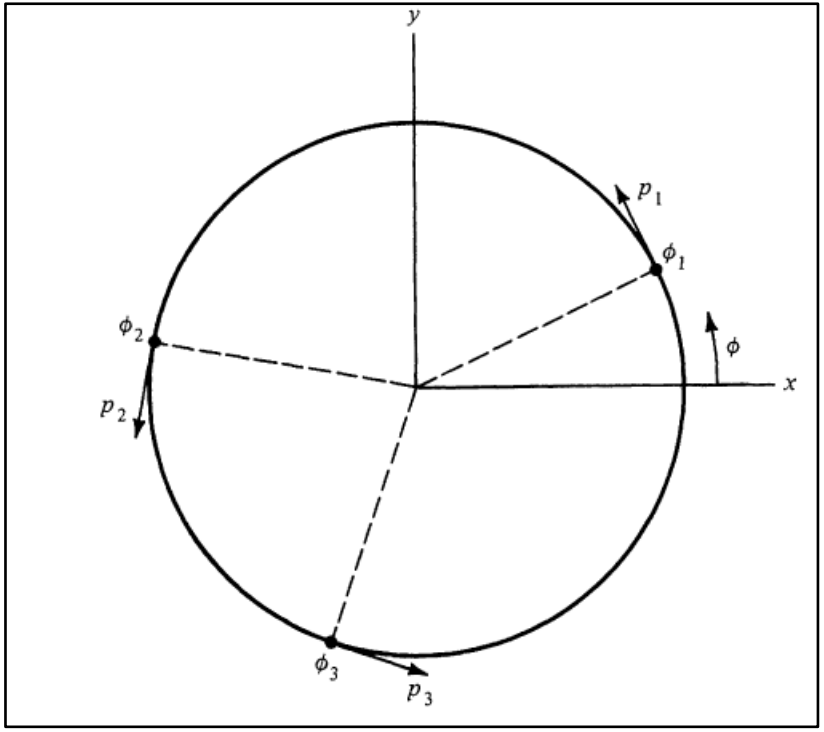
Integração numérica de sistemas Hamiltonianos

2023

Integração numérica de sistemas Hamiltonianos

- **Rede de Toda**
  - Três partículas se movimentando em um anel
  - Forças repulsivas exponencialmente decrescentes entre si



$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{-(\phi_1 - \phi_3)} + e^{-(\phi_2 - \phi_1)} + e^{-(\phi_3 - \phi_2)} - 3$$

Integração numérica de sistemas Hamiltonianos

- **Rede de Toda**

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{-(\phi_1 - \phi_3)} + e^{-(\phi_2 - \phi_1)} + e^{-(\phi_3 - \phi_2)} - 3$$

- Constantes de movimento

$$H \qquad\qquad P = p_1 + p_2 + p_3$$

Integração numérica de sistemas Hamiltonianos

- **Rede de Toda**

$$F_2 = P_1\phi_1 + P_2\phi_2 + (P_3 - P_1 - P_2)\phi_3$$

$$p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P} \qquad \mathcal{H} = H + \frac{\partial F_2}{\partial t}$$

$$\Downarrow$$

$$\mathcal{H} = \frac{1}{2}[P_1^2 + P_2^2 + (P_3 - P_1 - P_2)^2] + e^{-\Phi_1} + e^{-(\Phi_2 - \Phi_1)} + e^{\Phi_2} - 3$$

- **Rede de Toda**

$$F_2 = \frac{1}{4\sqrt{3}}[(p'_x - \sqrt{3}p'_y)\Phi_1 + (p'_x + \sqrt{3}p'_y)\Phi_2]$$

$$p'_x = 8\sqrt{3}p_x, \quad x' = x, \quad p'_y = 8\sqrt{3}p_y, \quad y' = y, \quad \bar{H} = \frac{H'}{\sqrt{3}}$$

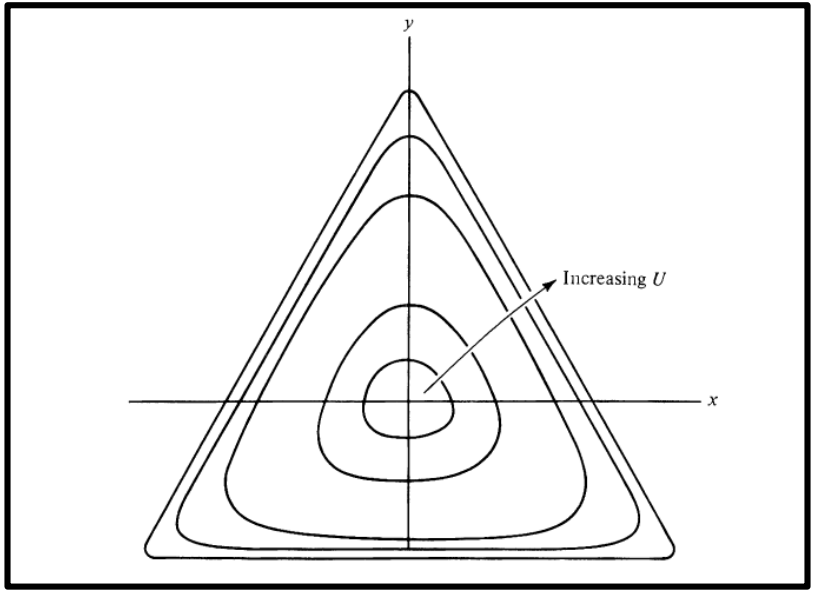
$$\Downarrow$$

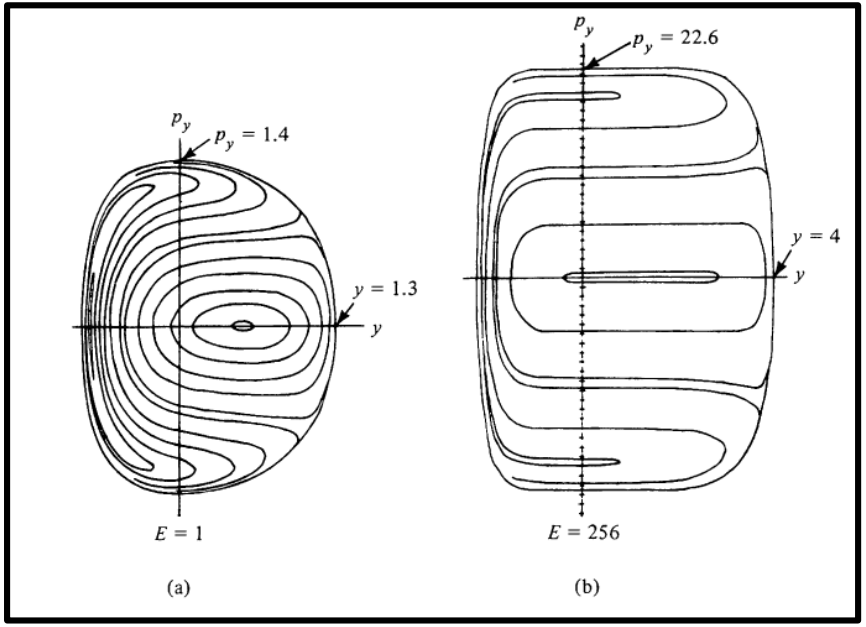
$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{24}\left(e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y}\right) - \frac{1}{8}$$

Integração numérica de sistemas Hamiltonianos

- **Rede de Toda**

$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{24}\left(e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y}\right) - \frac{1}{8}$$





Integração numérica de sistemas Hamiltonianos

- **Rede de Toda**

$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{24}\left(e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y}\right) - \frac{1}{8}$$

- Constantes de movimento

$$H \qquad\qquad P = p_1 + p_2 + p_3$$

$$I = 8p_x(p_x^2 - 3p_y^2) + (p_x + \sqrt{3}p_y)e^{2y-2\sqrt{3}x} - 2p_x e^{-4y} + (p_x - \sqrt{3}p_y)e^{2y+2\sqrt{3}x}$$

Integração numérica de sistemas Hamiltonianos

- **Rede de Toda**

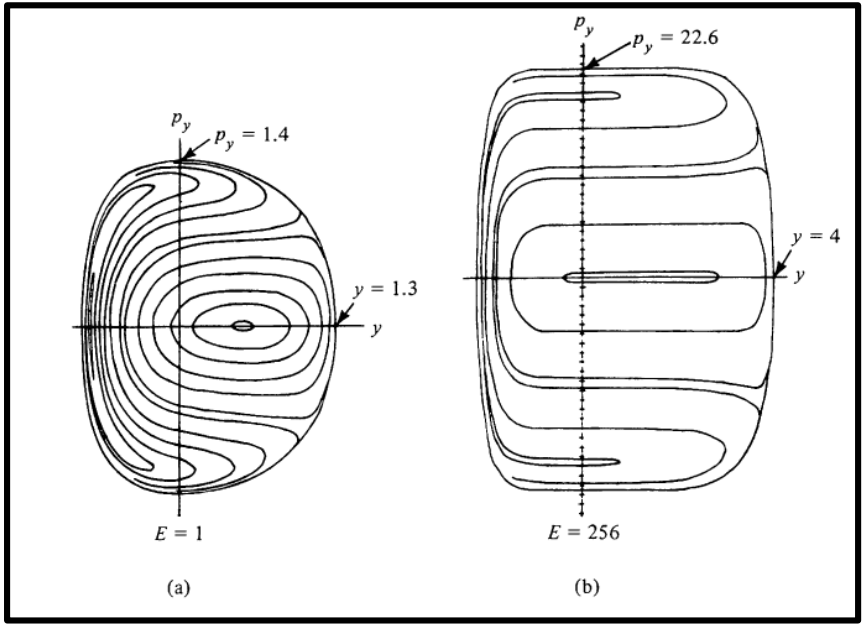
$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{24}\left(e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y}\right) - \frac{1}{8}$$

- 3 graus de liberdade
- 3 constantes de movimento

Sistema Integrável

⇓

Soluções regulares



Integração numérica de sistemas Hamiltonianos

- **Rede de Toda**

$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{24}\left(e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y}\right) - \frac{1}{8}$$

- Expandindo a Hamiltoniana e mantendo apenas os termos quadrados e cúbicos (exercício 10 da Lista 1)

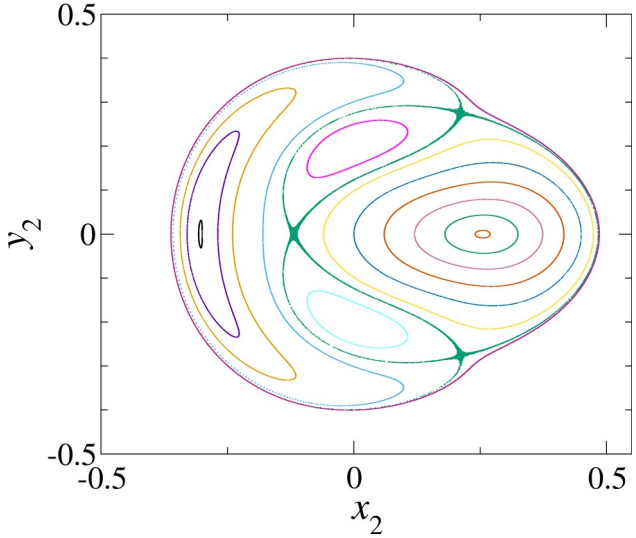
$$\bar{H}' = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + x^2 y - \frac{y^3}{3}$$

Hamiltoniana de Hénon-Heiles

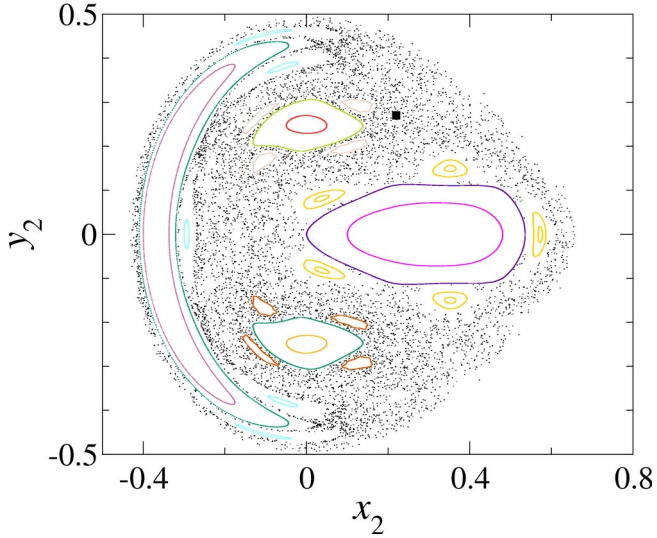
Integração numérica de sistemas Hamiltonianos

- **Hamiltoniana de Hénon-Heiles**

$$\bar{H}' = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + x^2 y - \frac{y^3}{3}$$



$E = 0.08333$



$E = 0.125$

- **Hamiltoniana de Hénon-Heiles**
  - Sistema não integrável
  - Comportamento caótico possível
- **Rede de Toda**
  - Sistema integrável
    - Simetria I
  - Apenas soluções regulares

- **Hamiltoniana de Hénon-Heiles**

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

$$\bar{H}' = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + x^2 y - \frac{y^3}{3}$$

$$\frac{dx}{dt} = p_x, \quad \frac{dp_x}{dt} = -x - 2xy$$

$$\frac{dy}{dt} = p_y, \quad \frac{dp_y}{dt} = -y^2 - y - x^2$$

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

- **Rede de Toda**

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{-(\phi_1 - \phi_3)} + e^{-(\phi_2 - \phi_1)} + e^{-(\phi_3 - \phi_2)} - 3$$

$$\frac{d\phi_1}{dt} = p_1, \quad \frac{dp_1}{dt} = e^{-(\phi_1 - \phi_3)} - e^{-(\phi_2 - \phi_1)}$$

$$\frac{d\phi_2}{dt} = p_2, \quad \frac{dp_2}{dt} = e^{-(\phi_2 - \phi_1)} - e^{-(\phi_3 - \phi_2)}$$

$$\frac{d\phi_3}{dt} = p_3, \quad \frac{dp_3}{dt} = e^{-(\phi_3 - \phi_2)} - e^{-(\phi_1 - \phi_3)}$$

Integração numérica

- Método de Euler

$$\dot{\mu} = \frac{d\mu}{dt} = f(\mu)$$

$$\mu = (\mu_1, \mu_2, ..., \mu_r) \quad \text{e} \quad f = (f_1, f_2, ..., f_r)$$

Integração numérica

- Método de Euler

$$\dot{\mu} = \frac{d\mu}{dt} = f(\mu) \quad \Rightarrow \quad \mu^{(n+1)} = \mu^{(n)} + f(\mu^{(n)})\Delta t$$

$$n \in \mathbb{Z},\ \mu^{(n)} = \mu(t_n),\ t_n = n\Delta t$$

Integração numérica

- Método de Euler

$$\dot{\mu} = \dot{q},\ \dot{p}$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$\Rightarrow$$

$$q^{(n+1)} = q^{(n)} + \left.\frac{\partial H}{\partial p}\right|_{(q^{(n)},p^{(n)})} \Delta t$$

$$p^{(n+1)} = p^{(n)} - \left.\frac{\partial H}{\partial q}\right|_{(q^{(n)},p^{(n)})} \Delta t$$

Integração numérica

- Método de Euler

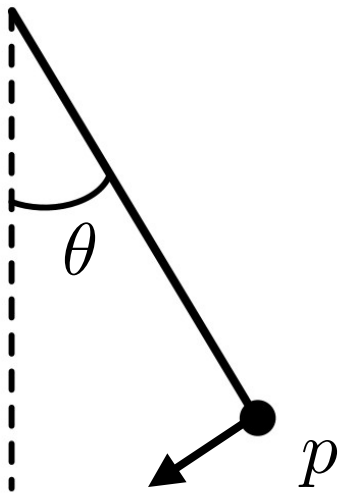
$$H = H(q, p, t)$$

$$q^{(n+1)} = q^{(n)} + \Delta t \left.\frac{\partial H}{\partial p}\right|_{(q^{(n)}, p^{(n)})}$$

$$p^{(n+1)} = p^{(n)} - \Delta t \left.\frac{\partial H}{\partial q}\right|_{(q^{(n)}, p^{(n)})}$$

Integração numérica

- Método de Euler - Exemplo



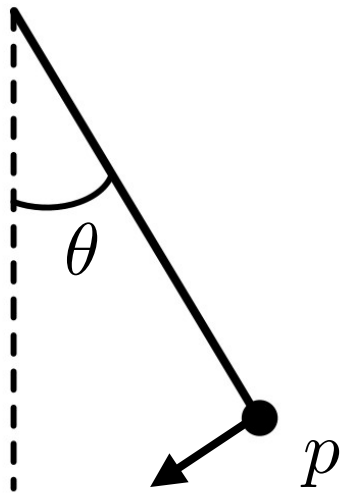
$$H = \frac{p^2}{2} - \cos\theta$$

$$\theta^{(n+1)} = \theta^{(n)} + p^{(n)}\Delta t$$

$$p^{(n+1)} = p^{(n)} - \sin\theta^{(n)}\Delta t$$

Integração numérica

- Método de Euler - Exemplo



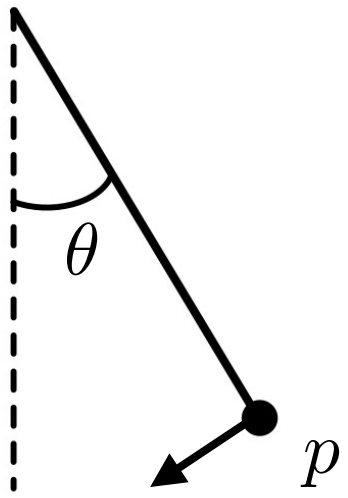
$$\theta^{(n+1)} = \theta^{(n)} + p^{(n)} \Delta t$$

$$p^{(n+1)} = p^{(n)} - \sin \theta^{(n)} \Delta t$$

$$\boxed{\theta^{(0)},\ p^{(0)},\ \Delta t}$$

Integração numérica

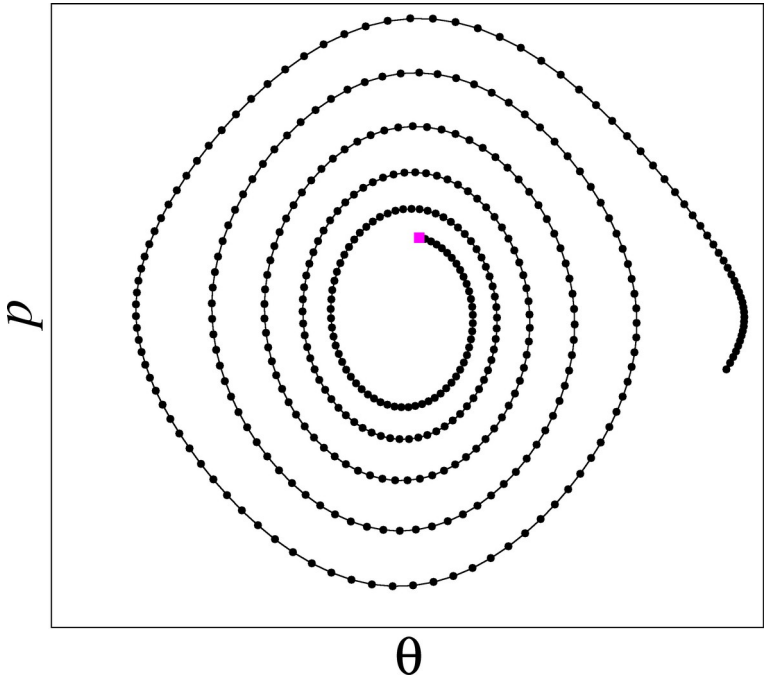
- Método de Euler - Exemplo



$$\theta^{(0)} = 0.1$$

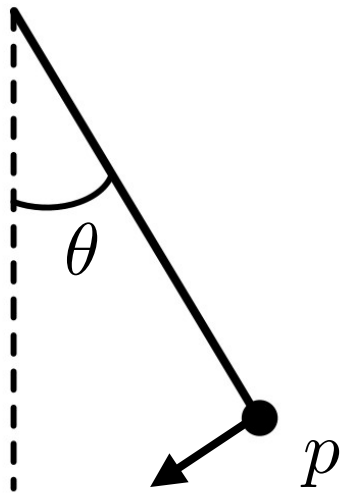
$$p^{(0)} = 0.5$$

$$\boxed{\Delta t = 10^{-1}}$$



Integração numérica

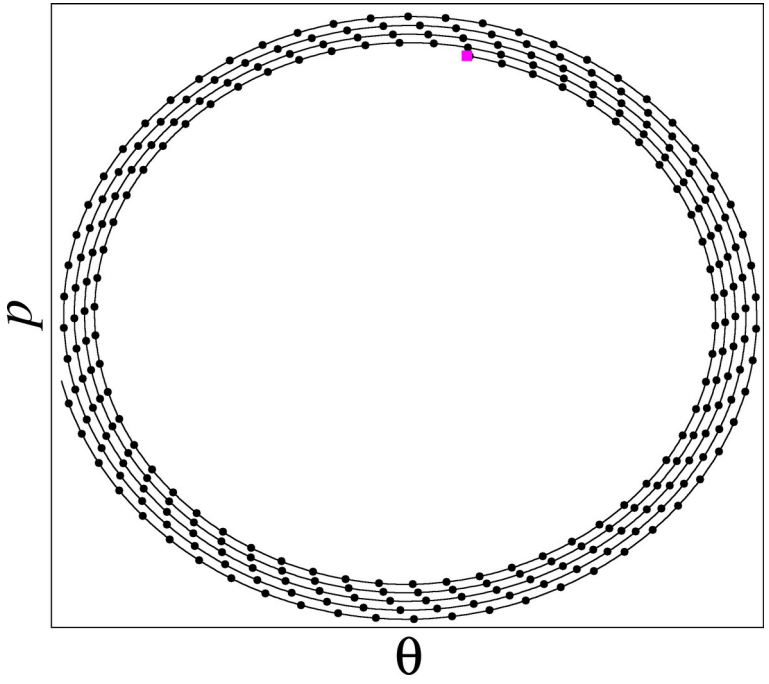
- Método de Euler - Exemplo



$$\theta^{(0)} = 0.1$$

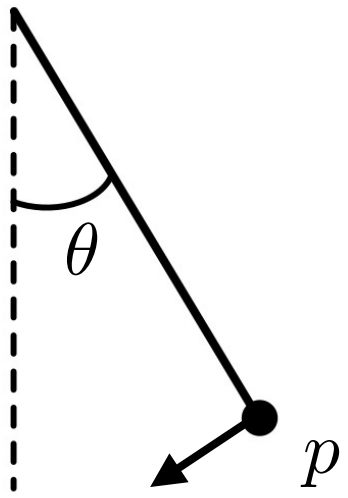
$$p^{(0)} = 0.5$$

$$\boxed{\Delta t = 10^{-2}}$$



Integração numérica

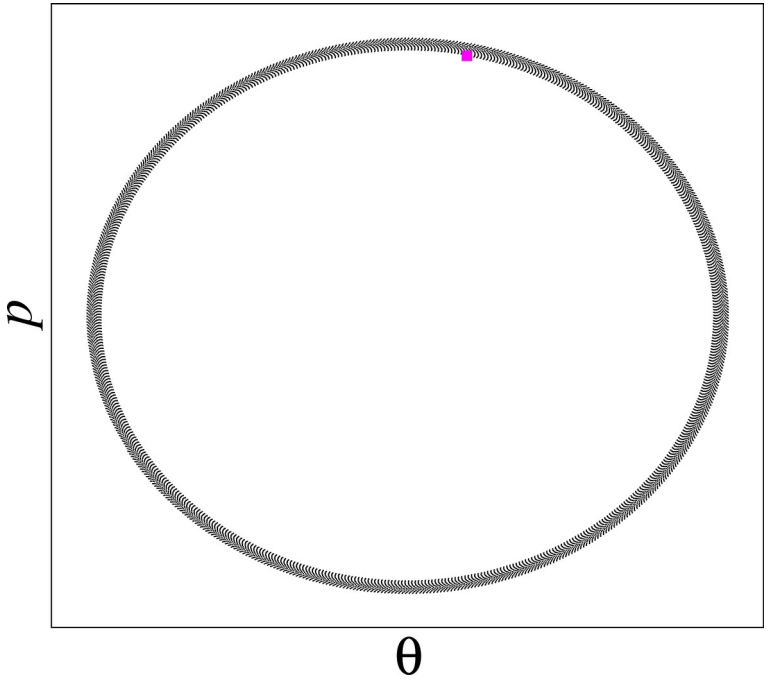
- Método de Euler - Exemplo



$$\theta^{(0)} = 0.1$$

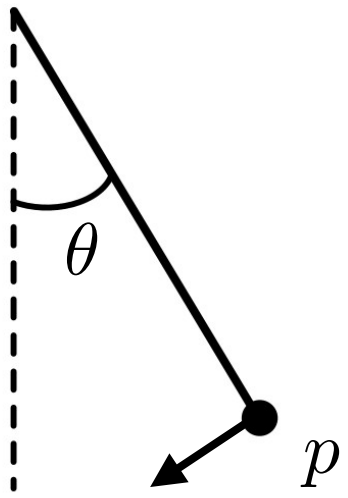
$$p^{(0)} = 0.5$$

$$\boxed{\Delta t = 10^{-3}}$$



Integração numérica

- Método de Euler - Exemplo



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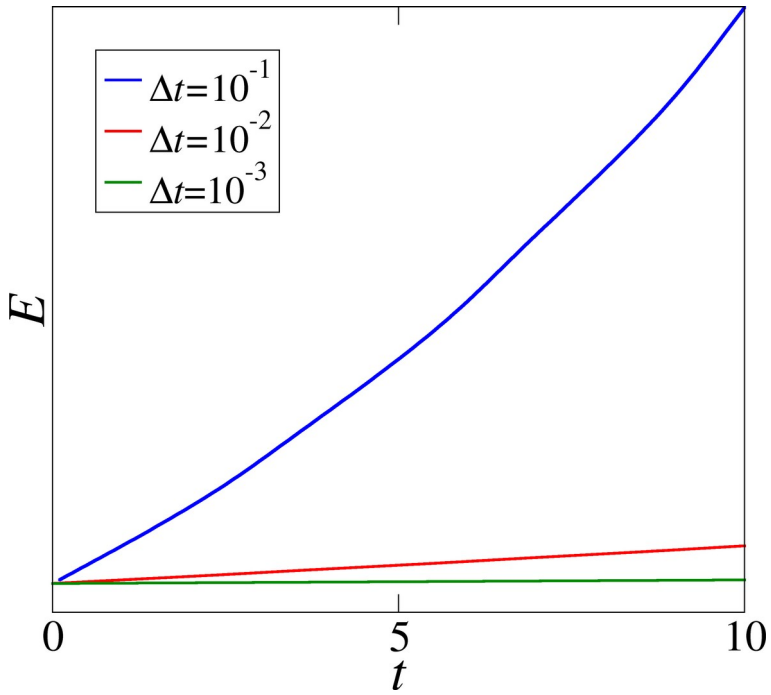
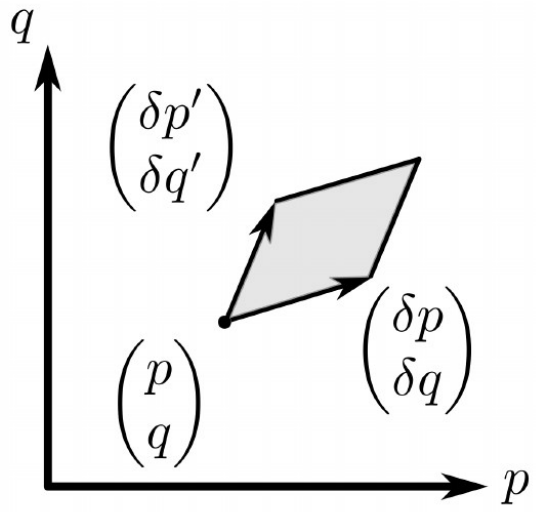


Integração numérica

- Método simplético de integração
  - Conservação da área simplética
  - Conservação da energia



Integração numérica

- Método simplético de integração

$$H = H(q, p, t)$$

$$q^{(n+1)} = q^{(n)} + \Delta t \left.\frac{\partial H}{\partial p}\right|_{\boxed{(q^{(n+1)}, p^{(n)})}}$$

$$p^{(n+1)} = p^{(n)} - \Delta t \left.\frac{\partial H}{\partial q}\right|_{\boxed{(q^{(n+1)}, p^{(n)})}}$$

Integração numérica

- Método de Euler simplético

$$q^{(n+1)} = q^{(n)} + \Delta t \left.\frac{\partial H}{\partial p}\right|_{\boxed{(q^{(n+1)}, p^{(n)})}}$$

$$p^{(n+1)} = p^{(n)} - \Delta t \left.\frac{\partial H}{\partial q}\right|_{\boxed{(q^{(n+1)}, p^{(n)})}}$$

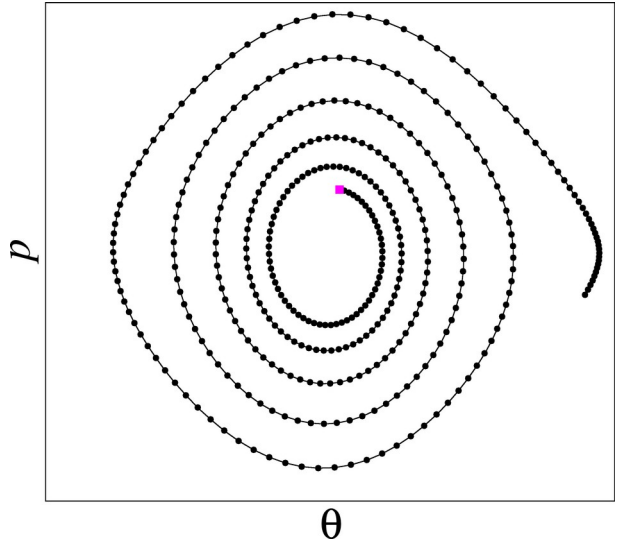
$$(p_n, q_n) \to (p_{n+1}, q_{n+1})$$

Transformação canônica

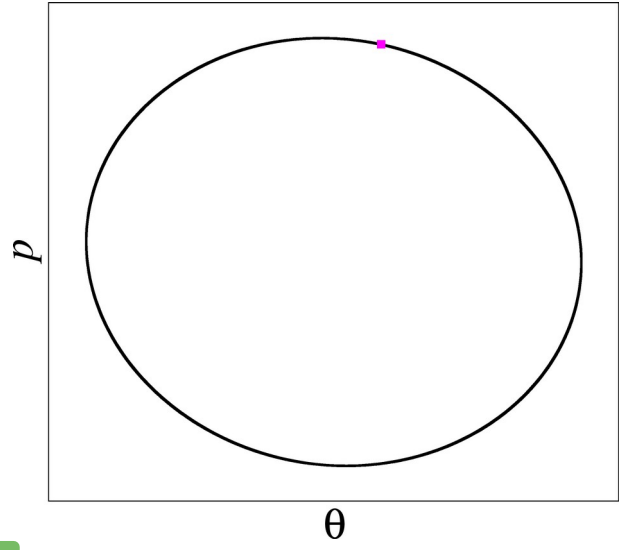
$$F_3(p_n, q_{n+1}) = -p_n q_{n+1} + hH(p_n, q_{n+1})$$

Integração numérica

- Método de Euler



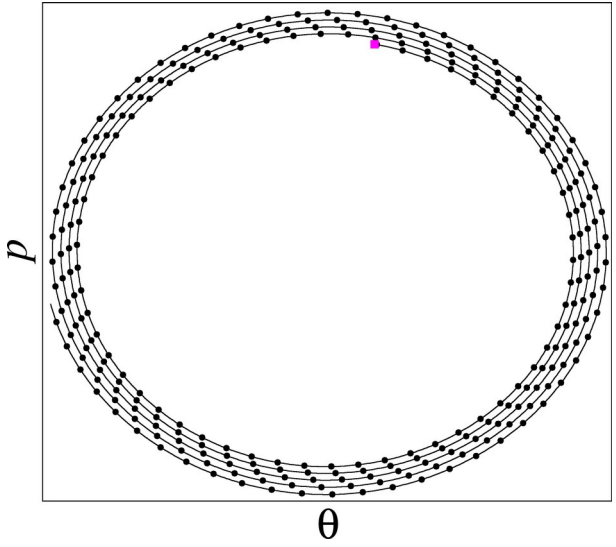
- Método de Euler simplético



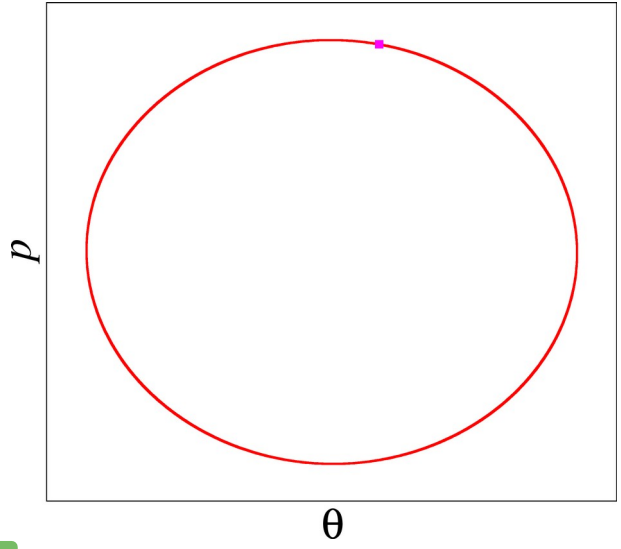
$$\Delta t = 10^{-1}$$

Integração numérica

- Método de Euler



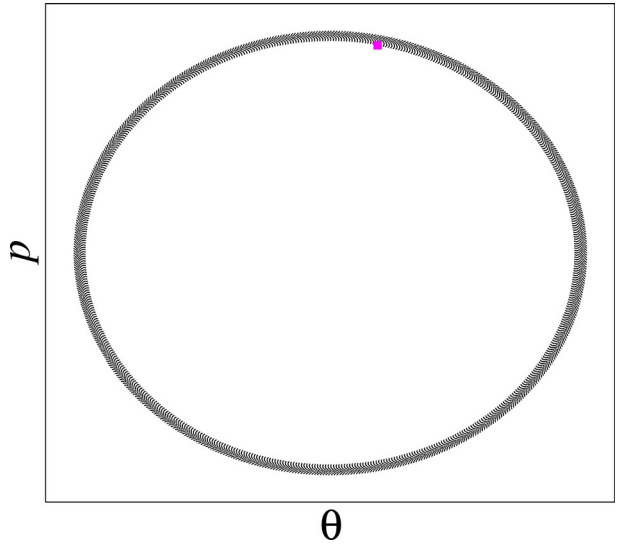
- Método de Euler simplético



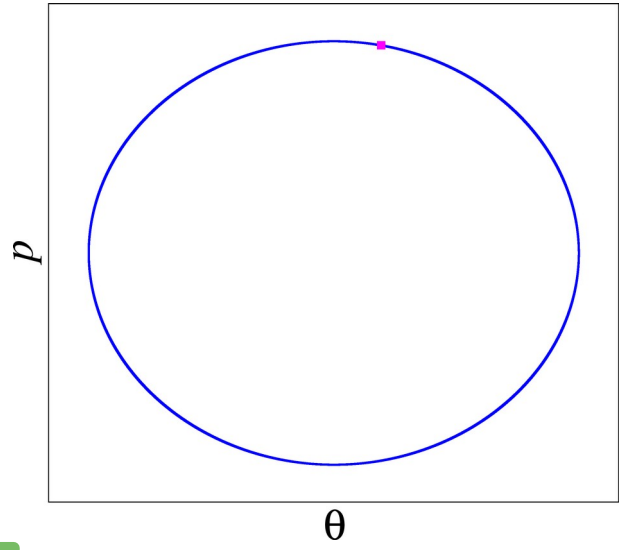
$$\Delta t = 10^{-2}$$

Integração numérica

- Método de Euler



- Método de Euler simplético



$$\Delta t = 10^{-3}$$

Integração numérica de sistemas Hamiltonianos

- **Hamiltoniana de Hénon-Heiles**

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + q_1^2 q_2 - \frac{1}{3} q_2^3$$

$$H = H(q, p, t)$$

$$q^{(n+1)} = q^{(n)} + \Delta t \left.\frac{\partial H}{\partial p}\right|_{(q^{(n+1)}, p^{(n)})}$$

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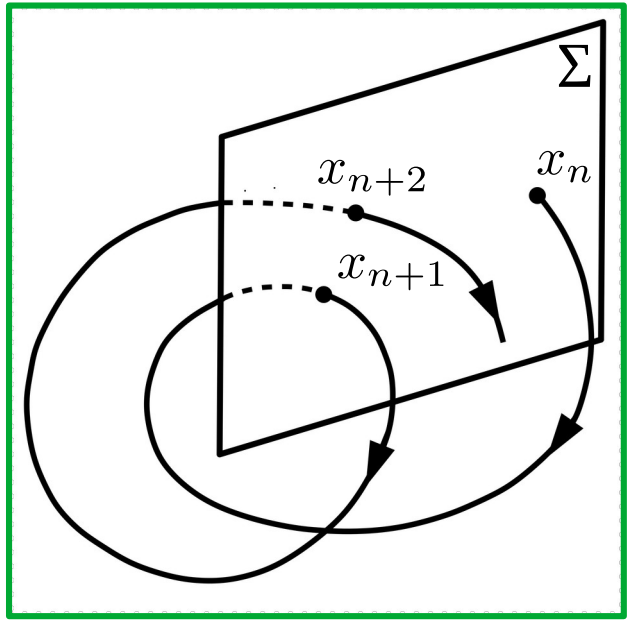
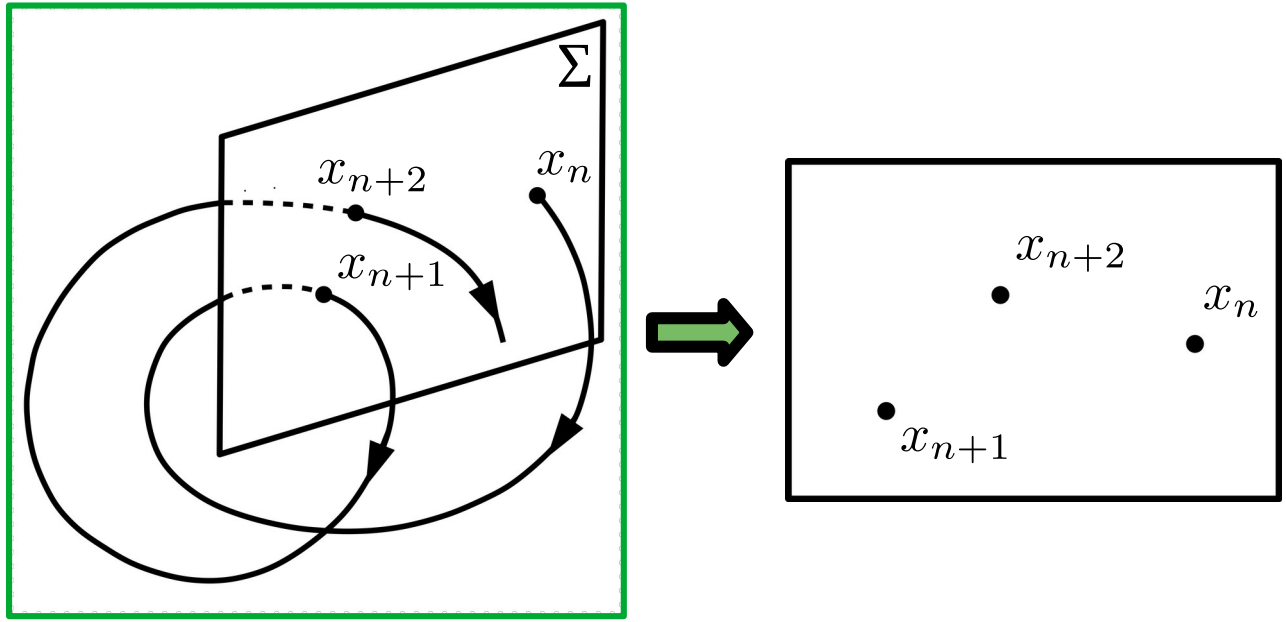
Integração numérica de sistemas Hamiltonianos

- Seção de Poincaré
  - Cálculo das interseções entre as trajetórias e uma superfície Σ



Integração numérica de sistemas Hamiltonianos

- Seção de Poincaré
  - Cálculo das interseções entre as trajetórias e uma superfície Σ



Integração numérica de sistemas Hamiltonianos

- Seção de Poincaré – Método de Hénon

$$\frac{d\mu_1}{dt} = f_1(\mu_1, \mu_2, ..., \mu_r)$$

$$\frac{d\mu_2}{dt} = f_2(\mu_1, \mu_2, ..., \mu_r)$$

$$\vdots$$

$$\frac{d\mu_r}{dt} = f_1(\mu_1, \mu_2, ..., \mu_r)$$

Sendo a superfície Σ definina pela variável dinâmica

$$\mu_r = \mu_r^s$$

Integração numérica de sistemas Hamiltonianos

- Seção de Poincaré – Método de Hénon

$$\frac{d\mu_1}{dt} = f_1(\mu_1, \mu_2, ..., \mu_r)$$

$$\frac{d\mu_2}{dt} = f_2(\mu_1, \mu_2, ..., \mu_r)$$

$$\vdots$$

$$\frac{d\mu_r}{dt} = f_1(\mu_1, \mu_2, ..., \mu_r)$$

Sendo a superfície Σ definina pela variável dinâmica

$$\mu_r = \mu_r^s$$

Uma trajetória atravessa a superfície Σ, em algum ponto entre $\mu^{(n)}$ e $\mu^{(n+1)}$ se

$$\left(\mu_r^{(n+1)} - \mu_r^s\right)\left(\mu_r^{(n)} - \mu_r^s\right) < 0$$

Integração numérica de sistemas Hamiltonianos

- Seção de Poincaré – Método de Hénon

Passo de integração

$$\Delta\mu_r = \mu_r^s - \mu_r^{(n)}$$