Interação entre Onda e Partícula

Lichtenberg, Regular and Chaotic Motion, Capítulo 2

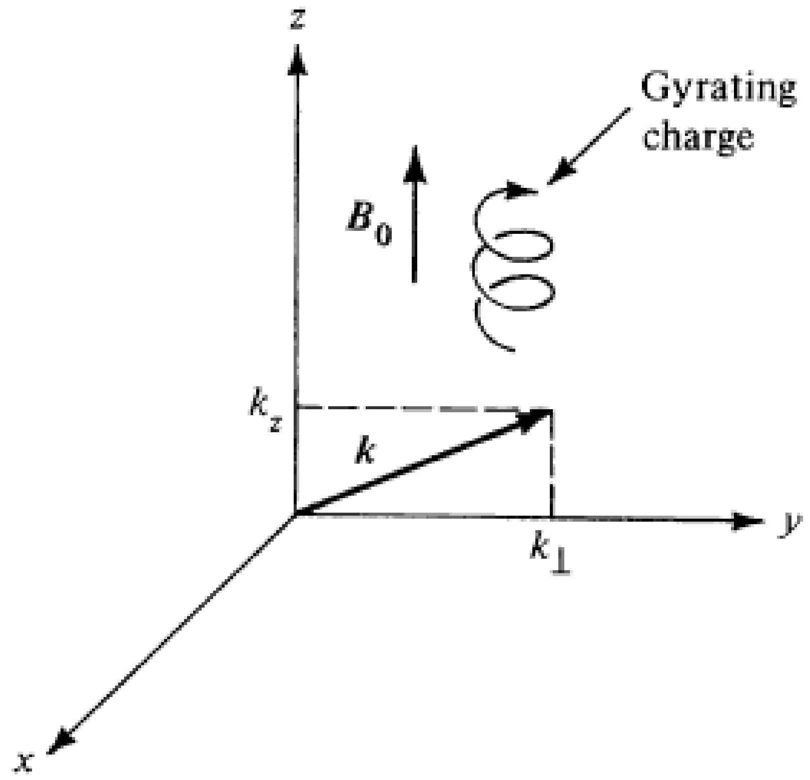


Figure 2.3. A charge moves in the field of a uniform static magnetic field $\boldsymbol{B}_0$ in the z-direction, and is perturbed by an electrostatic wave with amplitude Φ_0, frequency ω, and wavevector $\boldsymbol{k}$, with $\boldsymbol{k}$ lying in the y–z plane. In the absence of the wave, the charge gyrates perpendicular to $\boldsymbol{B}_0$ and moves uniformly along $\boldsymbol{B}_0$.

We first introduce action-angle variables for the unperturbed system. The unperturbed Hamiltonian is

$$H_{0p} = \frac{1}{2M}\left|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right|^2, \tag{2.2.53}$$

where M is the mass, e is the charge, c the velocity of light,

$$\boldsymbol{A}(\boldsymbol{x}) = -B_0 y\hat{x} \tag{2.2.54}$$

is the vector potential for a uniform field B_0, and

$$\boldsymbol{p} = M\mathbf{v} + \frac{e}{c}\boldsymbol{A} \tag{2.2.55}$$

is the canonical momentum conjugate to $\boldsymbol{x}$.

We transform to *guiding center variables* using the generating function

$$F_1 = M\Omega[\tfrac{1}{2}(y - Y)^2 \cot\phi - xY]. \tag{2.2.56}$$

Using (1.2.11), we obtain the new variables

$$\tan\phi = \frac{v_x}{v_y}, \tag{2.2.57a}$$

$$P_\phi = \frac{Mc}{e}\mu = \frac{1}{2}\frac{Mv_\perp^2}{\Omega} = \frac{1}{2}M\Omega\rho^2, \tag{2.2.57b}$$

$$Y = y + \rho\sin\phi, \tag{2.2.57c}$$

$$X = x - \rho\cos\phi, \tag{2.2.57d}$$

$$\Omega = \frac{eB_0}{Mc} \tag{2.2.58}$$

is the gyration frequency, μ is the magnetic moment, $v_\perp^2 = v_x^2 + v_y^2$, and $\rho = v_\perp/\Omega$ is the gyration radius. X and Y are the guiding center position, P_ϕ the angular momentum, and ϕ the gyration angle. The new momenta are P_ϕ, $M\Omega X$, and the linear momentum P_z, with their corresponding coordinates ϕ, Y, and z, respectively. The transformed Hamiltonian is found to be

$$H'_0 = \frac{P_z^2}{2M} + P_\phi \Omega. \tag{2.2.59}$$

Assuming a perturbation in the form of an electrostatic wave

$$\Phi = \Phi_0 \sin(k_z z + k_\perp y - \omega t) \tag{2.2.60}$$

with an electric field $\boldsymbol{E} = -\nabla\Phi$, we have in the guiding center coordinates

$$H'_1 = e\Phi_0 \sin(k_z z + k_\perp Y - k_\perp \rho) \sin(\phi - \omega t), \tag{2.2.61}$$

where

$$\rho(P_\phi) = \left(\frac{2P_\phi}{M\Omega}\right)^{1/2}. \tag{2.2.62}$$

Since

Since

$$H' = H'_0 + \epsilon H'_1 \tag{2.2.63}$$

is independent of the momentum $M\Omega X$, we have the corresponding coordinate $Y = \text{const.}$, and we shift z or t by a constant to eliminate the constant phase $k_\perp Y$ in (2.2.61). The nonlinearity in H' arises from the dependence of the phase on $\sin\phi$ and ρ. Because z and t appear only in the linear

Because z and t appear only in the linear combination $k_z z - \omega t$, time may be eliminated by transforming to the wave frame using the generating function

$$F_2 = (k_z z - \omega t)P_\psi + P_\phi \phi. \tag{2.2.64}$$

We obtain new variables P_ψ and ψ and new Hamiltonian H, using relations (1.2.13),

$$P_z = \frac{\partial F_2}{\partial z} = k_z P_\psi, \tag{2.2.65a}$$

$$\psi = \frac{\partial F_2}{\partial P_\psi} = k_z z - \omega t, \tag{2.2.65b}$$

and

$$\begin{aligned} H &= k_z^2 P_\psi^2/2M - P_\psi \omega + P_\phi \Omega + \epsilon e \Phi_0 \sin(\psi - k_\perp \rho \sin \phi) \\ &= E, \qquad \text{a const.} \end{aligned} \tag{2.2.66}$$

$$H = k_z^2 P_\psi^2/2M - P_\psi \omega + P_\phi \Omega + \epsilon e \Phi_0 \sin(\psi - k_\perp \rho \sin \phi)$$
$$= E, \qquad \text{a const.} \tag{2.2.66}$$

Here, as previously, ϵ is an arbitrary ordering parameter to be set equal to one at the end of the calculation. The harmonic resonances in the nonlinear forcing function can be exposed by expansion in a Bessel series to obtain

$$H = k_z^2 P_\psi^2/2M - P_\psi \omega + P_\phi \Omega + \epsilon e \Phi_0 \sum_m \mathscr{J}_m(k_\perp \rho) \sin(\psi - m\phi). \tag{2.2.67}$$

We have seen that it is necessary to stay sufficiently far from the unperturbed resonances that the Fourier amplitudes decrease faster than the near-resonant denominators. The Bessel coefficients $\mathscr{J}_m(k_\perp \rho)$ give the fall-off of the Fourier amplitudes. The unperturbed frequencies are obtained from of the Fourier amplitudes. The unperturbed frequencies are obtained from (2.2.67) as

$$\omega_\phi = \frac{\partial H_0}{\partial P_\phi} = \Omega, \tag{2.2.68a}$$

$$\omega_\psi = \frac{\partial H_0}{\partial P_\psi} = \frac{k_z^2}{M} P_\psi - \omega = k_z v_z - \omega. \tag{2.2.68b}$$

In first order the perturbation excites only resonances between the frequency ω_ψ and the various harmonics of ω_ϕ, so the resonance condition is

$$\omega_\psi - m\Omega = 0. \tag{2.2.69}$$

For $k_z = 0$, substituting for ω_ψ from (2.2.68b), we find that (2.2.69) becomes

$$\omega + m\Omega = 0. \tag{2.2.70}$$

For $k_z \neq 0$, solving (2.2.69) for P_ψ, we have the condition

$$P_\psi = \frac{M}{k_z^2}(\omega + m\Omega) \tag{2.2.71}$$

at resonance. We treat these resonant cases by secular perturbation theory in Section 2.4c.

Função Geratriz / Ângulo e Ação da H perturbada

$$S_1 = -e\Phi_0 \sum_m \mathscr{J}_m(k_\perp \bar{\rho}) \frac{\cos(\psi - m\phi)}{\omega + m\Omega}, \tag{2.2.73}$$

from which we obtain the old actions P_ψ and P_ϕ in terms of the new:

$$P_\psi = \bar{P}_\psi + \epsilon \frac{\partial S_1}{\partial \psi}$$

$$= \bar{P}_\psi + \epsilon e\Phi_0 \sum_m \mathscr{J}_m(k_\perp \bar{\rho}) \frac{\sin(\psi - m\phi)}{\omega + m\Omega}. \tag{2.2.74}$$

$\bar{P}_\phi$

Inverting, we have to first order

$$\bar{P}_\psi = P_\psi - \epsilon e\Phi_0 \sum_m \mathscr{J}_m(k_\perp \rho) \frac{\sin(\psi - m\phi)}{\omega + m\Omega}$$

$$= \text{const.} \tag{2.2.75}$$

Similarly, we obtain

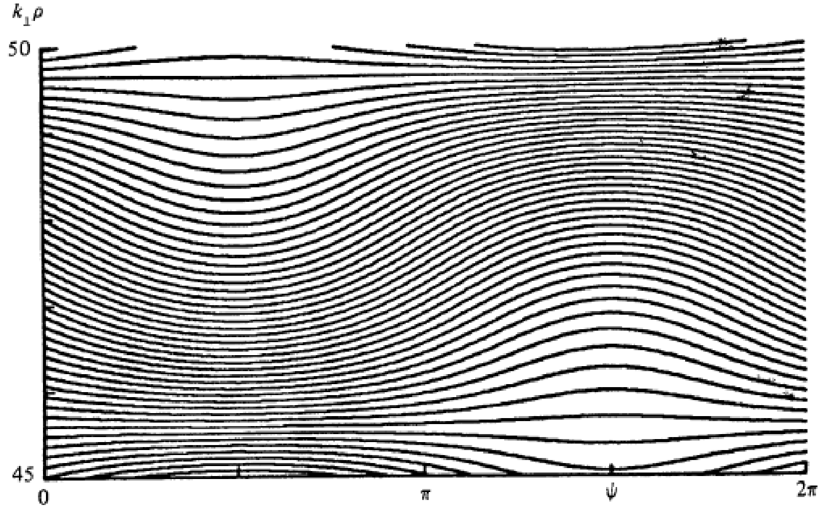
$$\bar{P}_\phi = P_\phi + \epsilon e\Phi_0 \sum_m m\mathscr{J}_m(k_\perp \rho) \frac{\sin(\psi - m\phi)}{\omega + m\Omega}$$

$$= \text{const.}, \tag{2.2.76}$$

$$\bar{\rho} = \rho(\bar{J}_\phi) \qquad \rho(P_\phi)$$

H depende das ações

$$\bar{P}_\psi \qquad \bar{P}_\phi$$



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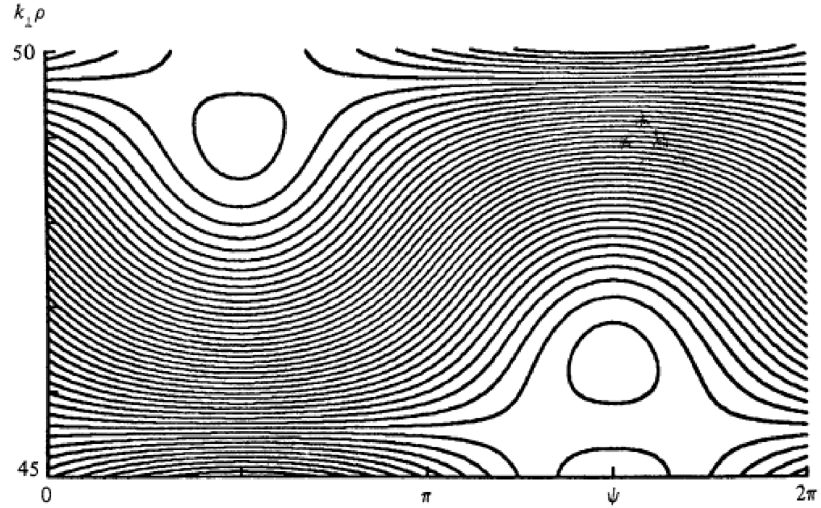
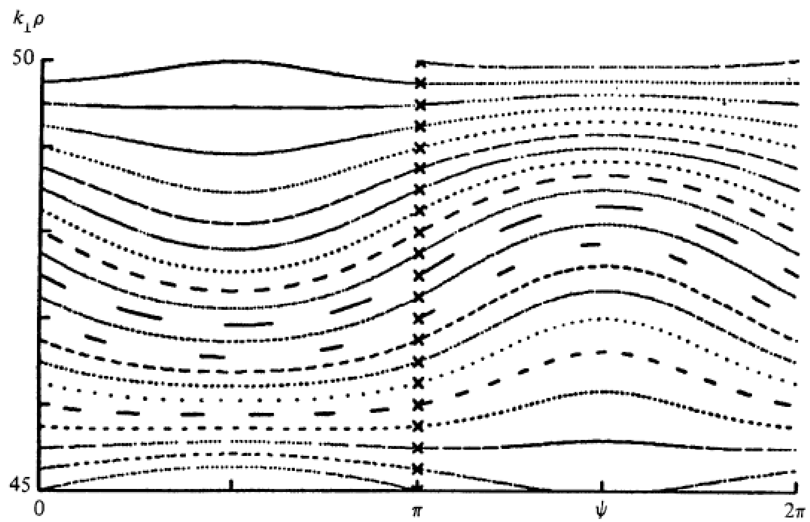
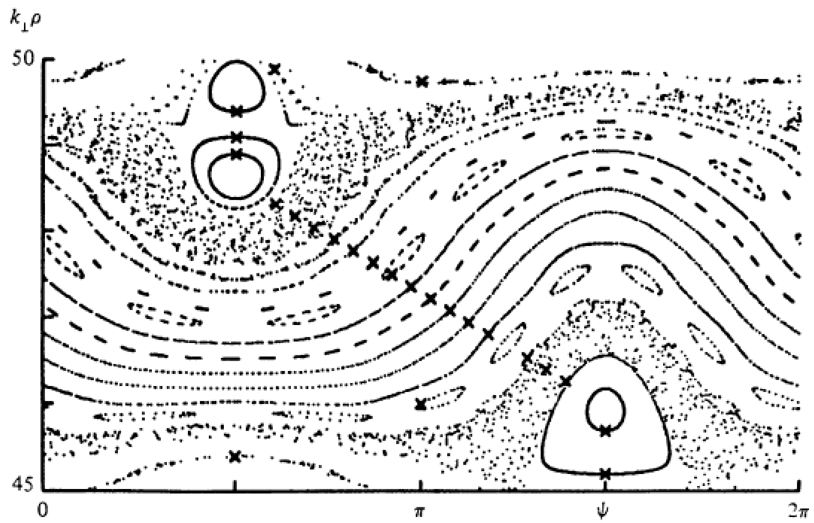


(b)

Figure 2.4. The nonresonant invariant curves of $k_\perp \rho$ versus ψ in a surface of section $\phi = \pi$ for off-resonant interaction of a gyrating particle interacting with a perpendicular propagating wave. The ratio of the applied frequency to the gyrofrequency is $\omega/\Omega = 30.11$. (a) Low wave amplitude—no trapping; (b) higher amplitude—with trapping (after Karney, 1977).



(a)



(b)

Figure 2.5. Exact trajectories computed in a surface of section $\phi = \pi$, with parameters as in Fig. 2.4. The x's mark the initial conditions (after Karney, 1977).

$$H = H_0(J_1, J_2) + \epsilon V(J_1, J_2, \theta_1, \theta_2)$$

$$H = H_0(J_1, J_2) + \epsilon \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} V_{n_1,n_2}(J_1, J_2) \cos(n_1\theta_1 + n_2\theta_2)$$

$$G(\mathcal{J}_1, \mathcal{J}_2, \theta_1, \theta_2) = \mathcal{J}_1\theta_1 + \mathcal{J}_2\theta_2 + \epsilon \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} g_{n_1,n_2}(\mathcal{J}_1, \mathcal{J}_2) \sin(n_1\theta_1 + n_2\theta_2)$$

where g_{n_1,n_2} will be determined below. The generating function in Eq. (2.2.9) generates a canonical transformation from the set of action-angle variables, $(J_1, J_2, \theta_1, \theta_2)$, to a new set of canonical action-angle variables, $(\mathcal{J}_1, \mathcal{J}_2, \Theta_1, \Theta_2)$, via the following equations [Goldstein 1980]:

$$J_i = \frac{\partial G}{\partial \theta_i} = \mathcal{J}_i + \epsilon \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} n_i g_{n_1,n_2} \cos(n_1\theta_1 + n_2\theta_2) \tag{2.2.10}$$

and

$$\Theta_i = \frac{\partial G}{\partial \mathcal{J}_i} = \theta_i + \epsilon \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{\partial g_{n_1,n_2}}{\partial \mathcal{J}_i} \sin(n_1\theta_1 + n_2\theta_2). \tag{2.2.11}$$

The new Hamiltonian, $H'(\mathcal{J}_1, \mathcal{J}_2, \Theta_1, \Theta_2)$, is obtained from Eq. (2.2.8) by solving Eqs. (2.2.10) and (2.2.11) for (J_i, θ_i) as a function of $(\mathcal{J}_i, \Theta_i)$ and plugging into Eq. (2.2.8). If we do that and then expand $H'(\mathcal{J}_1, \mathcal{J}_2, \Theta_1, \Theta_2)$ in a Taylor series in the small parameter ϵ, we find

$$
\begin{aligned}
H'(\mathcal{J}_1, \mathcal{J}_2, \Theta_1, \Theta_2) \\
&= H'_0(\mathcal{J}_1, \mathcal{J}_2) + \epsilon \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} (n_1\omega_1 + n_2\omega_2) g_{n_1,n_2} \cos(n_1\Theta_1 + n_2\Theta_2) \\
&\quad + \epsilon \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} V_{n_1,n_2}(\mathcal{J}_1, \mathcal{J}_2) \cos(n_1\Theta_1 + n_2\Theta_2) + O(\epsilon^2), \quad (2.2.12)
\end{aligned}
$$

where

$$\omega_i = \frac{\partial H'_o}{\partial \mathcal{J}_i}.$$

Now remove terms of order ϵ by choosing

$$g_{n_1,n_2} = -\frac{V_{n_1,n_2}(\mathcal{J}_1, \mathcal{J}_2)}{(n_1\omega_1 + n_2\omega_2)}.$$

Then

$$H'(\mathcal{J}_1, \mathcal{J}_2, \Theta_1, \Theta_2) = H'_o(\mathcal{J}_1, \mathcal{J}_2) + O(\epsilon^2)$$

and

$$J_i = \mathcal{J}_i - \epsilon \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{n_i V_{n_1,n_2} \cos(n_1\Theta_1 + n_2\Theta_2)}{(n_1\omega_1 + n_2\omega_2)} + O(\epsilon^2).$$

varies linearly in time. This is the hope. However, there is a catch. For any of this to have meaning, we must have

$$|n_1\omega_1 + n_2\omega_2| \gg \epsilon V_{n_1,n_2}. \tag{2.2.17}$$

But the condition in Eq. (2.2.17) breaks down when internal nonlinear resonances occur and cause the perturbation expansion to diverge. Poincaré showed that it is a general property of perturbation expansions of this type that they can be expected to diverge.

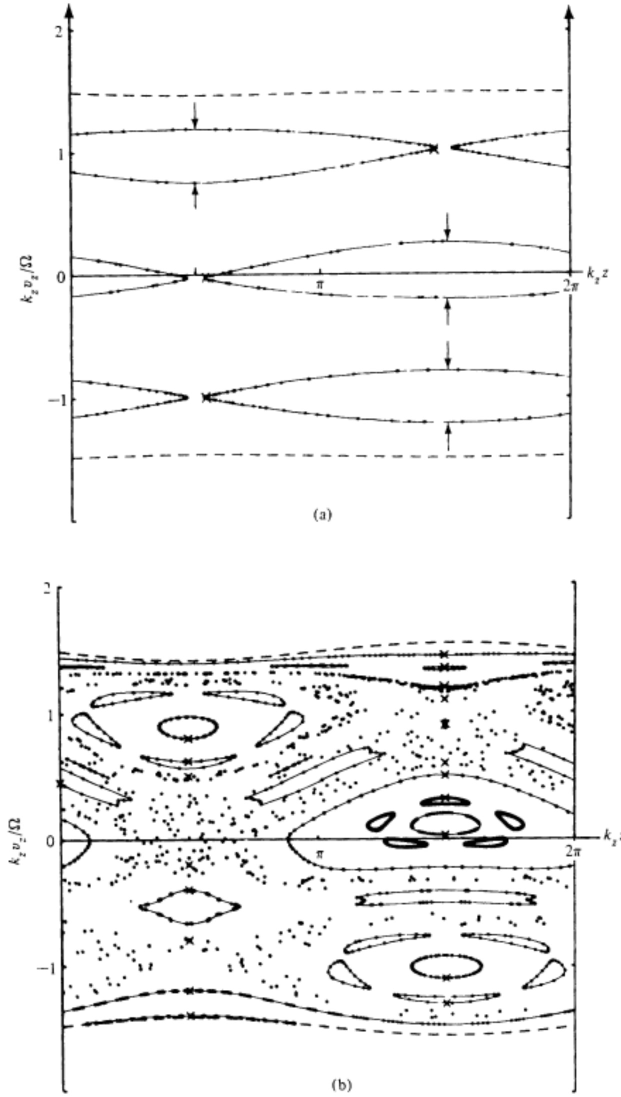


Figure 2.10. Surface of section plots $k_z v_z/\Omega \propto P_\psi$ versus $k_z z = \psi$ for oblique angle ($k_z \neq 0$), wave-particle resonance. Three resonances are shown, calculated by integration of Hamilton's equations. (a) Weak perturbation. The initial conditions, indicated by the x's, were chosen to yield trajectories very close to the separatrices. The points representing the trajectories have been connected with hand-drawn curves. The wave amplitude is given by $k_z^2 e\Phi_0/M\Omega^2 = 0.025$. (b) Strong perturbation. The wave amplitude is increased to 0.1 (after Smith and Kaufman, 1975).

$$H = k_z^2 P_\psi^2/2M - P_\psi \omega + P_\phi \Omega + \epsilon e \Phi_0 \sum_m \mathscr{J}_m(k_\perp \rho) \sin(\psi - m\phi). \quad (2.2.67)$$

$$\omega_\phi = \frac{\partial H_0}{\partial P_\phi} = \Omega, \quad (2.2.68a)$$

$$\omega_\psi = \frac{\partial H_0}{\partial P_\psi} = \frac{k_z^2}{M} P_\psi - \omega = k_z v_z - \omega. \quad (2.2.68b)$$

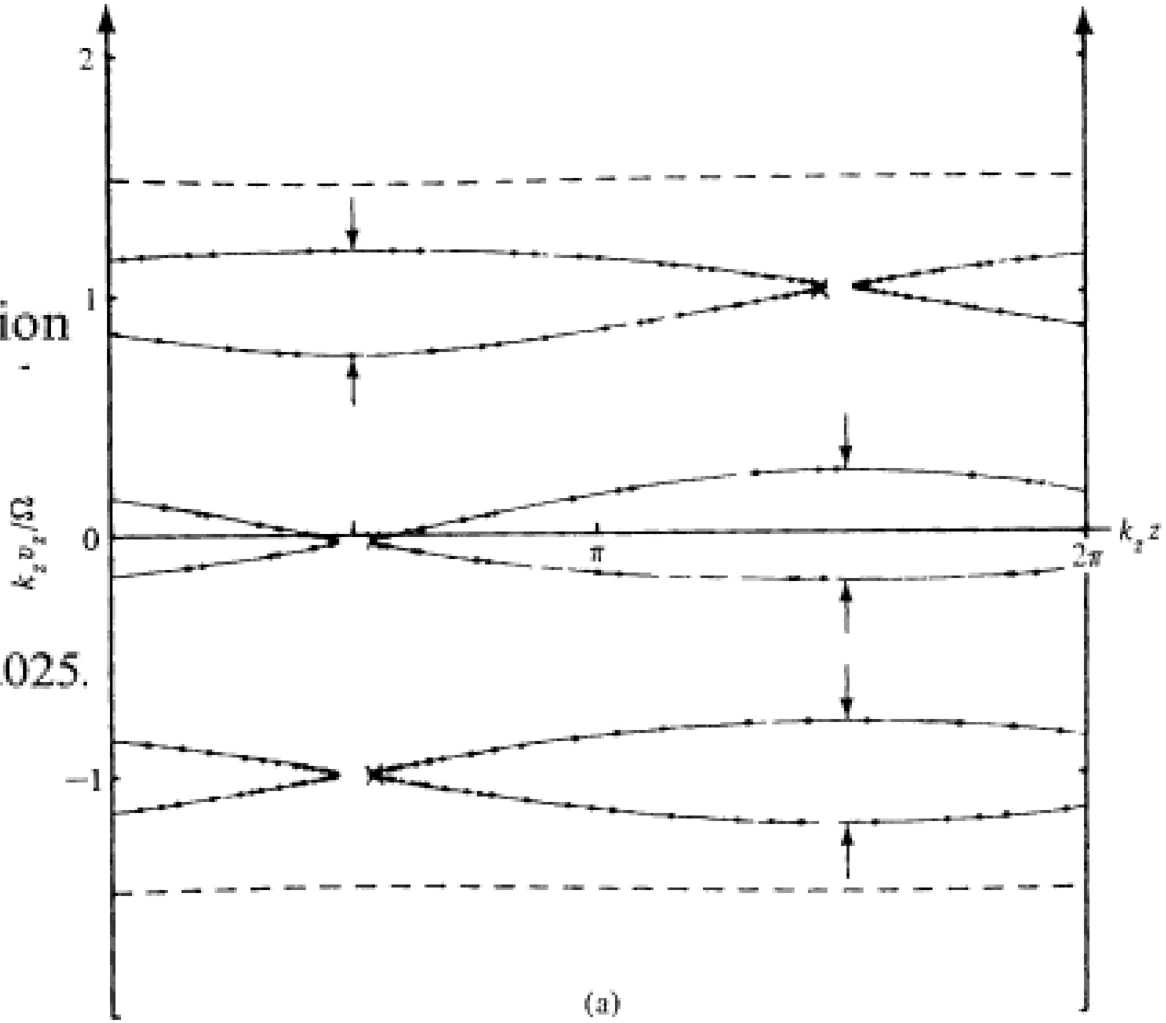
$$\omega_\psi - m\Omega = 0$$

For $k_z \neq 0$, solving (2.2.69) for P_ψ, we have the condition

$$P_\psi = \frac{M}{k_z^2}(\omega + m\Omega)$$

Weak perturbation

$k_z^2 e\Phi_0/M\Omega^2 = 0.025.$



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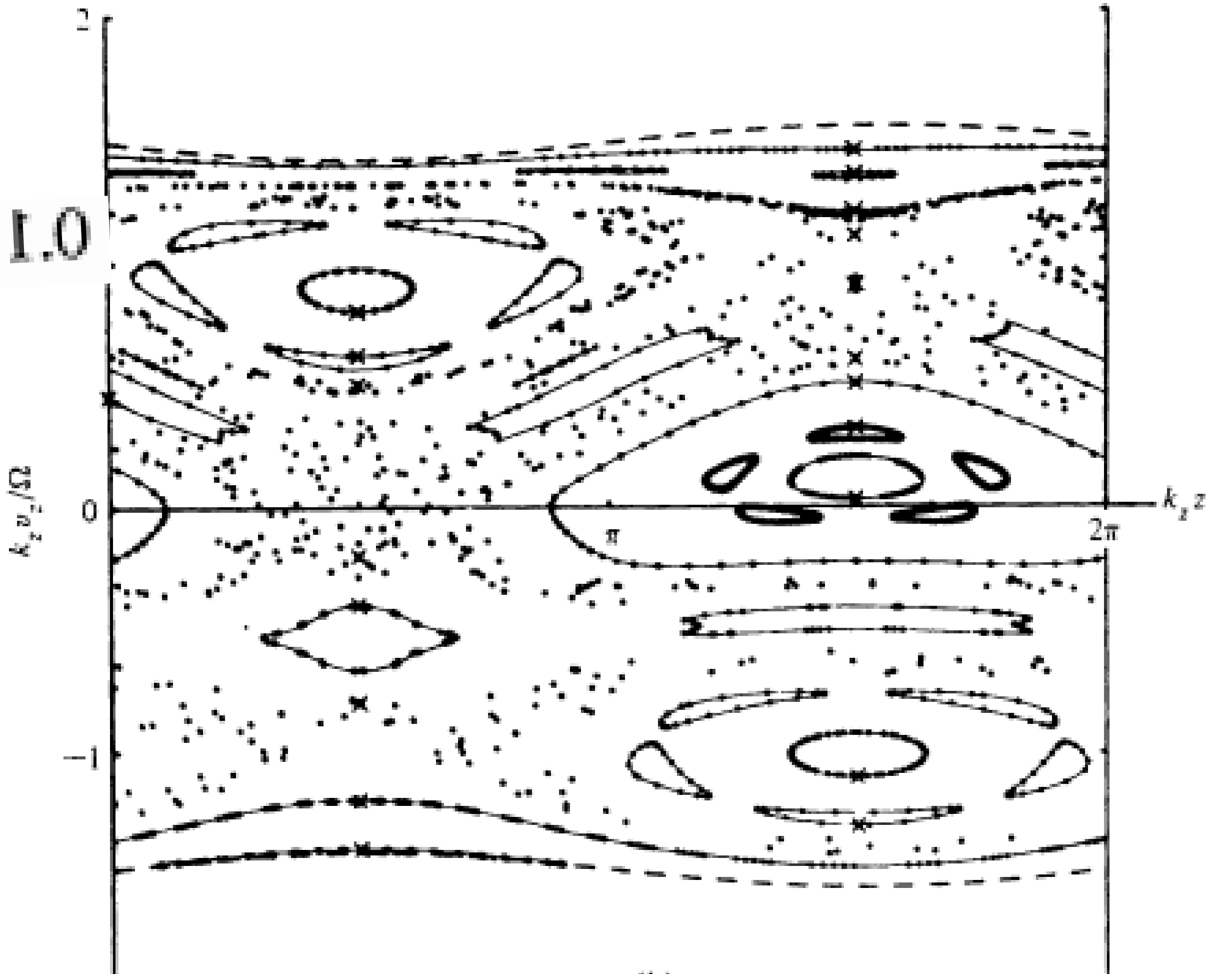
Strong perturbation

$k_z^2 e\Phi_0/M\Omega^2 = 1.0$



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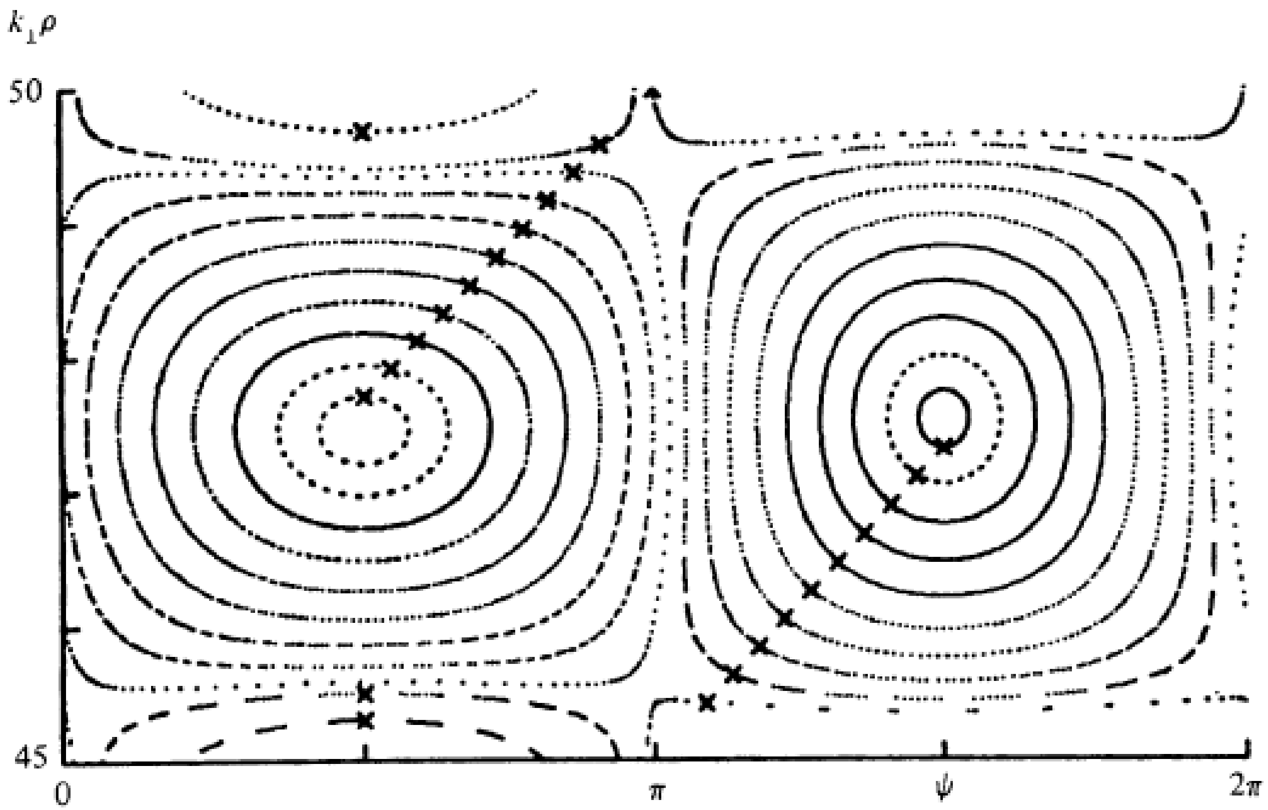
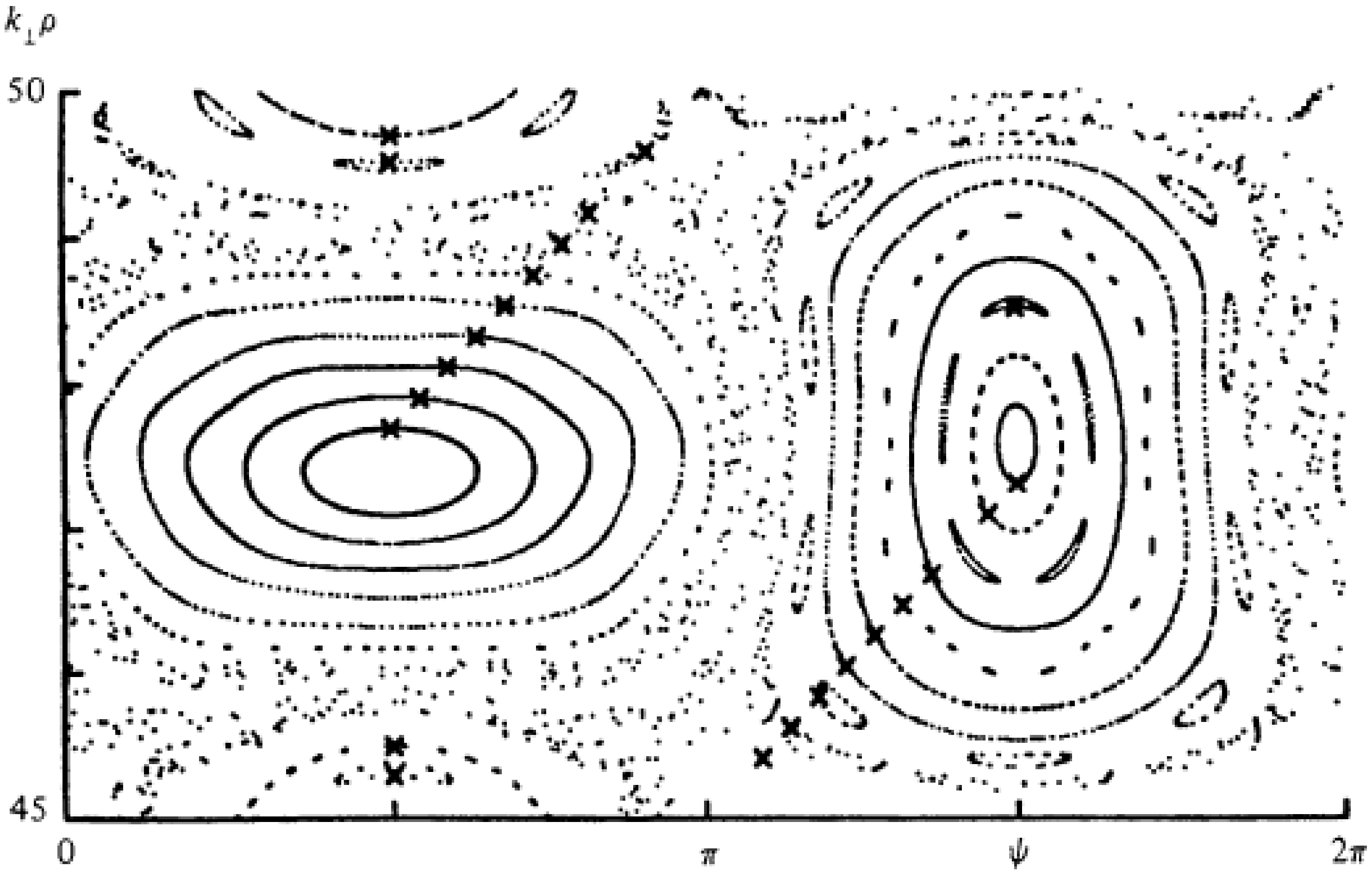


(a)

Figure 2.11. Surface of section plots $k_\perp \rho$ versus ψ for perpendicular ($k_z = 0$) wave-particle resonance, $\omega/\Omega = 30$, calculated by integration of Hamilton's equations. (a) For small perturbation strength; (b) for larger perturbation strength. The x's mark the initial conditions (after Karney, 1977).



(b)

Figure 2.11. Surface of section plots $k_\perp \rho$ versus ψ for perpendicular ($k_z = 0$) wave-particle resonance, $\omega/\Omega = 30$, calculated by integration of Hamilton's equations. (a) For small perturbation strength; (b) for larger perturbation strength. The x's mark the initial conditions (after Karney, 1977).