

8. Sistemas Não-Integráveis

PGF 5005 - Mecânica Clássica

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(Referências principais: 2 artigos de Reichl e Zheng,
em Physical Review A, 1984)

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Equação de Duffing (potencial duplo simétrico perturbado periódicamente)

L. E. Reichl, W. M. Zheng, Phys. Rev. A 29, 2186 (1984)

L. E. Reichl, W. M. Zheng, Phys. Rev. A 30, 1068 (1984)

O *poço de potencial duplo* é aplicado para investigar um grande número de fenômenos em Mecânica Clássica e em Mecânica Quântica.

Apresentamos, a seguir, um sistema hamiltoniano unidimensional com *poço duplo de potencial simétrico harmônico*.

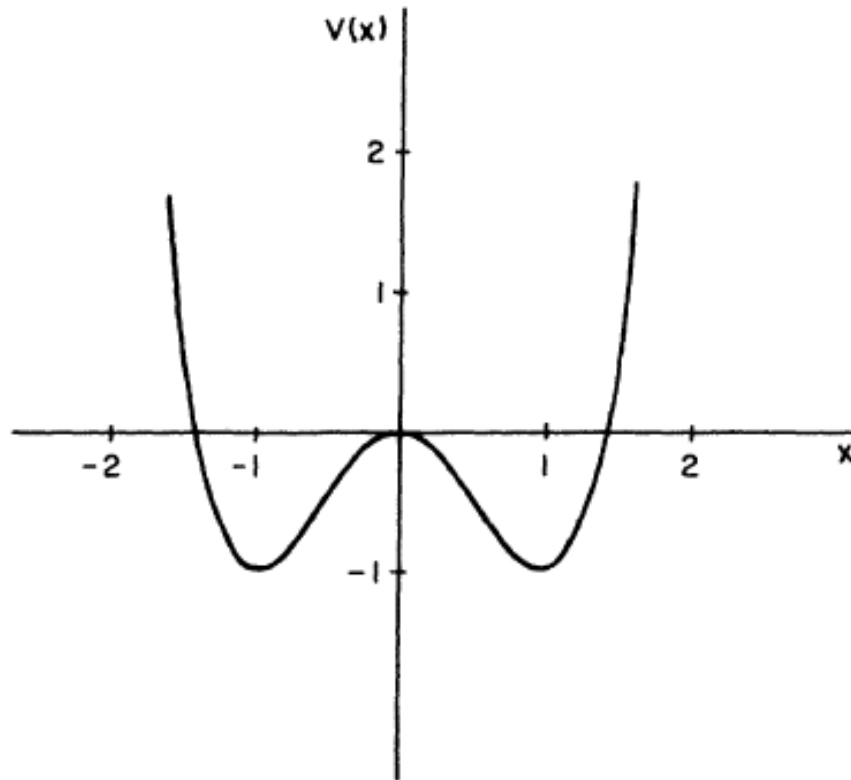
II. UNPERTURBED DOUBLE-WELL SYSTEM

We will first consider a particle of mass $m=2$ constrained to move along the x axis in a double-well potential

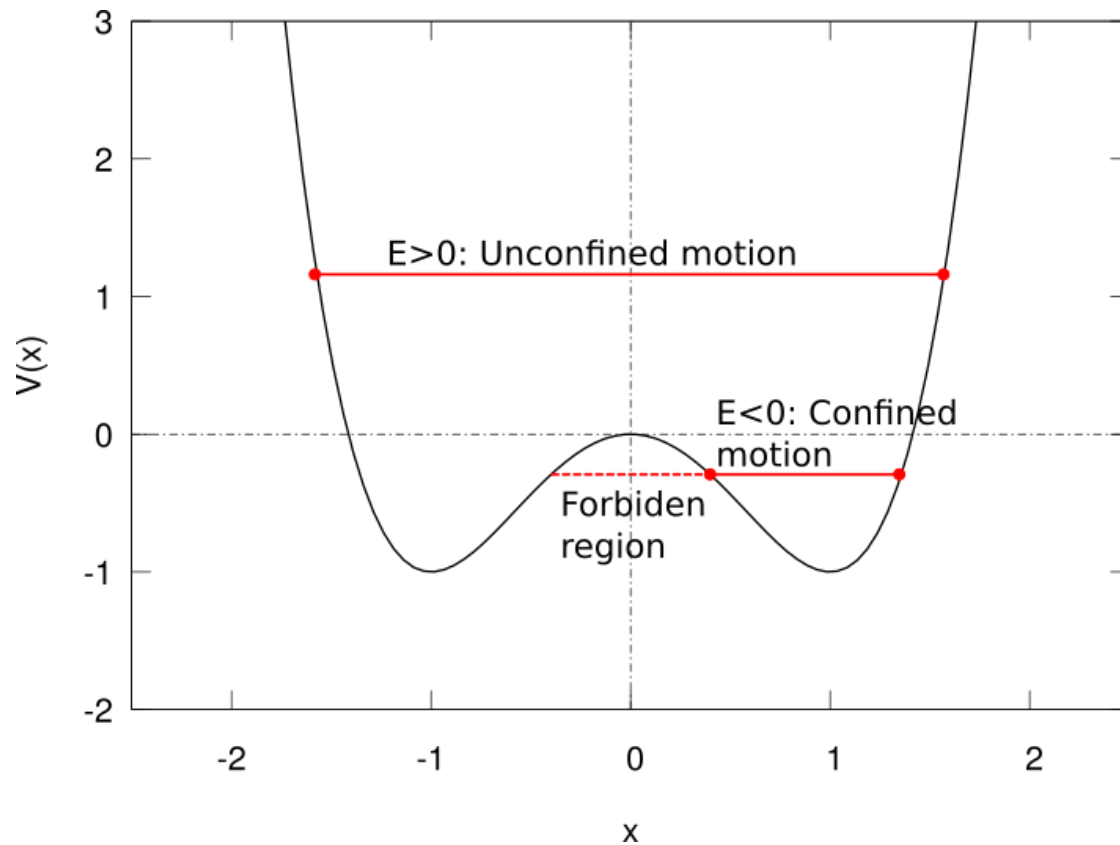
$$V(x) = -2x^2 + x^4. \quad (2.1)$$

The Hamiltonian for this system is

$$H = \frac{1}{4}p^2 - 2x^2 + x^4 = E_0, \quad (2.2)$$

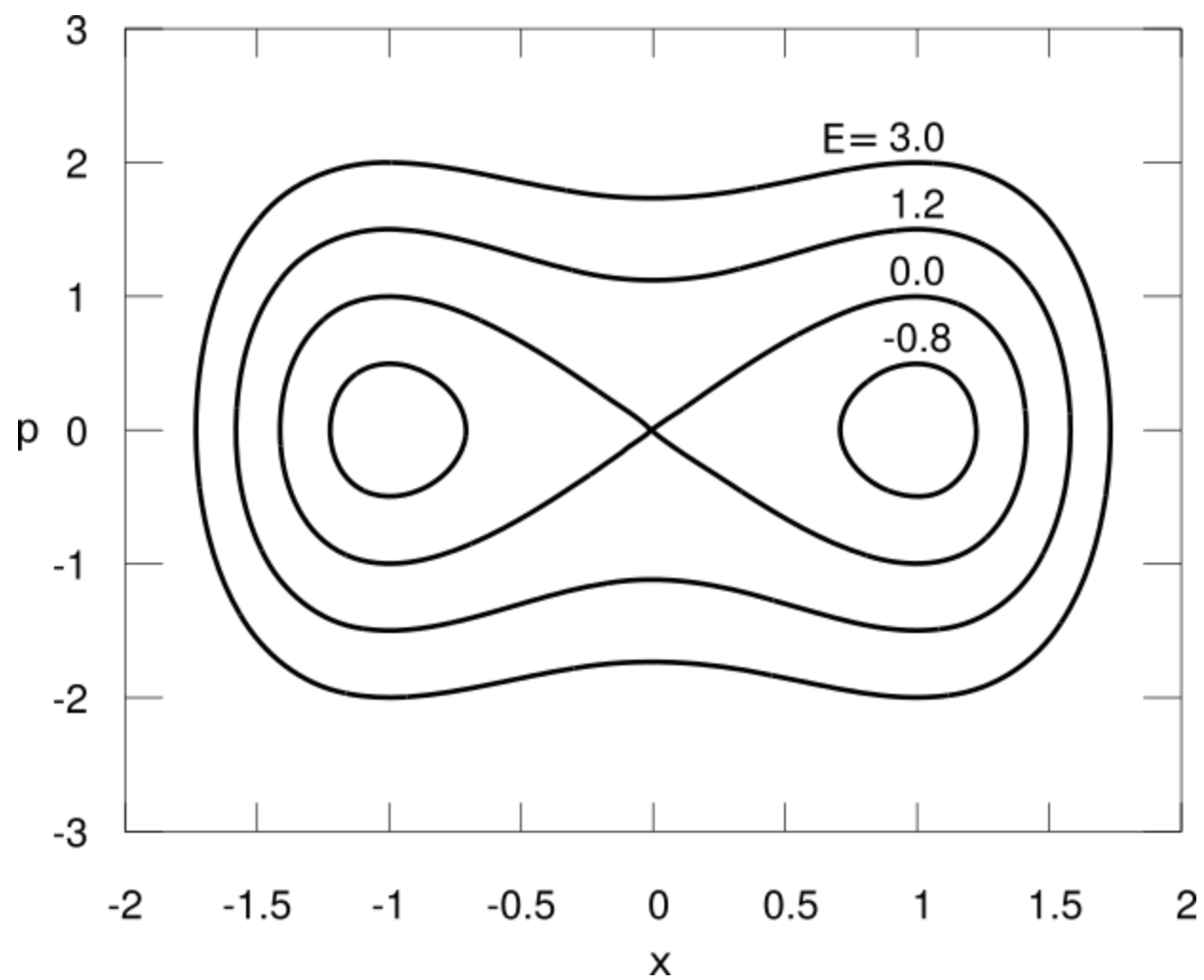


II – Double-Well Systems



$$V(x) = -2x^2 + x^4$$

$$H(x, p) = \frac{1}{4}p^2 - 2x^2 + x^4$$



- **Caso integrável $\varepsilon = 0$**

A partícula se movimenta com energia $H_0 = E_0$.

$E_0 < 0 \rightarrow$ partícula presa em um dos dois poços, não ultrapassa a barreira.

$E_0 > 0 \rightarrow$ partícula pode ir de um lado para o outro ultrapassando a barreira.

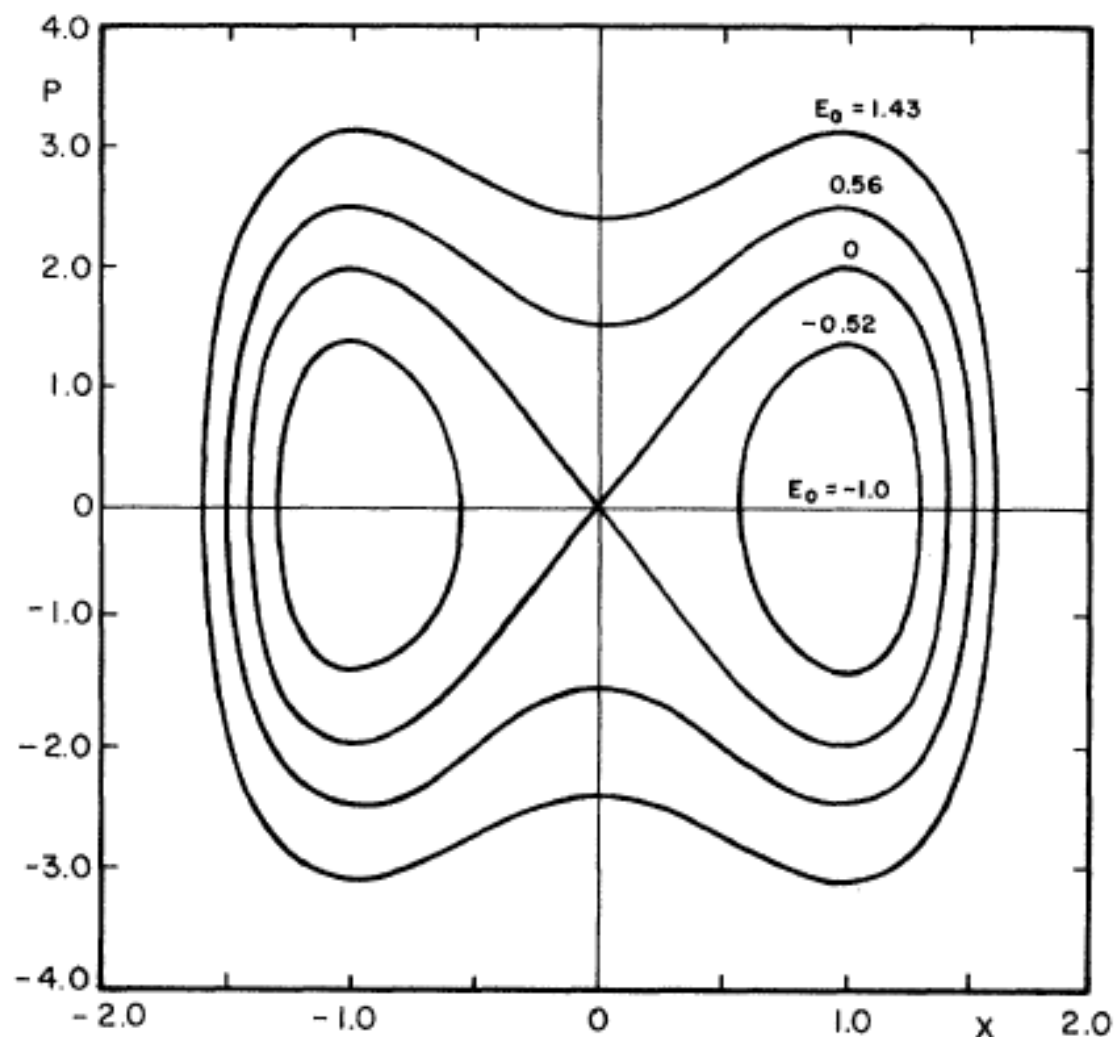


FIG. 2. Plot of phase-space trajectories for particle of mass $m=2$ trapped in the quartic potential $V(x)=-2x^2+x^4$. Trajectories are plotted for energies $E_0=-1$, -0.52 , 0.0 , 0.56 , and 1.43 . The trajectory $E_0=0.0$ is the separatrix and separates trapped motion from untrapped motion.

Partícula aprisionada ($E_o < 0$).

Quando $E_o < 0$ o momento p da partícula é dado por

$$p = \pm [2m(E_o + kx^2/2 - x^4)]^{1/2} = \pm [2m(f^2 - x^2)(x^2 - e^2)]^{1/2},$$

$$f^2 = k/4 + (k^2/16 + E_o)^{1/2} \quad e \quad e^2 = k/4 - (k^2/16 + E_o)^{1/2}.$$

É fácil vermos que $x_1 = e$ e $x_2 = f$ são os pontos entre os quais a partícula oscila dentro do poço onde está aprisionada.

A variável de ação J é dada por

$$\begin{aligned} J &= (1/2\pi) \oint p \, dx = (1/\pi) \int_e^f dx [2m(f^2 - x^2)(x^2 - e^2)]^{1/2} \\ &= 2(2m)^{1/2} f [kE(\kappa^2)/4 - e^2 K(\kappa^2)]/(3\pi), \end{aligned}$$

$\kappa^2 = (f^2 - e^2)/f^2$. $K(\kappa^2)$ e $E(\kappa^2)$ são as integrais elípticas de primeira e segunda espécies, respectivamente.

Caso não integrável, $\varepsilon \neq 0$

Hamiltoniano H , denominado de *Hamiltoniana de Duffing*, é dada por

$$H(p,x,t) = H_0 + \varepsilon x \cos(\omega_0 t) \quad \text{onde } H_0 = p^2/2m - kx^2/2 + x^4 .$$

Nessas condições a Hamiltoniana não é constante; E_0 e J não são constantes de movimento.

Temos de resolver numericamente as equações de Hamilton:

$$dx/dt = \partial H / \partial p \quad \text{e} \quad dp/dt = - \partial H / \partial x$$

Integração numérica

Para obter cada figura a seguir, os autores escolheram $\omega_o = 1,6$ e $1,8$

A partícula estava inicialmente dentro do poço direito com

$x_0 = 0.24$, $p_0 = 0$ e energia E_o .

Os cálculos numéricos são feitos para tempos discretizados $t_k = 2\pi k / \omega_o$ onde $k = 0, 1, 2, \dots$. Notemos que o primeiro ponto (x_1, p_1) a ser graficado é obtido através da integração numérica usando a condição inicial (x_o, p_o) ; o segundo ponto (x_2, p_2) é obtido usando (x_1, p_1) como condição inicial e assim, sucessivamente.

O nome *gráfico* ou *mapa estroboscópico* vem do fato de se observar o sistema em instantes discretos t_k . Notemos que esse mapa corresponde a um *mapa de Poincaré*.

Deterministic Chaos Theory: Basic Concepts

Teoria do Caos Determinístico: Conceitos Básicos

Mauro Cattani^{*1}, Iberê Luiz Caldas¹, Silvio Luiz de Souza², Kelly Cristiane Iarosz¹

$$\ddot{x} + \beta \dot{x} - x + \gamma x^3 = F \cos(\omega t)$$

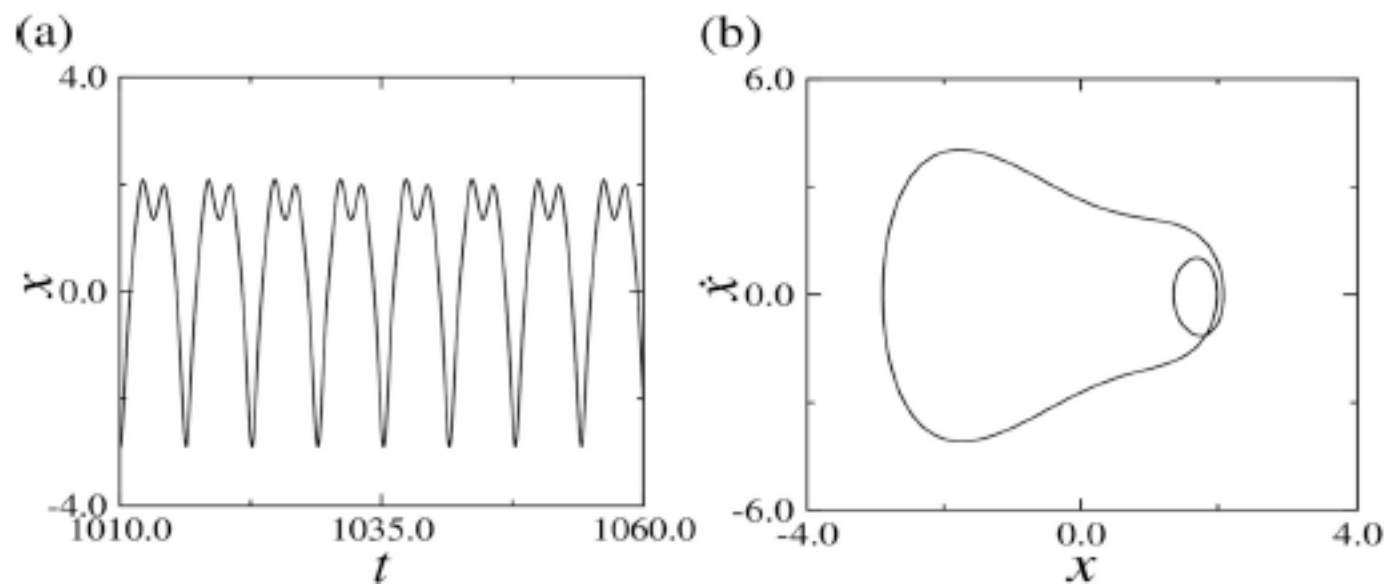


Figure 1: (a) Periodic time evolution of x , from a numerical solution of the Duffing equation for parameters $\beta=0.05$, $\gamma=1.0$, $\omega=1.0$, $F=4.0$. (b). Orbit in phase space for the same solution.

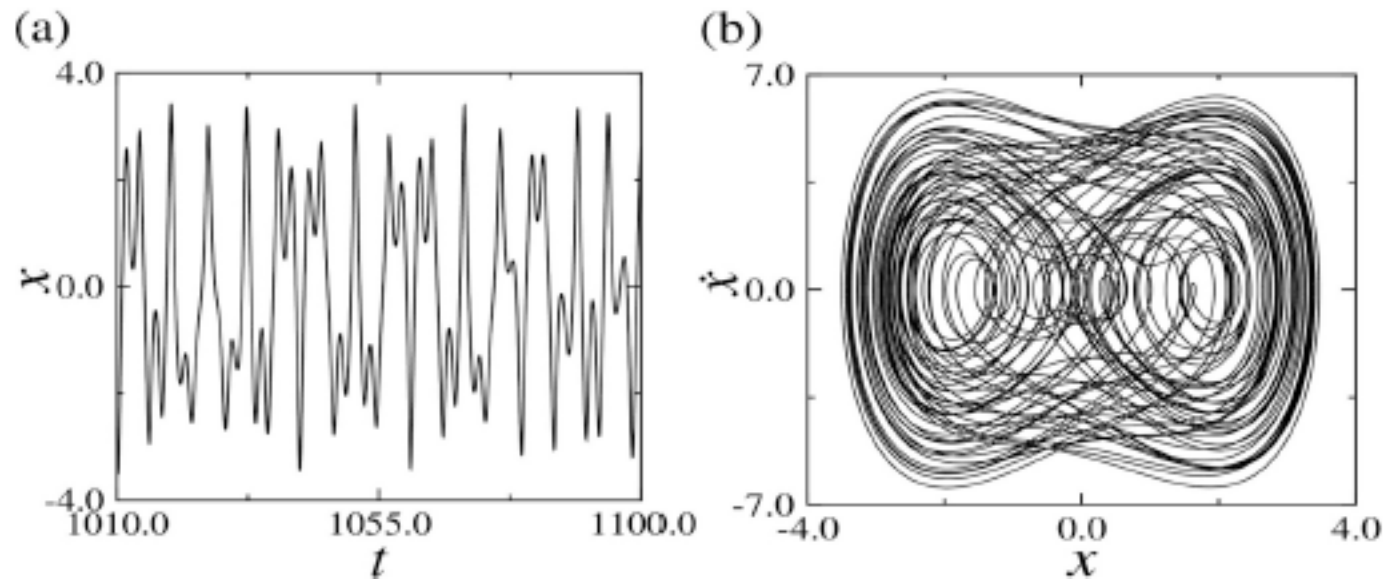


Figure 2: (a) Chaotic time evolution of x , from a numerical solution of the Duffing equation for parameters $\beta=0.05$, $\gamma=1.0$, $\omega=1.0$, $F=6.0$. (b). Orbit in phase space for the same solution.

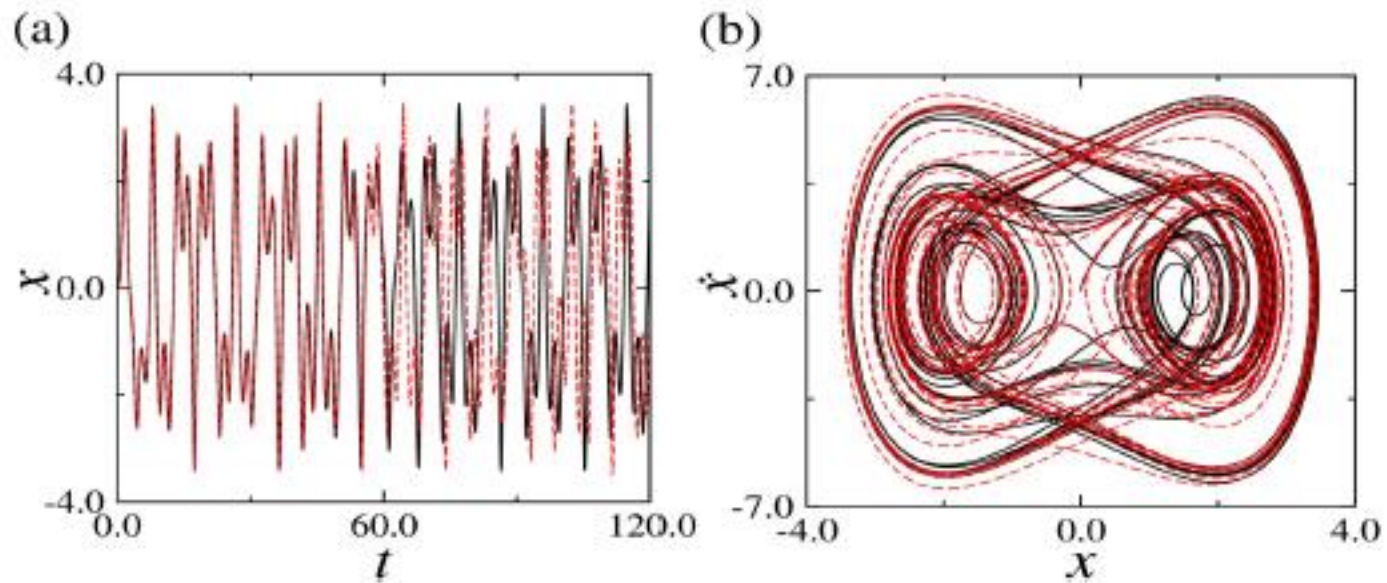


Figure 3: Sensitivity dependence on initial conditions. Chaotic time evolution of two solutions for the same parameters of Fig. 2, for initial conditions $(x_0, x'_0, t_0) = (0.0, 0.0, 0.0)$ for the black solid line and $(x_0, x'_0, t_0) = (0.001, 0.0, 0.0)$ for the red dashed line.

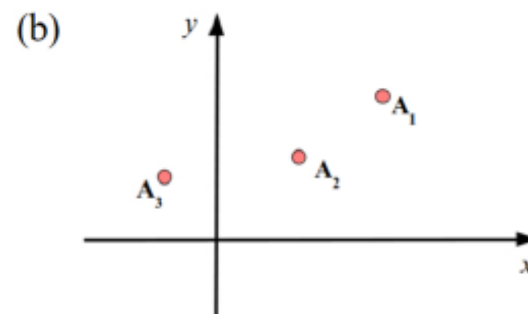
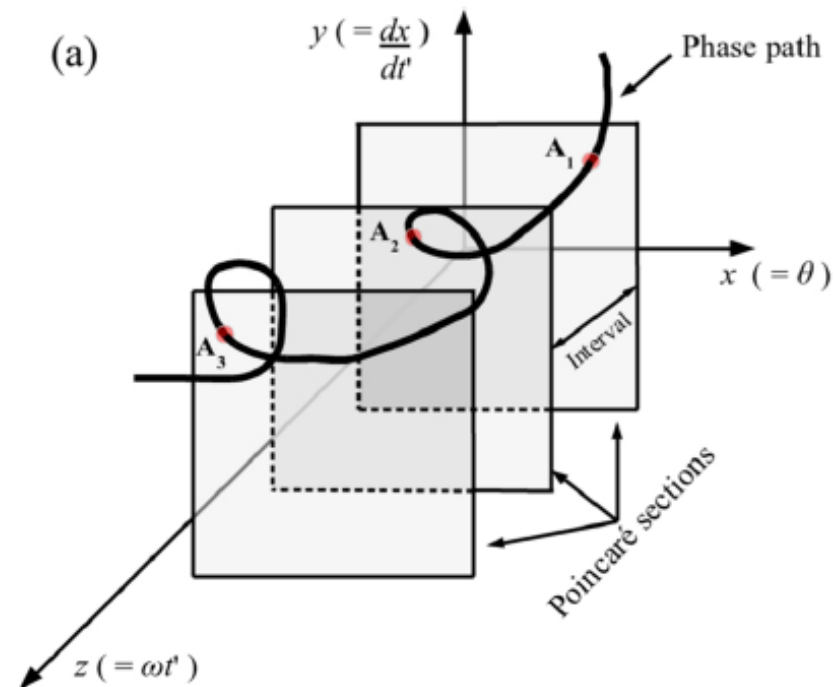
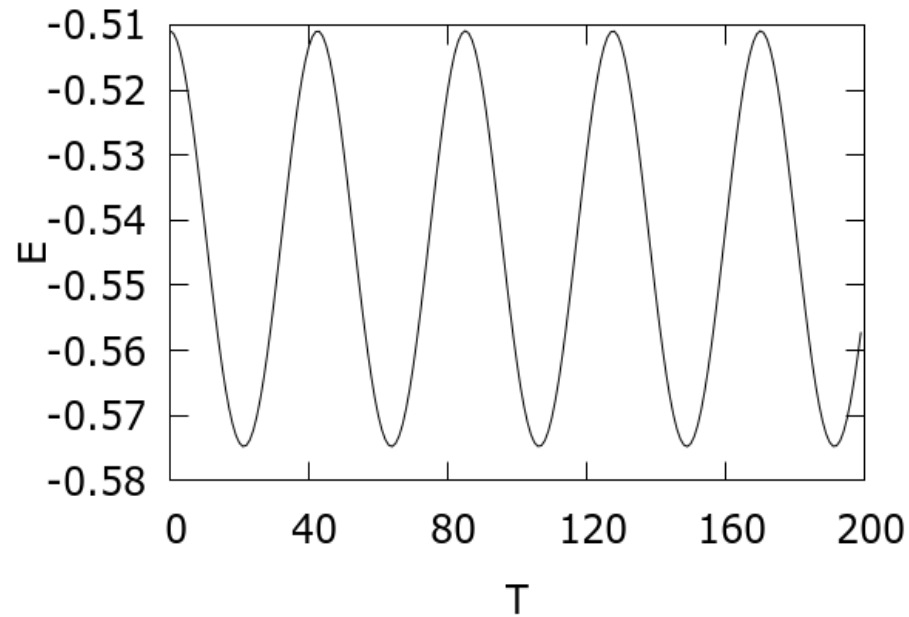
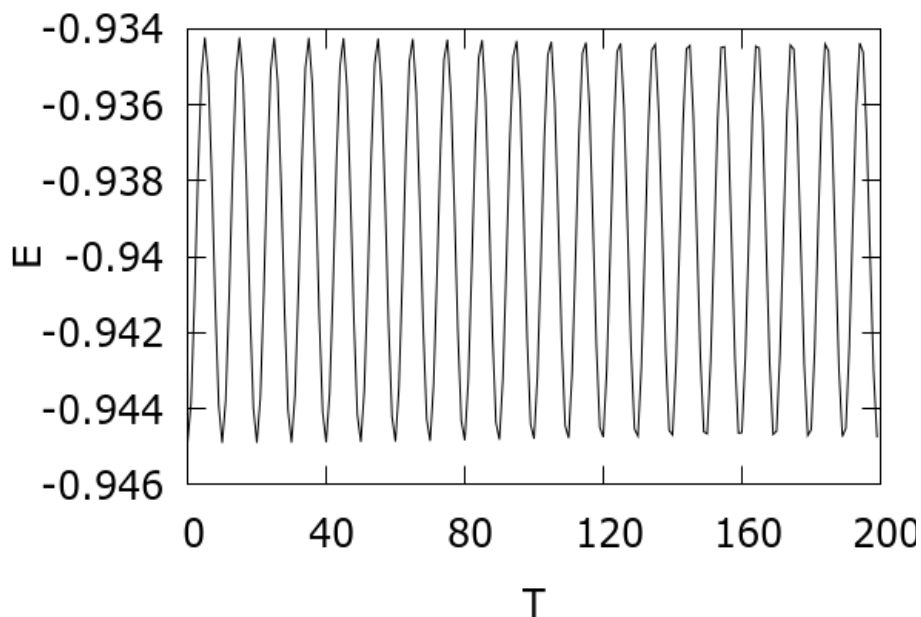
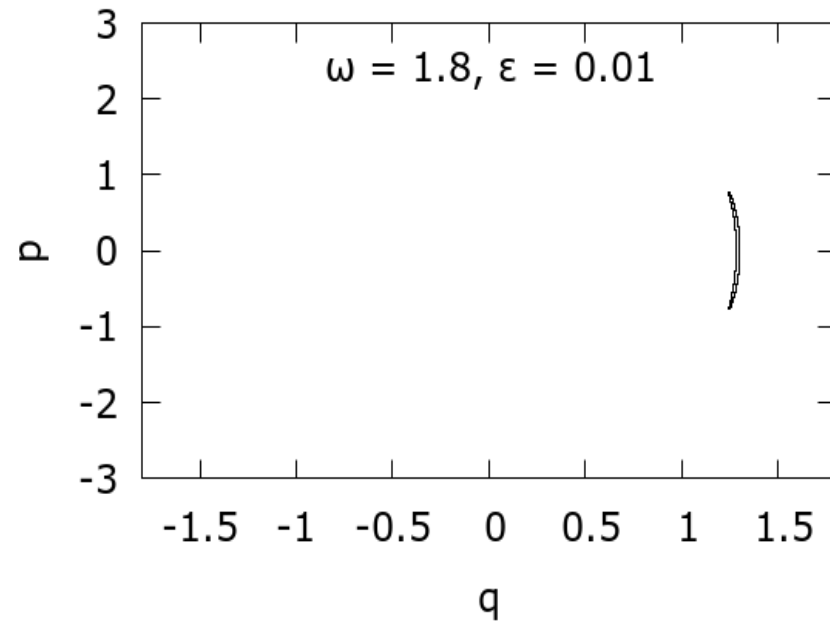
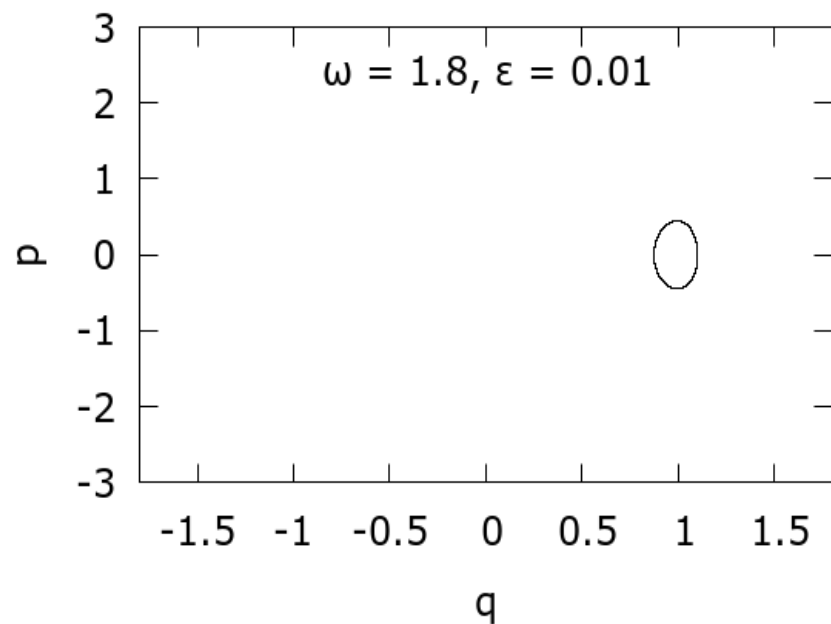
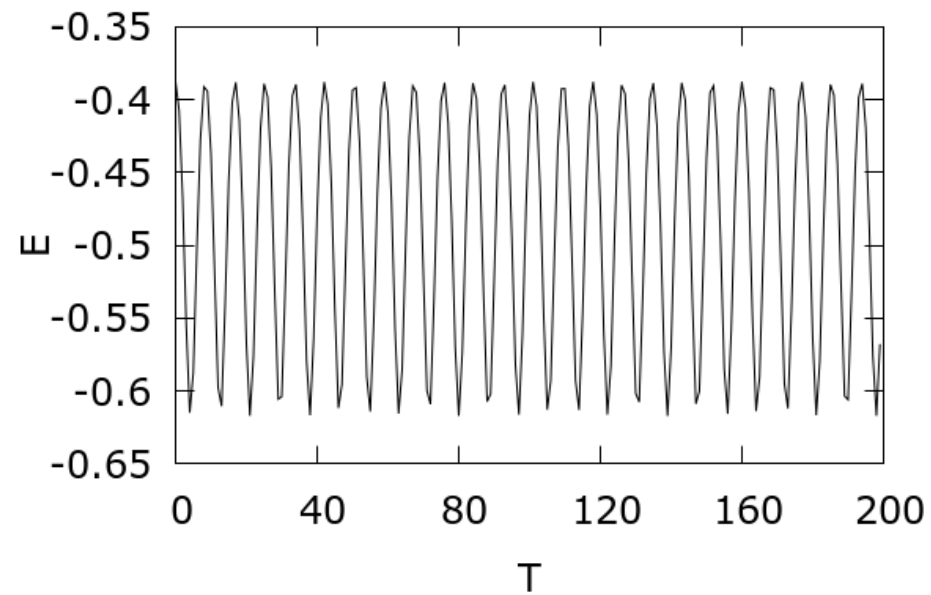
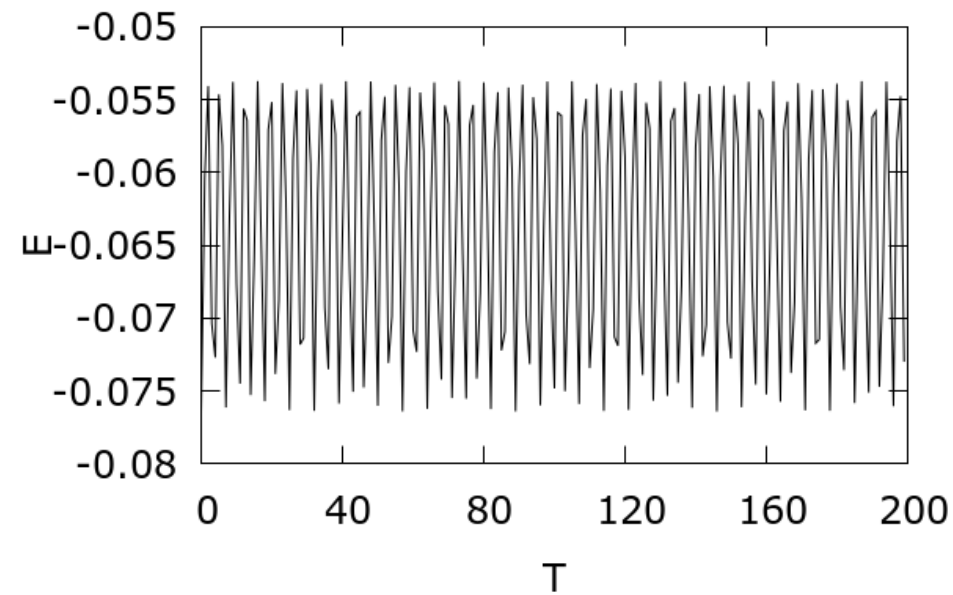
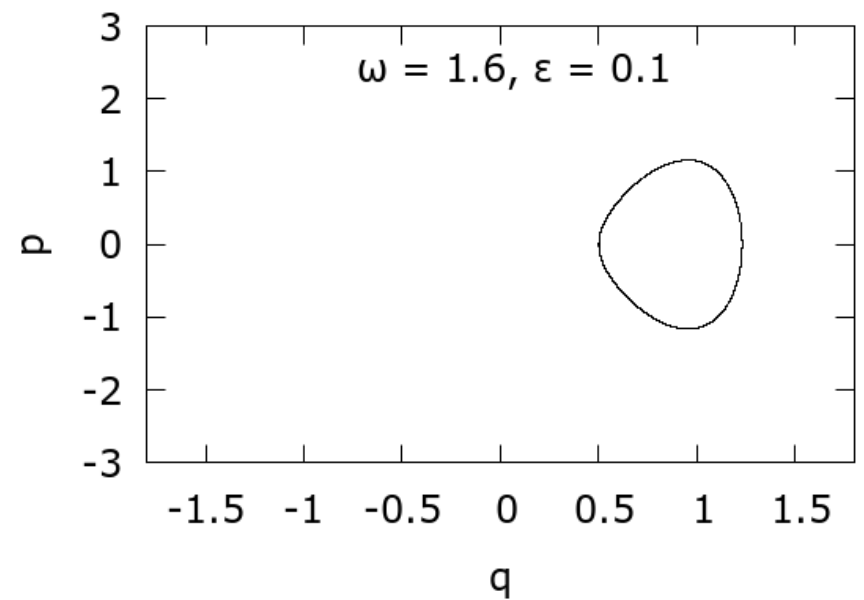
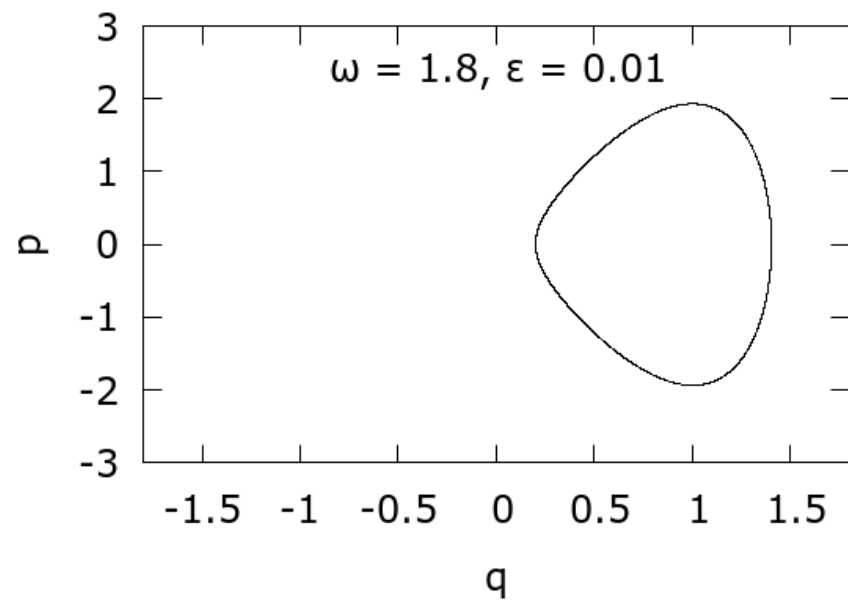


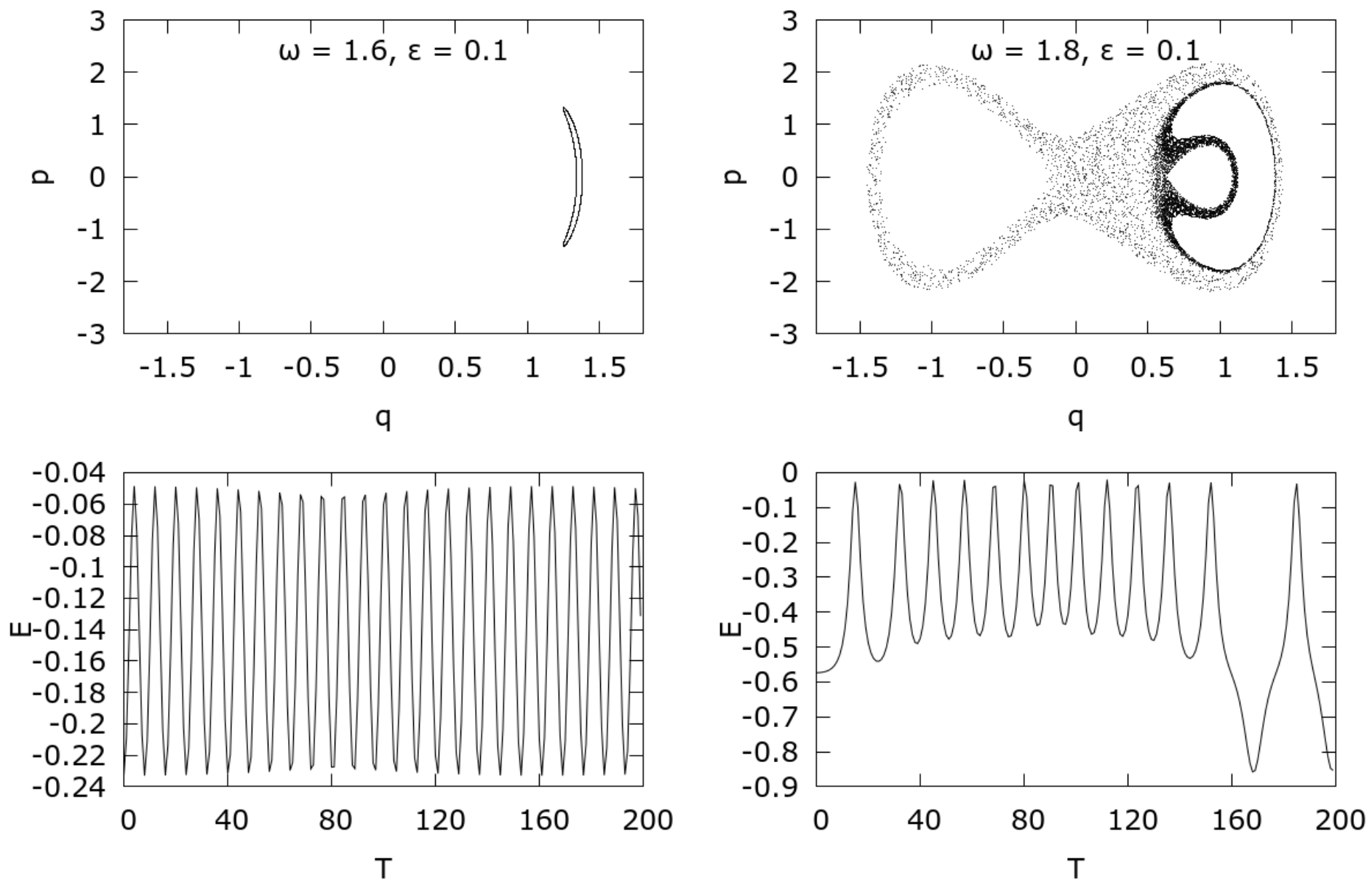
Figure 4: Technique invented by Poincaré to represent the phase space diagrams. The parallel planes are stroboscopic sections of the motion. The path pierces these planes at the points A_1 , A_2 , A_3 . (b) Points A_1 , A_2 , A_3 , projected on the plane (x, y) .



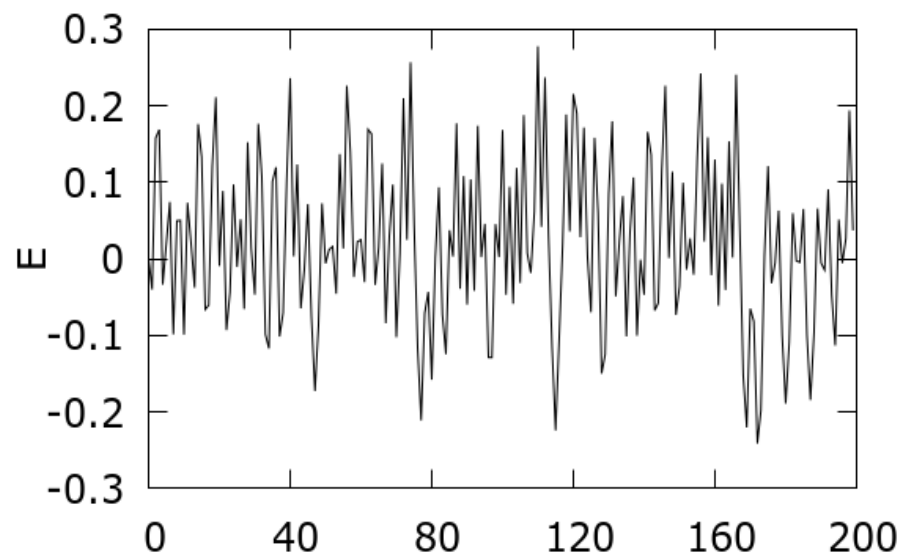
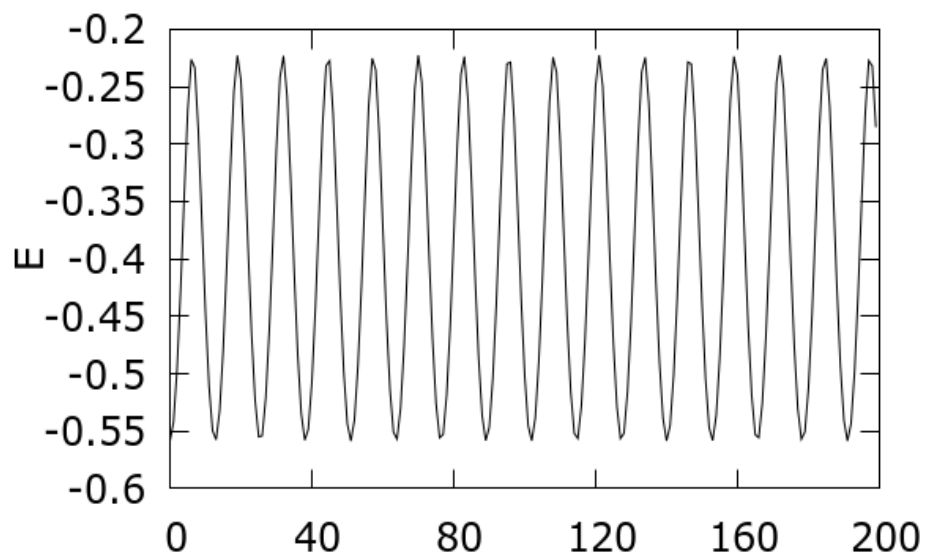
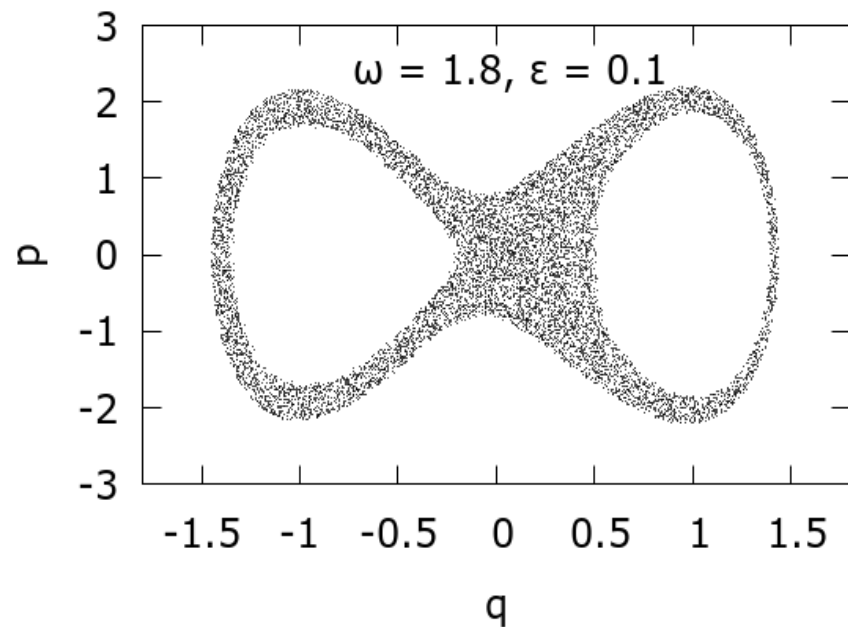
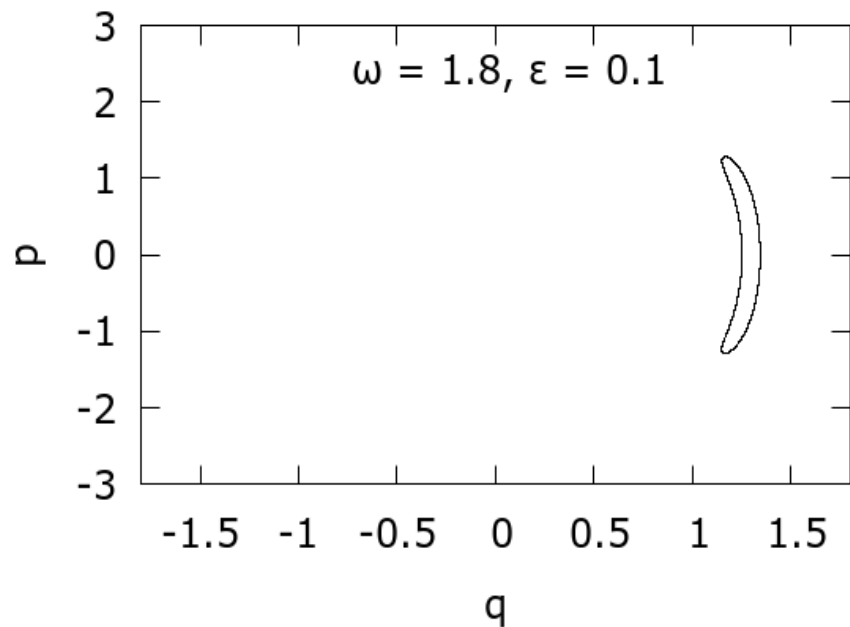
Figs. 3, 4: Strobe plots (x, p) for $\varepsilon = 0.01, \omega = 1.8$ (up). Corresponding energies for 200 cycles (down) Initial energies $E_0 = -0.94$ (left), -0.51 (right).



Figs. 5, 6: Strobe plots (x, p) for $\varepsilon = 0.01$, $\omega = 1.8$ and $\varepsilon = 0.1$, $\omega = 1.6$ (up). Corresponding energies for 200 cycles (down) Initial energies $E_0 = -0.076$ (left), -0.38 (right).



Figs. 7, 8: Strobe plots (x, p) for $\varepsilon = 0.1, \omega = 1.6$ and $\varepsilon = 0.1, \omega = 1.8$ (up). Corresponding energies for 200 cycles (down) Initial energies $E_0 = -0.23$ (left), -0.57 (right).



Figs. 9, 10: Strobe plots (x, p) for $\varepsilon = 0.1$, $\omega = 1.8$ (up). Corresponding energies E for 200 cycles (down) Initial energies $E_0 = -0.55$ (left), 0.0 (right).