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Amplitude Instability and Ergodic Behavior for Conservative Nonlinear Oscillator Systems*

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Sistema Quase Integrável

$$H = H_0 + V$$

H quase integrável

$$H = H_0(J_1, J_2) + V(J_1, J_2, \varphi_1, \varphi_2)$$

Variáveis de ação/ângulo

$$H = H_0(J_1, J_2) + f_{mn}(J_1, J_2)$$

Expansão de V em
série de Fourier

$$\times \cos(m\varphi_1 + n\varphi_2) + \cdots,$$

Teoria da Perturbação

tori. As illustration, let us seek to eliminate the explicit angle-dependent term in Hamiltonian (3) by introducing the canonical transformation generated[12] by

$$F = \mathcal{J}_1\varphi_1 + \mathcal{J}_2\varphi_2 + B_{mn}(\mathcal{J}_1, \mathcal{J}_2)\sin(m\varphi_1 + n\varphi_2), \tag{4}$$

where $(\mathcal{J}_i, \theta_i)$ are the transformed action-angle variables and $B_{mn}(\mathcal{J}_1, \mathcal{J}_2)$ is to be determined. We note that if $B_{mn} = 0$, we have the identity transformation $J_i = \mathcal{J}_i$ and $\varphi_i = \theta_i$.

Introducing the canonical transformation generated by Eq. (4) into Hamiltonian (3), we obtain

$$H = H_0(\mathcal{J}_1, \mathcal{J}_2) + \{[m\omega_1(\mathcal{J}_1, \mathcal{J}_2) + n\omega_2(\mathcal{J}_1, \mathcal{J}_2)] \times B_{mn}(\mathcal{J}_1, \mathcal{J}_2) + f_{mn}(\mathcal{J}_1, \mathcal{J}_2)\} \cos(m\theta_1 + n\theta_2) + \cdots, \tag{5}$$

where $\omega_i(\mathcal{J}_1, \mathcal{J}_2) = \partial H_0(\mathcal{J}_1, \mathcal{J}_2)/\partial \mathcal{J}_i$ and where we have explicitly retained only the lowest-order terms. We may now eliminate the given angle-dependent term, provided we set

$$B_{mn}(\mathcal{J}_1, \mathcal{J}_2) = -\frac{f_{mn}(\mathcal{J}_1, \mathcal{J}_2)}{m\omega_1(\mathcal{J}_1, \mathcal{J}_2) + n\omega_2(\mathcal{J}_1, \mathcal{J}_2)}, \tag{6}$$

A transformação canônica existe com a condição

$$|m\omega_1(J_1, J_2) + n\omega_2(J_1, J_2)| \ll |f_{mn}(J_1, J_2)|$$

Ressâncias Isoladas

$$H = H_0(J_1, J_2) + f_{mn}(J_1, J_2)\cos(m\varphi_1 + n\varphi_2)$$

$I = nJ_1 - mJ_2$ Segunda constante de movimento

$$H_0 = J_1 + J_2 - J_1^2 - 3J_1 J_2 + J_2^2$$

Hamiltoniana não perturbada

Coordenadas cartesianas

$$q_i = (2J_i)^{1/2} \cos\varphi_i$$

$$p_i = -(2J_i)^{1/2} \sin\varphi_i$$

$$J_i = \tfrac{1}{2}(p_i^2 + q_i^2)$$

J constante $\rightarrow$ círculos no plano pxq

$$\omega_1 = 1 - 2J_1 - 3J_2,$$

$$\omega_2 = 1 - 3J_1 + 2J_2$$

Frequências

$$0 \leq E \leq \frac{3}{13}$$

Thus the unperturbed level curves in the (q_2, p_2) plane, hereafter called the J_2 plane, are concentric circles centered on the origin since J_2 is a constant. Similarly points on the level curves in the (q_1, p_1) plane or J_1 plane, defined by $q_2 = 0$, $p_2 \geqslant 0$ (or equivalently $\varphi_2 = \frac{1}{2} 3\pi$), also lie on concentric circles. These circular level curves in

Ressonância 2/2

$$H = H_0(J_1, J_2) + \alpha J_1 J_2 \cos(2\varphi_1 - 2\varphi_2)$$

$$I = J_1 + J_2$$

H, I constantes de movimento

H –integrável

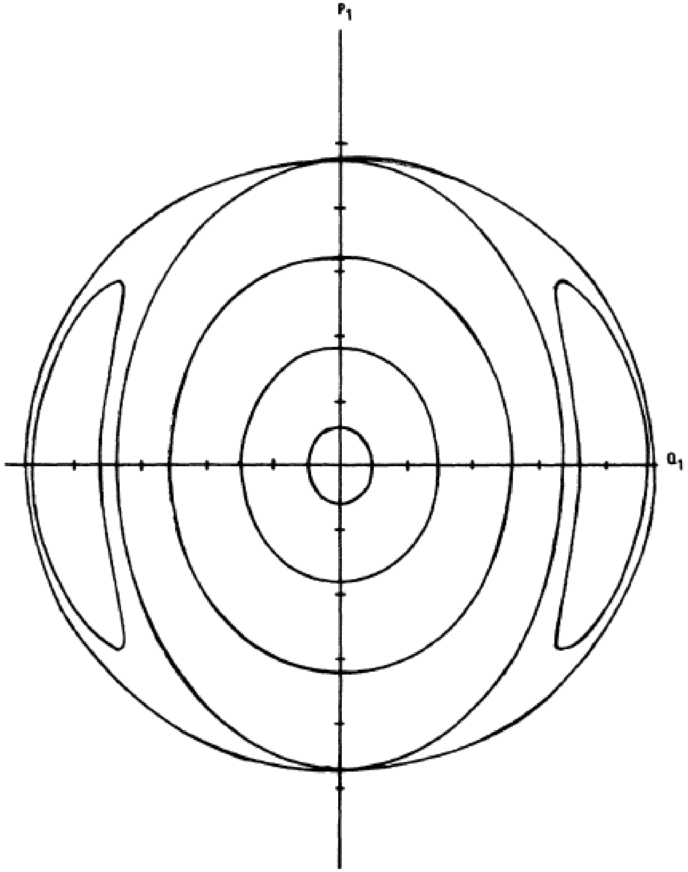

FIG. 6. Typical level curves for an isolated, 2–2 resonance computed algebraically.

Curvas de nível com I cte.

Ilhas surgem para E=0.
Aumentando E, ilhas se afastam do centro e aumentam sua largura.

allowed energies $0 \leq E \leq \frac{3}{13}$. As the energy increases from zero, the 2-2 resonance zone moves out from the origin and increases in width.

Ressonância 2/3

$$H = H_0(J_1, J_2) + \beta J_1 J_2^{3/2} \cos(2\varphi_1 - 3\varphi_2)$$

H - integrável

$$I = 3J_1 + 2J_2$$

H, I constantes de movimento

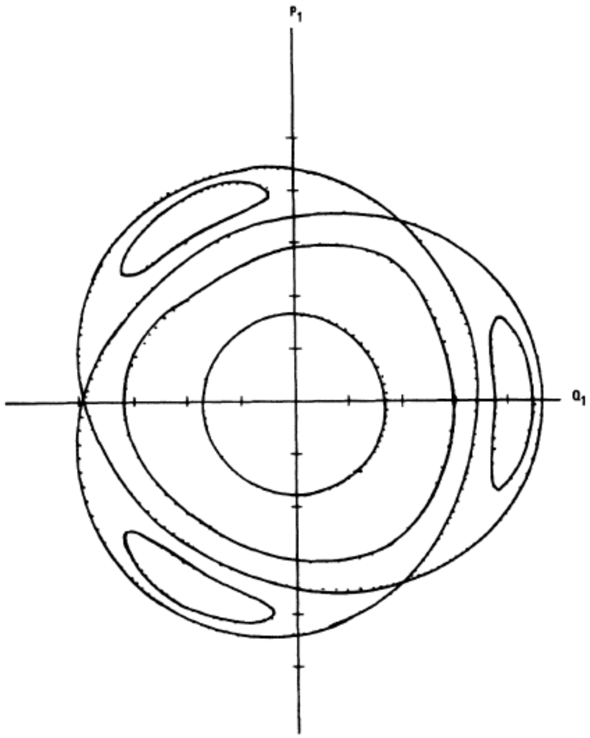

FIG. 7. Typical level curves for an isolated, 3–2 resonance computed algebraically. The dots represent points computed using Eq. (25); the curves were drawn in freehand. The chain of three islands first appears at the origin for $E = 0.08$. All the widths of the islands including this one increase with increasing energy.

$$0{,}16 \leq E$$

Curvas de nível com I cte.

Ilhas surgem para E=0,16.
Aumentando E, ilhas se afastam do centro e aumentam sua largura.

Ressonância Dupla
2/2 e 2/3

$$H = H_0(J_1, J_2) + \alpha J_1 J_2 \cos(2\varphi_1 - 2\varphi_2)$$
$$+ \beta J_1 J_2^{3/2} \cos(2\varphi_1 - 3\varphi_2) \, ,$$

$$\alpha = \beta = 0.02$$

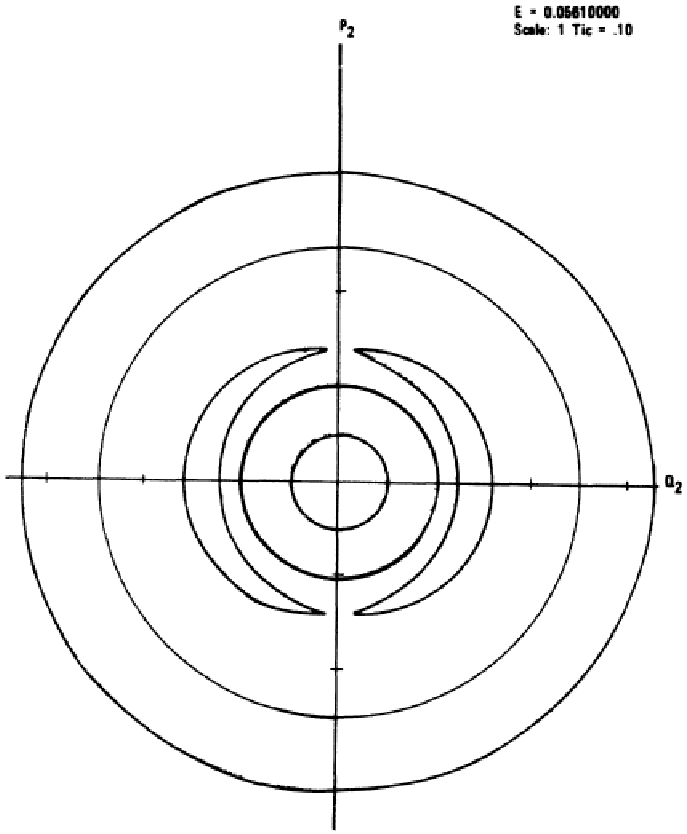

FIG. 8. Typical level curves for the 2-2, 2-3, doubly resonant Hamiltonian for energies below the appearance of the 2-3 resonance. Note the similarity to Fig. 6.

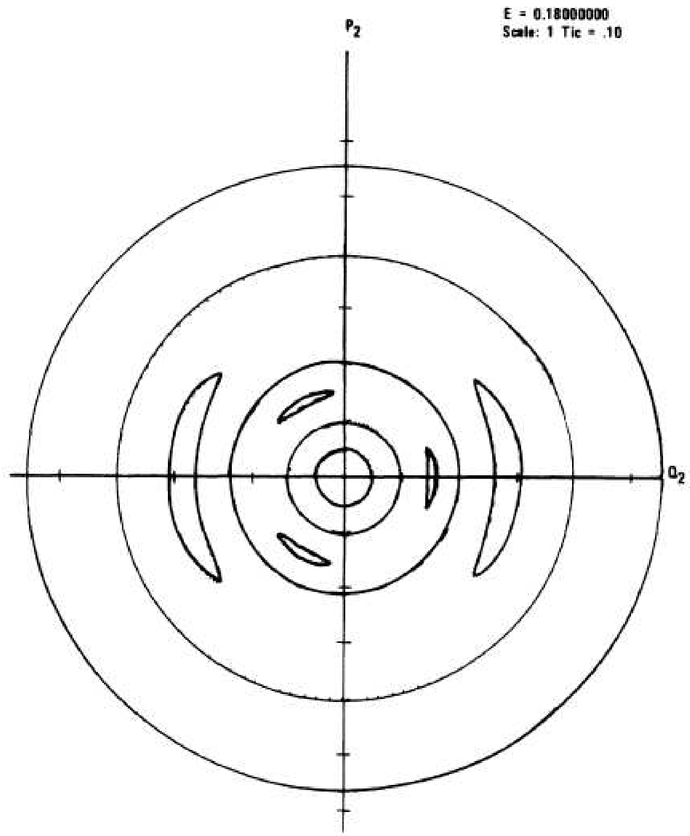

FIG. 9. Typical level curves for the 2-2, 2-3, doubly resonant Hamiltonian for energies yielding widely separated 2-2 and 2-3 resonances.

E = 0,0561
E < 0,23 → ilhas 2/2
E < 0,16 → não há ilhas 2/3

E = 0,1800

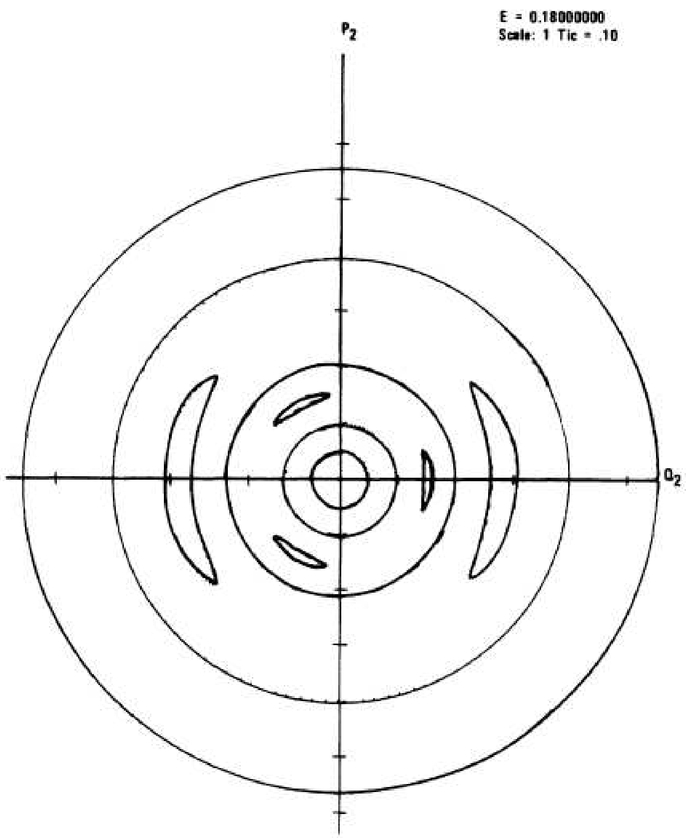

FIG. 9. Typical level curves for the 2–2, 2–3, doubly resonant Hamiltonian for energies yielding widely separated 2–2 and 2–3 resonances.

E = 0,1800

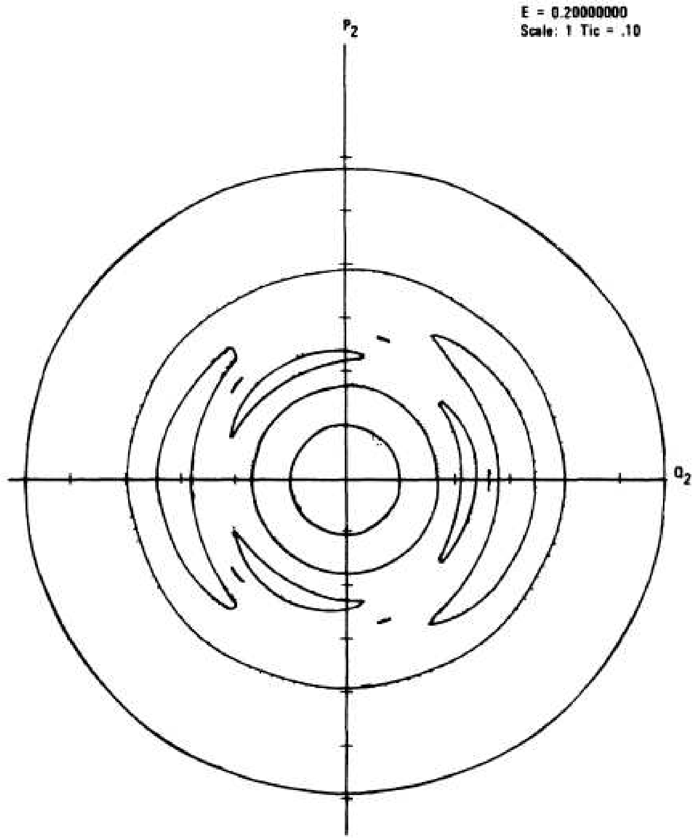

FIG. 12. Level curves for the 2–2, 2–3, doubly resonant Hamiltonian at $E = 0.20$, slightly below the predicted overlap energy. The dots between the 2–2 and 2–3 crescents are part of a chain of five islands. A chain of seven islands, not shown, has also been found in this region.

E = 0,2000

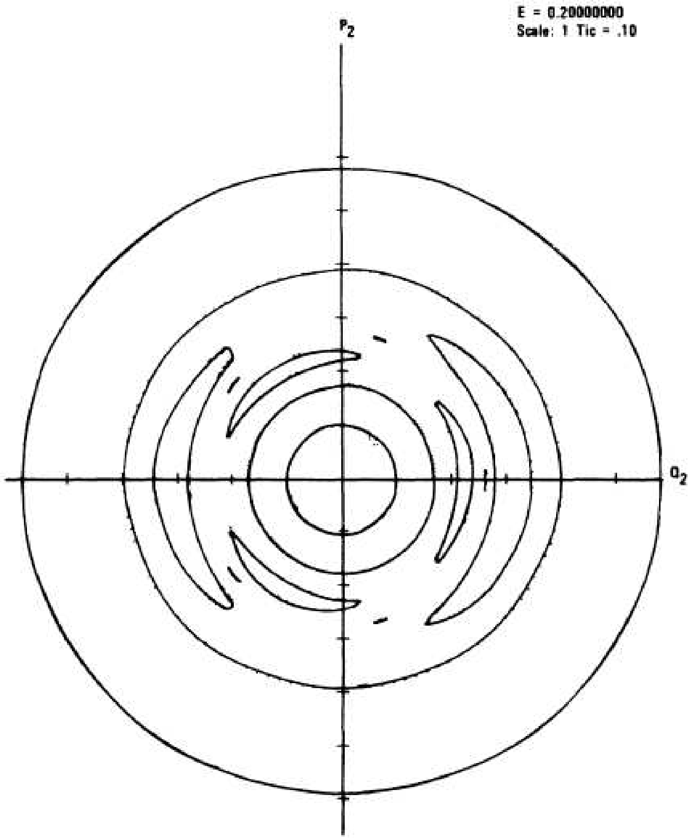

FIG. 12. Level curves for the 2–2, 2–3, doubly resonant Hamiltonian at $E=0.20$, slightly below the predicted overlap energy. The dots between the 2–2 and 2–3 crescents are part of a chain of five islands. A chain of seven islands, not shown, has also been found in this region.

E = 0,2000

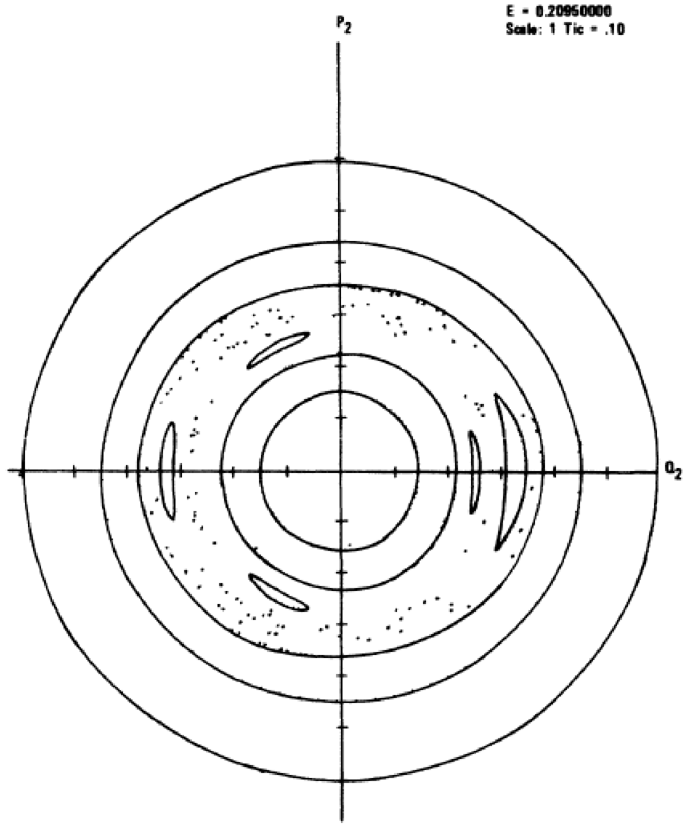

FIG. 11. Level curves for the 2–2, 2–3, doubly resonant Hamiltonian at the energy predicted for initial overlap of the 2–2 and 2–3 resonance zones.

E = 0,2095

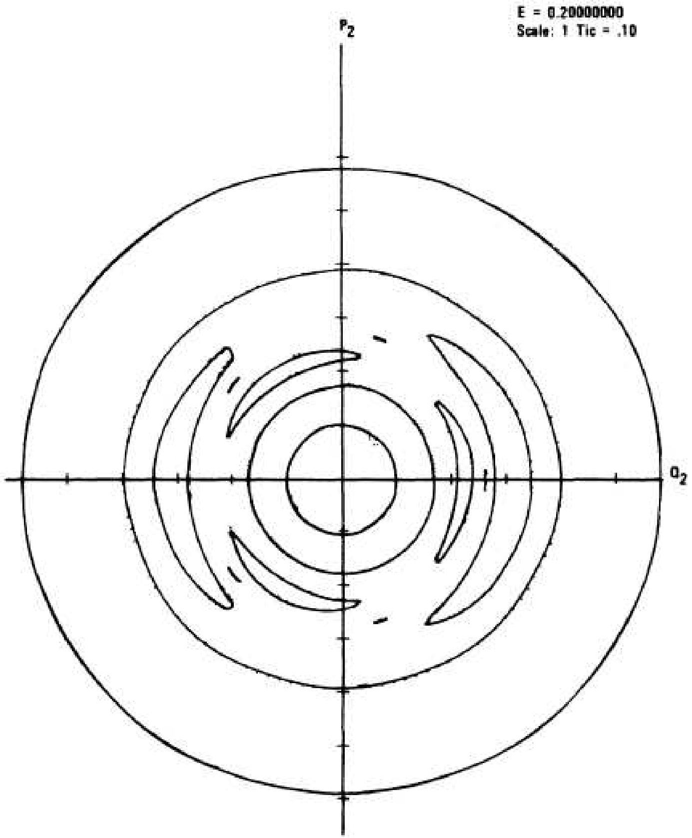

FIG. 12. Level curves for the 2–2, 2–3, doubly resonant Hamiltonian at $E = 0.20$, slightly below the predicted overlap energy. The dots between the 2–2 and 2–3 crescents are part of a chain of five islands. A chain of seven islands, not shown, has also been found in this region.

E = 0,2000

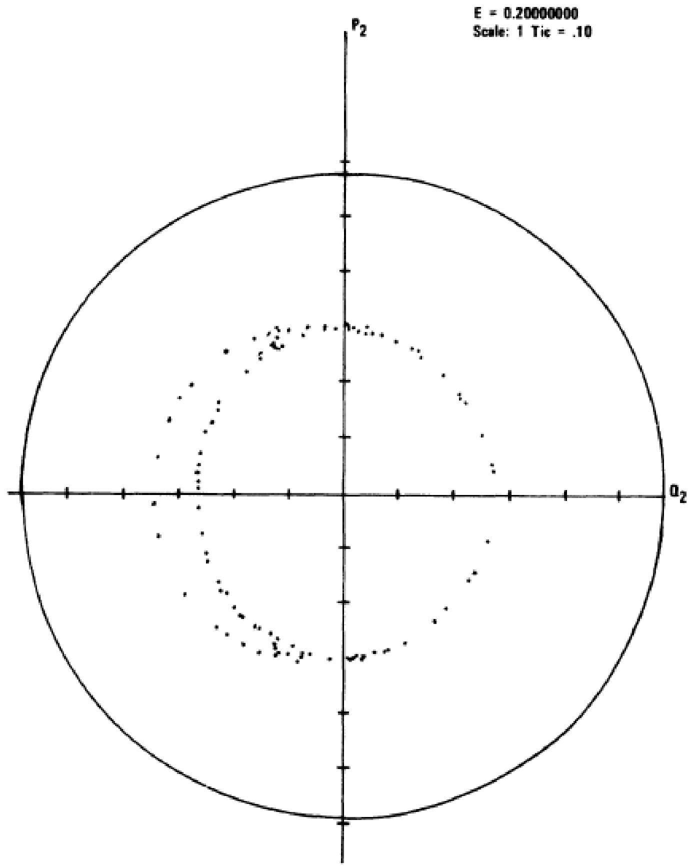

FIG. 13. This figure is a continuation of Fig. 12 and shows that a small zone of instability exists at energies below the predicted 2–2, 2–3 overlap.

E = 0,200

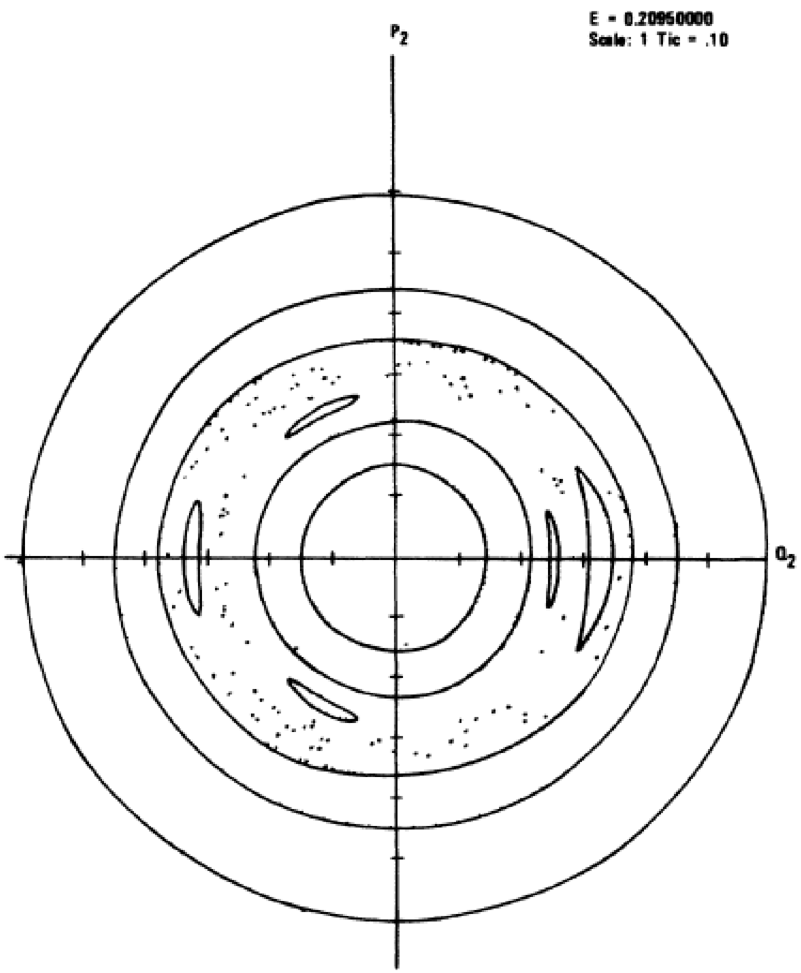

FIG. 11. Level curves for the 2–2, 2–3, doubly resonant Hamiltonian at the energy predicted for initial overlap of the 2–2 and 2–3 resonance zones.

E = 0,2096

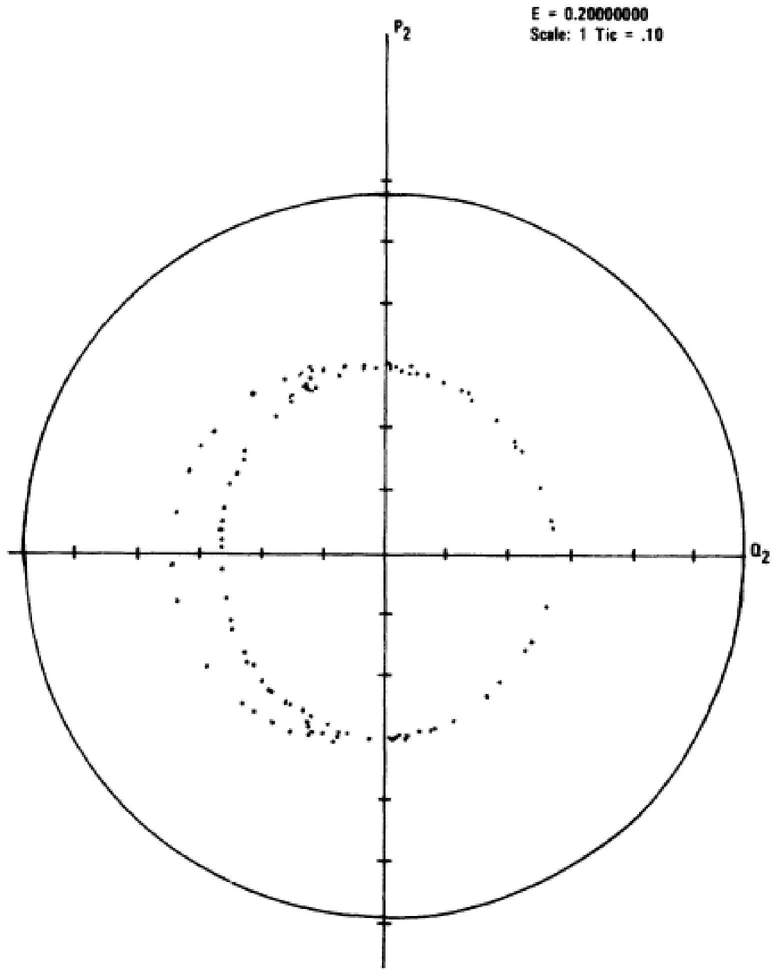

FIG. 13. This figure is a continuation of Fig. 12 and shows that a small zone of instability exists at energies below the predicted 2–2, 2–3 overlap.

E = 0,200

Ressonâncias Secundárias

Ressonâncias secundárias

Cadeias com 5 e 7 ilhas com

$4\omega_1 = 5\omega_2$ e $6\omega_1 = 57\omega_2$

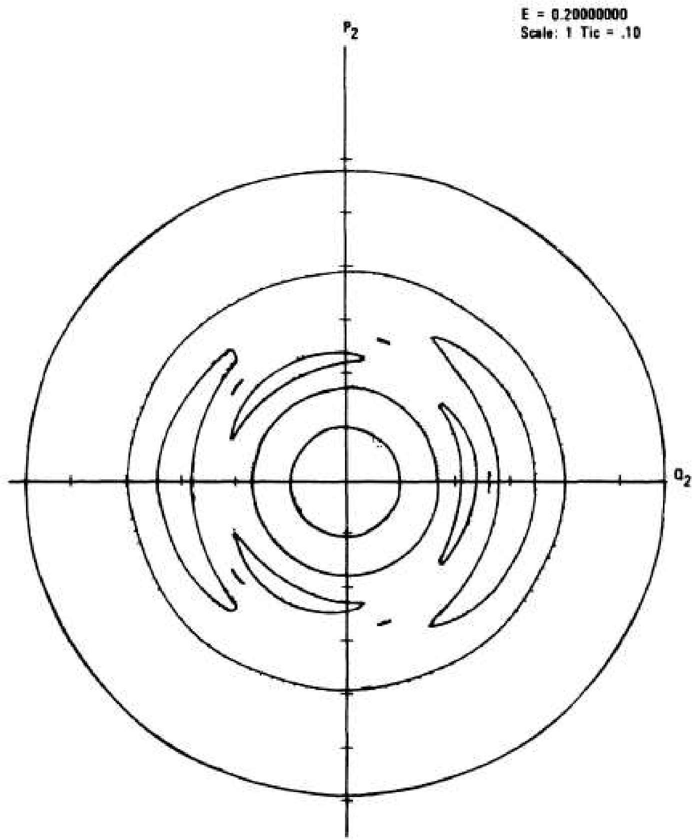

FIG. 12. Level curves for the 2–2, 2–3, doubly resonant Hamiltonian at $E = 0.20$, slightly below the predicted overlap energy. The dots between the 2–2 and 2–3 crescents are part of a chain of five islands. A chain of seven islands, not shown, has also been found in this region.

Estimativa do valor crítico de E para a colisão das ilhas.

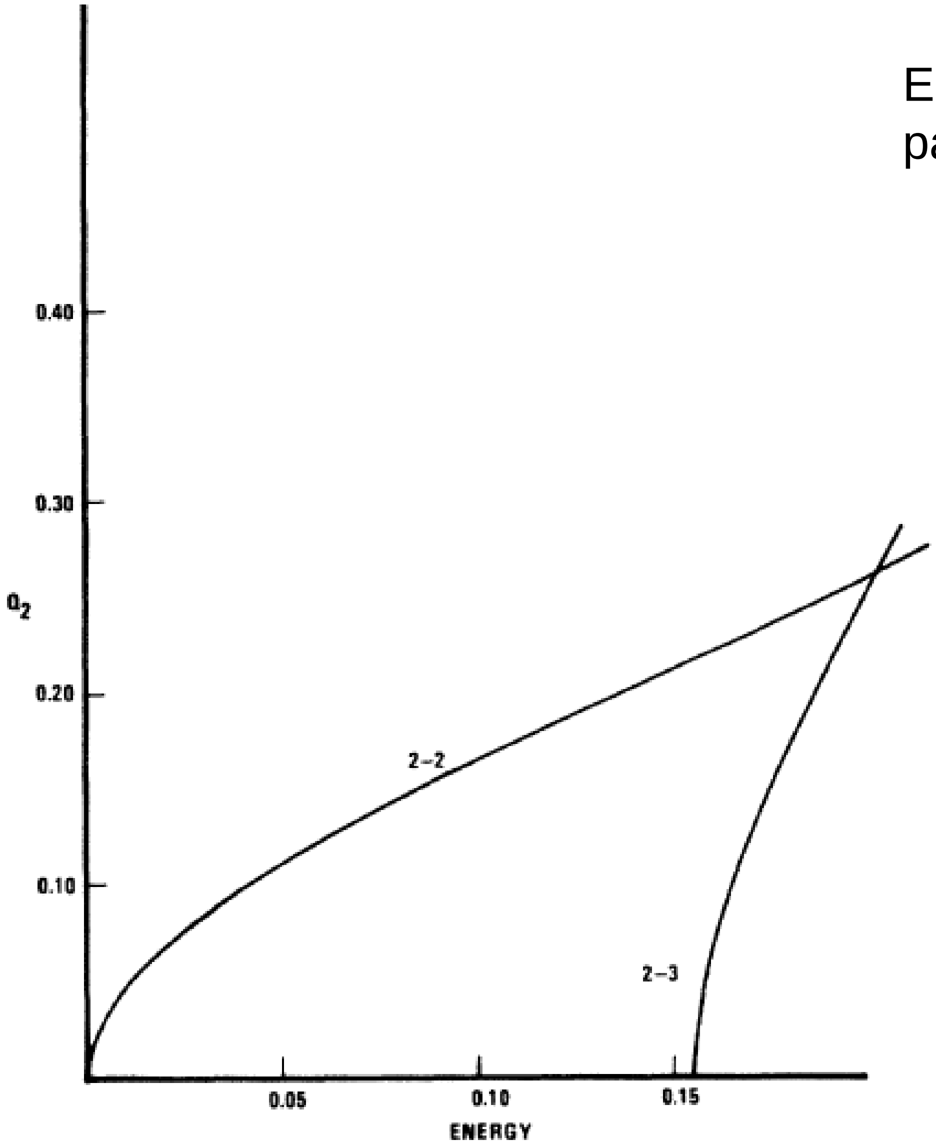

FIG. 10. q_2-axis intercepts of the inner 2–2 and the outer 2–3 separatrices are plotted as a function of total energy.

Ressonâncias 2/2 e 3/2

$$H = H_0(J_1, J_2) + \alpha J_1 J_2 \cos(2\varphi_1 - 2\varphi_2)$$
$$+ \beta J_1^{3/2} J_2 \cos(3\varphi_1 - 2\varphi_2) \quad ,$$

$\alpha = 0.95$ and $\beta = 0.25$

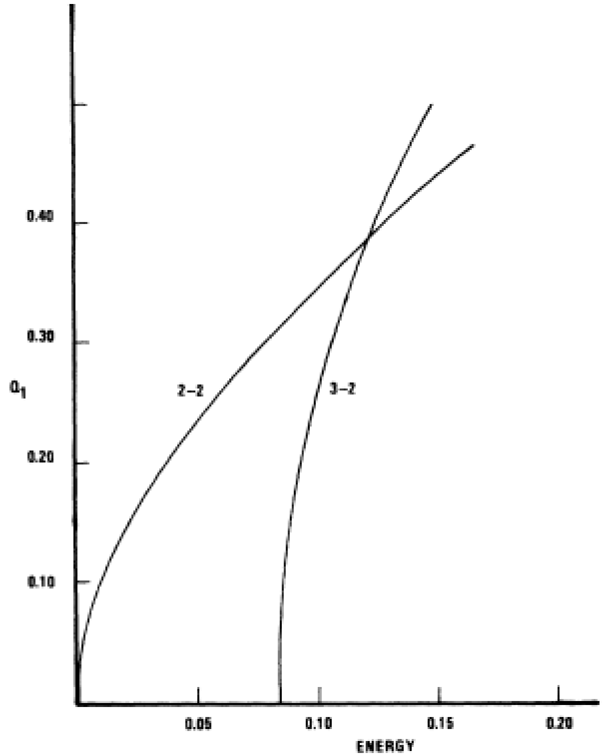

FIG. 14. q_1-axis intercepts of the 2-2 and 3-2 separatrices are plotted as a function of total energy.

Curvas para determinar o valor crítico de E (encontro das ilhas)

Ressonâncias secundárias são identificaads paar valores menores de E.

Ressonância Secundária 5/4

$5\omega_1 = 4\omega_2$

(nas duas figuras)

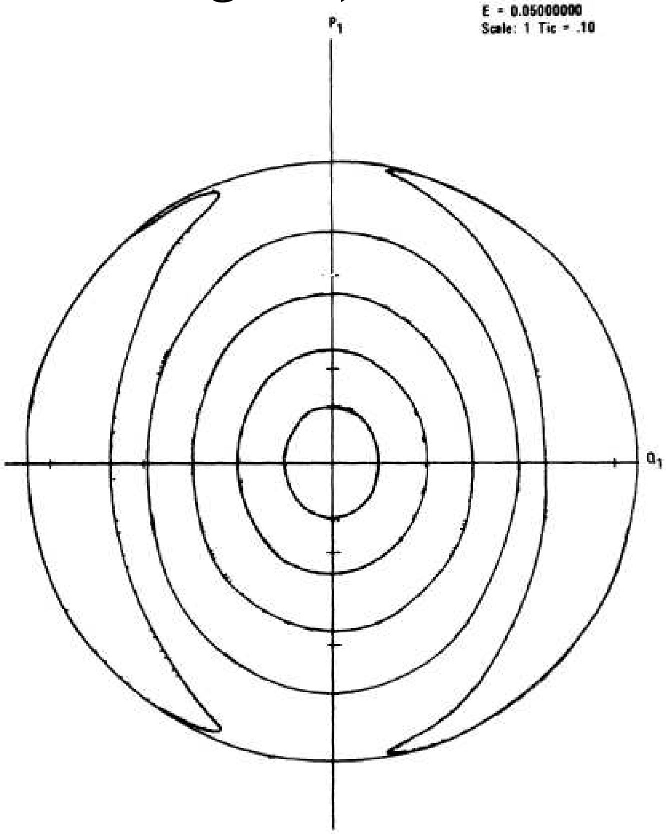

FIG. 15. Typical level curves for the 2-2, 3-2, doubly resonant Hamiltonian at low energy. Even at this low energy, a chain of five islands, not shown, has been detected.

E = 0,050

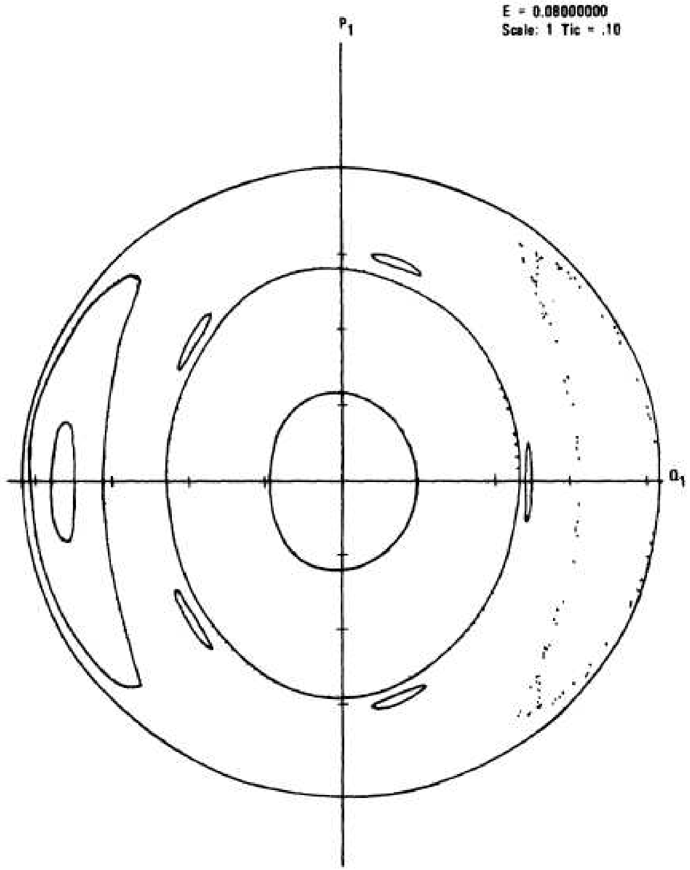

FIG. 16. Level curves for the 2-2, 3-2, doubly resonant Hamiltonian showing that a chain of islands and a zone of instability occur even before the 3-2 resonance appears.

E = 0,080

Aumento da Região Caótica com E

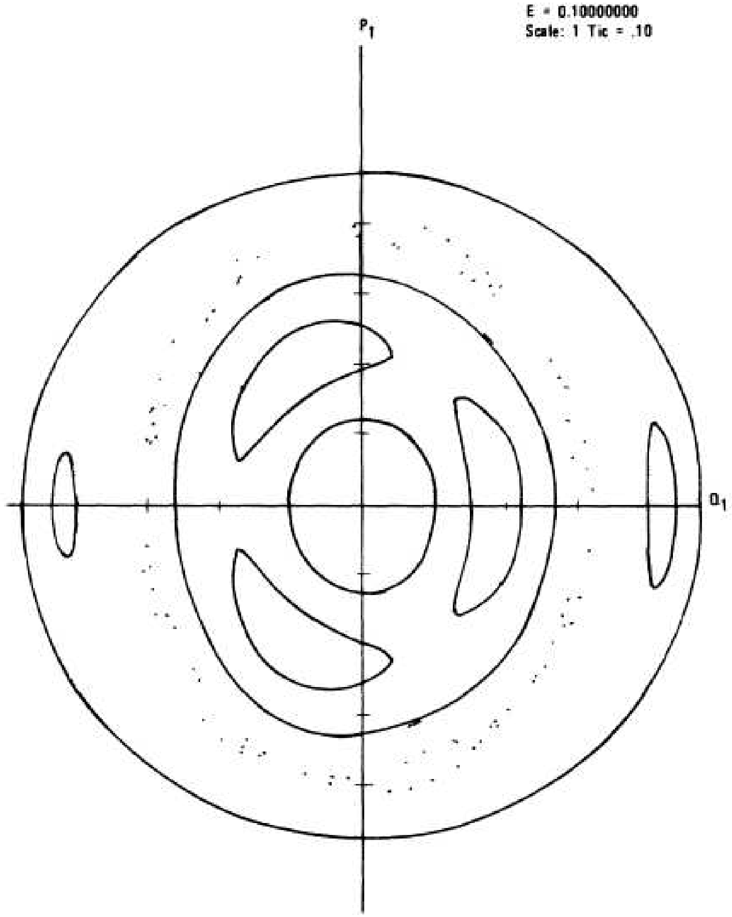

FIG. 17. Level curves for the 2-2, 3-2, doubly resonant Hamiltonian showing the increase of the instability zone with energy.

E = 0,100

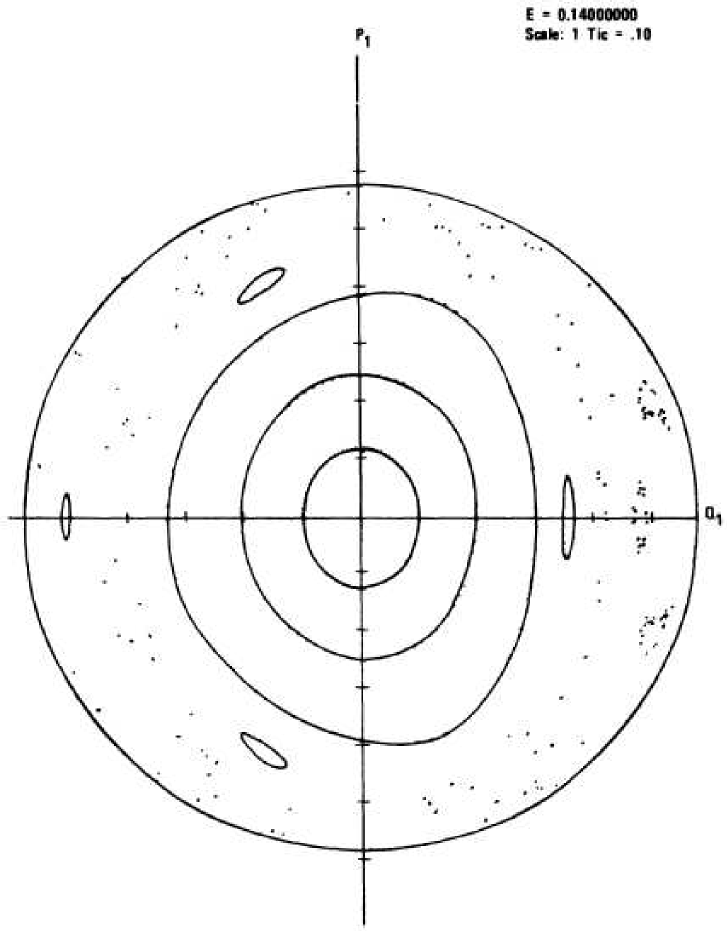

FIG. 18. Level curves for the 2-2, 3-2, doubly resonant Hamiltonian showing the 3-2 resonance as it moves into the ever-increasing zone of instability. The ragged looking chain of three islands in the right of the diagram represents a single level curve.

E =0,140