as terminals for the reception and transmission of nerve stimuli, respectively [16].

<sup>1</sup> xmgrace, Matlab, and Origin.



**Fig. 1** Schematic representation of the basic structure of a neuron.

A neuron exhibits complex dynamics as a result of nonlinear interactions in cellular processes [17]. When the neurons are coupled, a type of collective dynamics can arise and it is known as synchronisation [18]. By means of mathematical and computational tools it is possible to simulate the action potential of a neuron [1] or even a set of them. In section (4), Fig. 2 displays some of the possible behaviours of the action potential when the neuronal membrane dynamics is simulated by the Hindmarsh-Rose model [? ].

Neurons can be coupled to each other or to other cells of the nervous system [19], the contact zones of neurons with other neurons are called synapses. The synapses can be electrical or chemical [20]. In chemical synapses, impulse transmission involves the release of neurotransmitters or chemical mediators from presynaptic cells to postsynaptic cells [21]. In the electrical synapses, the impulse passes directly from one cell to another [22]. Synapses allow the propagation of information, in the form of a pulse of electric potential, through neurons for the purpose of performing functions [14, 23].

## 4 Hindmarsh-Rose model

### 4.1 The neuron model

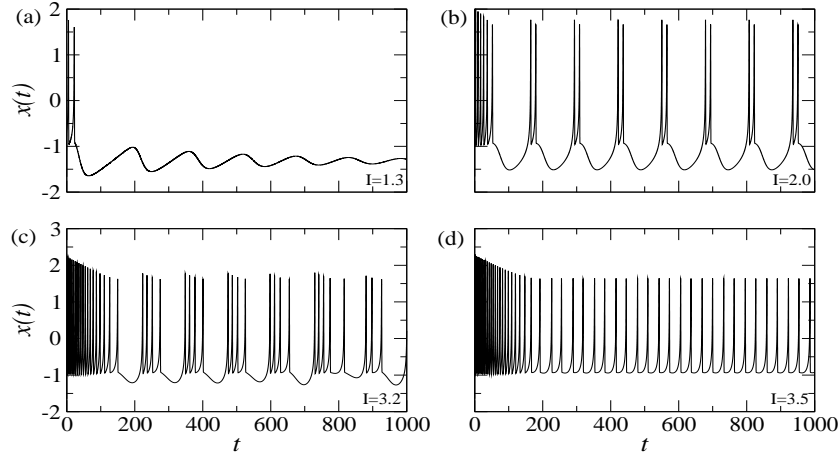
The HR model [? ] was developed from a modification of the equations of FitzHugh and Nagumo (FN) [24, 25]. The FN model is derived from simplifications in the neuronal equations of Hodgkin and Huxley [26]. The researchers Hindmarsh and Rose refined the model (FN) in order to obtain chaotic firings [4]. This modification generated bursts separated by time intervals [27]. The model is given by

$$\begin{aligned} \frac{dx}{dt} &= y - ax^3 + bx^2 - z + I, \\ \frac{dy}{dt} &= c - dx^2 - y, \\ \frac{dz}{dt} &= r(s(x - x_0) - z), \end{aligned} \quad (4.1)$$

where  $x$  is the membrane potential and  $x(t) = \frac{dx}{dt}$  represents the variation of the potential in time.  $y$  and  $z$  are the ion exchange by means of slow and slow transport channels, respectively. The parameters  $a$ ,  $b$ ,  $c$ , and  $d$  control the behaviour and frequency of neuronal firing [28].  $I$  is the intensity of the current injected into the system. The parameter  $r$  controls the velocity of variation of  $z$ , while the parameter  $s$  governs the adaptation intensity and  $x_0$  defines the resting potential.

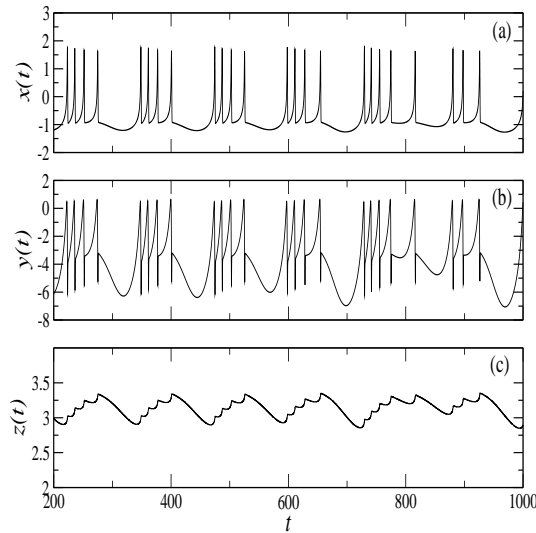
Figure 2 exhibits the temporal evolution of the membrane potential for  $a = 1$ ,  $b = 3$ ,  $c = 1$ ,  $d = 5$ ,  $r = 0.006$ ,  $s = 4$ , and  $x_0 = -1.6$  [4]). We observe that the increase in  $I$  modifies the structure of the spikes. In Fig. 2(a) we observe a different firing pattern when compared to Fig. 2(b). Figure 2(c)

displays an irregular firing pattern. Therefore,  $I$  modifies the dynamic behaviour of the HR model. In Fig. 2(d), the spikes describe a periodic behaviour.



**Fig. 2** Temporal evolution of the membrane potential. We consider  $a = 1, b = 3, c = 1, d = 5, r = 0.006, s = 4,$  and  $x_0 = -1.6$  for (a)  $I = 1.3,$  (b)  $I = 2.0,$  (c)  $I = 3.2,$  and (d)  $I = 3.5.$

Figure 3 shows the temporary evolution of (a)  $x,$  (b)  $y,$  and (c)  $z$  for  $I = 3.2.$  The membrane potential is represented in 3(a). Figures 3(b) and 3(c) exhibit the fast and slow transport channels, respectively. The spike starts approximately when a change in the slope of  $z$  occurs,  $\frac{dz}{dt} < 0$  becomes  $\frac{dz}{dt} > 0.$



**Fig. 3** Time evolution of the HR model. The membrane potential is represented in (a). In (b) and (c) we show the fast and slow transport channels, respectively. We consider  $a = 1, b = 3, c = 1, d = 5, r = 0.006, s = 4, x_0 = -1.6,$  and  $I = 3.2.$

## 4.2 Two coupled neurons

We study the dynamic behaviour of two neurons coupled by means of a master-slave configuration. In 1990, Pecora and Carroll [7] presented a form of unidirectional coupling, the master-slave. When the coupling is unidirectional, the dynamics of the slave system is altered by the influence of the master system. A representation of the unidirectional coupling is illustrated in Fig. 4.

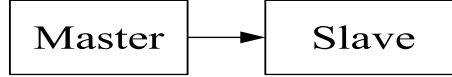


Fig. 4 Schematic representation of a master-slave coupling.

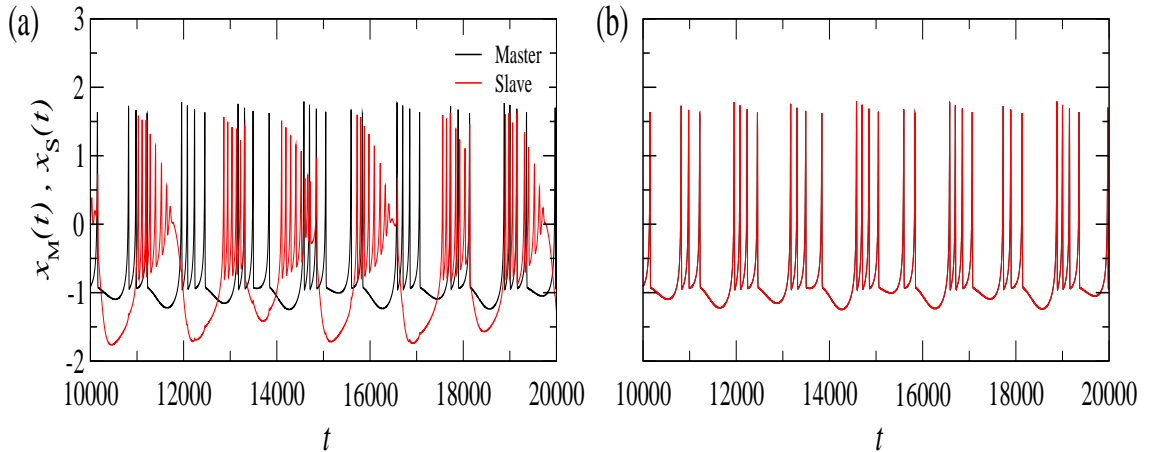
Two neurons connected via a one-way coupling in the master-slave configuration can be expressed mathematically through two sets of Eqs. (4.1), where M and S identify the master and slave neurons, respectively,

$$\begin{aligned}
 \frac{dx_M}{dt} &= y_M - ax_M^3 + bx_M^2 - z_M + I_M, \\
 \frac{dy_M}{dt} &= c - dx_M^2 - y_M, \\
 \frac{dz_M}{dt} &= r(s(x_M - x_0) - z_M).
 \end{aligned} \tag{4.2}$$

$$\begin{aligned}
 \frac{dx_S}{dt} &= y_S - ax_S^3 + bx_S^2 - z_S + I_S + \varepsilon(x_M - x_S), \\
 \frac{dy_S}{dt} &= c - dx_S^2 - y_S, \\
 \frac{dz_S}{dt} &= r(s(x_S - x_0) - z_S).
 \end{aligned} \tag{4.3}$$

$\varepsilon$  is a number in the interval  $[0, 1]$  that represents the coupling strength. The main term in the sets of equations is  $\frac{dx}{dt}$  of the slave neuron. In this configuration, the master neuron influences the slave neuron. The uncoupled neurons have a chaotic behaviour for  $I = 3.2$  [4, 27].

In Fig. 5, we show the behaviour of the membrane potential of the master neuron ( $x_M(t)$  in black) and the slave neuron ( $x_S(t)$  in red) for two different values of  $\varepsilon$ . In Fig. 5(a), we observe that there is no synchronisation between the master ( $x_M(t)$ ) and the slave ( $x_S(t)$ ) neurons for  $\varepsilon = 0.2$ . For  $\varepsilon = 0.95$  both neurons in the master-slave configuration have synchronised spikes, as shown in Fig. 5(b). Figure 5(a) exhibits state variables that do not overlap as occur in 5(b). The synchronous behaviour occurs due to an increase of  $\varepsilon$ . The synchronisation is achieved as a response of the action of an external force. The master neuron adjusts the slave neuron through the coupling term.



**Fig. 5** Action potentials of the master ( $x_M(t)$  in black) and the slave ( $x_S(t)$  in red) neurons for  $I = 3.2$ , (a)  $\varepsilon = 0.2$ , (b)  $\varepsilon = 0.95$ .

## 5 Results and discussions

### 5.1 Synchronisation

Synchronisation is the phenomenon that shows the correlation of the dynamics of different systems when they present some form of interaction [29], being observed in natural systems [30] and defined as an adjustment of oscillator rhythms<sup>2</sup> due to a interaction between they [7, 29, 30].

In the literature, it is found theories that describe the mechanisms in networks of neurons, such as full synchronisation [12], phase synchronisation [29] and generalised synchronisation [3]. In full synchronisation, due to strong interaction between two or a number  $N$  of identical oscillators, the states can coincide. It is considered a neuron as a neuronal oscillator with identical or unlike characteristics. By identical oscillators, it is understood that they present the same dynamic and equal set of parameters.

Chaotic synchronisation in neuronal networks is a subject that has generated several discussions in neuroscience. Investigations have shown that the synchronisation is related to Parkinson's disease, epilepsy and autism [33]. The dynamics of a system is chaotic if it is aperiodic, limited in space and sensitive to initial conditions [34]. The complete synchronisation occurs due to the strong interaction between  $N \geq 2$  ( $N$  can be understood as the number of neurons) systems with the same dynamics and equal sets of parameters. In particular, in this work, we show spike synchronisation of neurons, when the neurons have simultaneous firings, as shown in Fig. 5(b).

### 5.2 Lyapunov exponents

The dynamic behaviour in the HR model (Fig. 3 and Fig. 5) can depend on the initial conditions. In a dynamic system, when a very small difference between two initial conditions yields distinct results, the system is considered to be chaotic. One way to identify chaotic behaviour is by calculating the Lyapunov exponents. The Lyapunov exponents give the rate of the average with which the trajectories approach or move away in phase space. When the evolution of two very close trajectories, for a

<sup>2</sup> From classical mechanics: a harmonic oscillator corresponds to a system that, when removed from its position of equilibrium, presents a restoring force proportional to the displacement in space [31, 32].

given set of parameters, are aperiodic and diverge exponentially, the system is chaotic. The Lyapunov exponent of the one-dimensional network composed of  $N$  elements can be found by means of the Eq. 5.1, where we calculate the logarithms of the eigenvalues of the matrix product of  $N$  Jacobian matrices,

$$\lambda_k = \lim_{t \rightarrow \infty} \frac{1}{t} |DF^t \cdot v_k|, \quad (5.1)$$

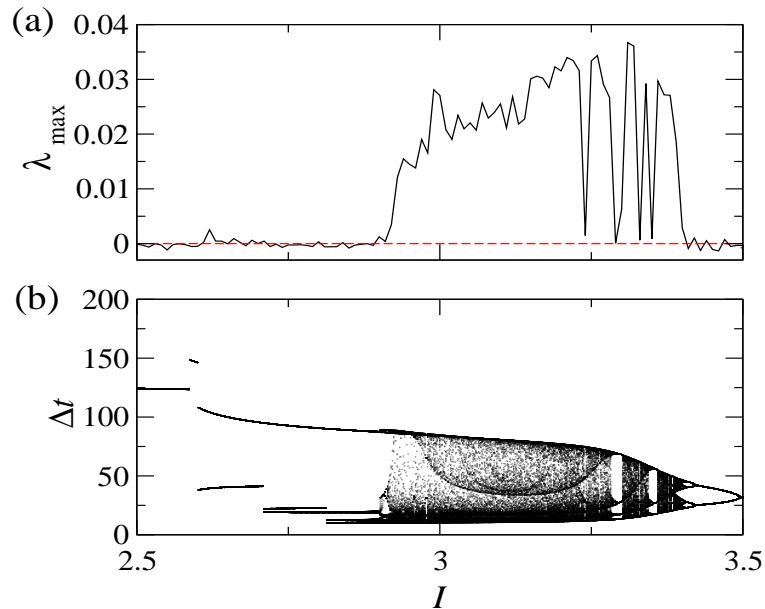
where  $\lambda_k$  are the values that the Lyapunov exponents can assume of  $k = 1, 2, \dots, N$ ,  $DF^t$  is the Jacobian matrix

$$\begin{bmatrix} -3ax_1^2 + 2bx_1 & 1 & -1 & 0 & 0 & 0 \\ -2dx_1 & -1 & 0 & 0 & 0 & 0 \\ rs & 0 & -r & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & -3ax_2^2 + 2bx_2 - \varepsilon & 1 & -1 \\ 0 & 0 & 0 & -2dx_2 & -1 & 0 \\ 0 & 0 & 0 & rs & 0 & -r \end{bmatrix}, \quad (5.2)$$

at the  $th$  instant of time and  $v$  corresponds the direction tangent to the trajectory in the phase space.

For a system with  $N$  equations, the spectrum of Lyapunov exponents is formed by  $N$  exponents, sorted in decreasing order,  $\lambda_1 > \lambda_2 > \dots > \lambda_N$ . One or more positive Lyapunov exponents indicate that the dynamics of the system is chaotic.  $\lambda_k > 0$  implies a chaotic attractor [8]. The Lyapunov exponents characterise the stability of the solutions of the studied model. Stability is an important feature in the sense of the ability of the model to return to the state of equilibrium. Figure 6(a) shows the maximum Lyapunov exponent for different values of  $I$ .

The structure of intervals between neuronal spikes is a form in which neurons can encode information [9]. We compute the bifurcation diagram showing the intervals between consecutive spikes given by  $\Delta = t_{n+1} - t_n$  as a function of current  $I$ . The bifurcation diagram provides information about the system according to the variation of  $I$ , as shown in Fig. 6(b).



**Fig. 6** (a) The largest exponent of the Lyapunov as a function of  $I$ . (b) Bifurcation diagram of the interval between spikes as a function of  $I$ .

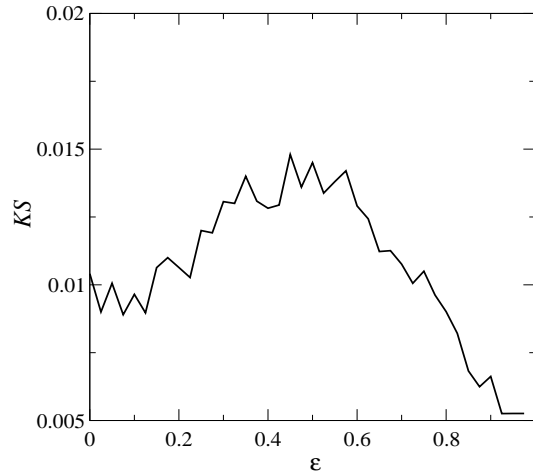
### 5.3 Kolmogorov-Sinai entropy

Through the Lyapunov exponents we calculate the Kolmogorov-Sinai entropy (KS) [10]. In Eq. (5.3), we present the definition of the entropy that can be understood as the rate which the information of the initial conditions is lost. For chaotic systems the entropy assumes positive values, while for non-chaotic systems it assumes a value equal to zero [36].

$$KS = \frac{1}{n} \sum_{i=1}^{i=n} \lambda_i, \quad (5.3)$$

where  $\lambda_i > 0, \forall_i$ . The Wolf algorithm [37] is one of the forms used to calculate numerically the Lyapunov exponents of a set of differential equations.

We calculate the KS entropy as a function of  $\varepsilon$  in the interval  $[0 \leq \varepsilon \leq 1]$  for  $I = 3.2$ . There is a value  $\varepsilon = 0.45$  for which the KS entropy reaches a maximum value (0.0148). We observe in Fig. 7 that by increasing the coupling intensity, the KS entropy oscillates around 0.0125 and in the sequence becomes smaller and close to zero, indicating that the system goes to a regime of chaos synchronisation. Chaos synchronisation is identified by the second largest Lyapunov exponent equal to zero (transverse direction of synchronisation) [38].



**Fig. 7** The KS entropy as a function of  $\varepsilon$  for  $I = 3.2$ .

## 6 Conclusions

We investigated the synchronisation phenomenon of two Hindmarsh-Rose neurons coupled according to a master-slave configuration. We verified that the synchronisation of the spikes depends on the coupling intensity values. Chaos synchronisation occurs when the system has a strong coupling intensity. In addition, we calculated the Kolmogorov-Sinai entropy (KS). This entropy decreases when the intensity of the coupling increases, showing that it can also be used as a diagnosis of chaos synchronisation. The results presented are important not only for understanding the dynamics of neurons unidirectionally coupled, but also for the study of synchronisation of chaos.



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