# II- Mapas Bidimensionais

Referência: *Chaos*, K. Alligood, T. D. Sauer, J. A. Yorke; Springer (1997).

# 1 - Novas Características Dinâmicas

- Além de pontos fixos, há pontos de selas.
- Ponto de sela: contração em uma direção e expansão na outra.

- Mapa de Poincaré bidimensional de uma órbita tridimensional.

# Atratores, Repulsores e Pontos de Sela

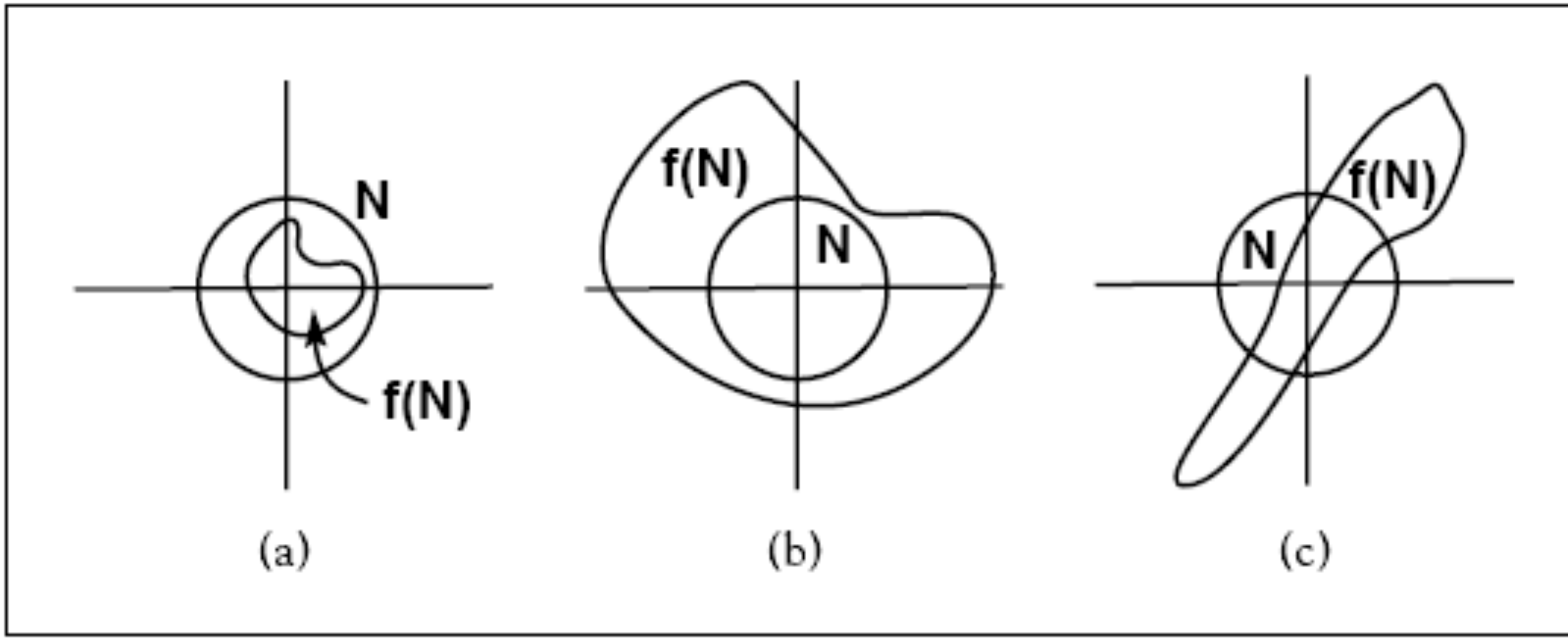



**Figure 2.8 Local dynamics near a fixed point.**
The origin is (a) a sink, (b) a source, and (c) a saddle. Shown is a disk $N$ and its iterate under the map $\mathbf{f}$.

(Alligood et al.
Chaos...)

# 2 - Mapa de Hénon

Hénon (Comm. Math. 50, 69, 1976) introduziu o mapa

$(x_{n+1}, y_{n+1}) = f\,(x_n, y_n) = (a - x^2 + b\,y,\ x)$

a, b : parâmetros de controle

Para a = 1,28 e b = -0,3 e $(x_0, y_0) = (0, 0)$,

trajetória converge para órbita com período 2

Os pontos iniciais convergem para essa orbita ou para $x \to \infty$

Bacias de atração variam com a, b

Para a = 1,28 e b = -0,3 ; fronteira das bacias é contínua

Para a = 1,40 e b = -0,3 ; fronteira das bacias é fractal

# Mapa de Hénon
# b = -0.3

a = 1.28

a= 1.4

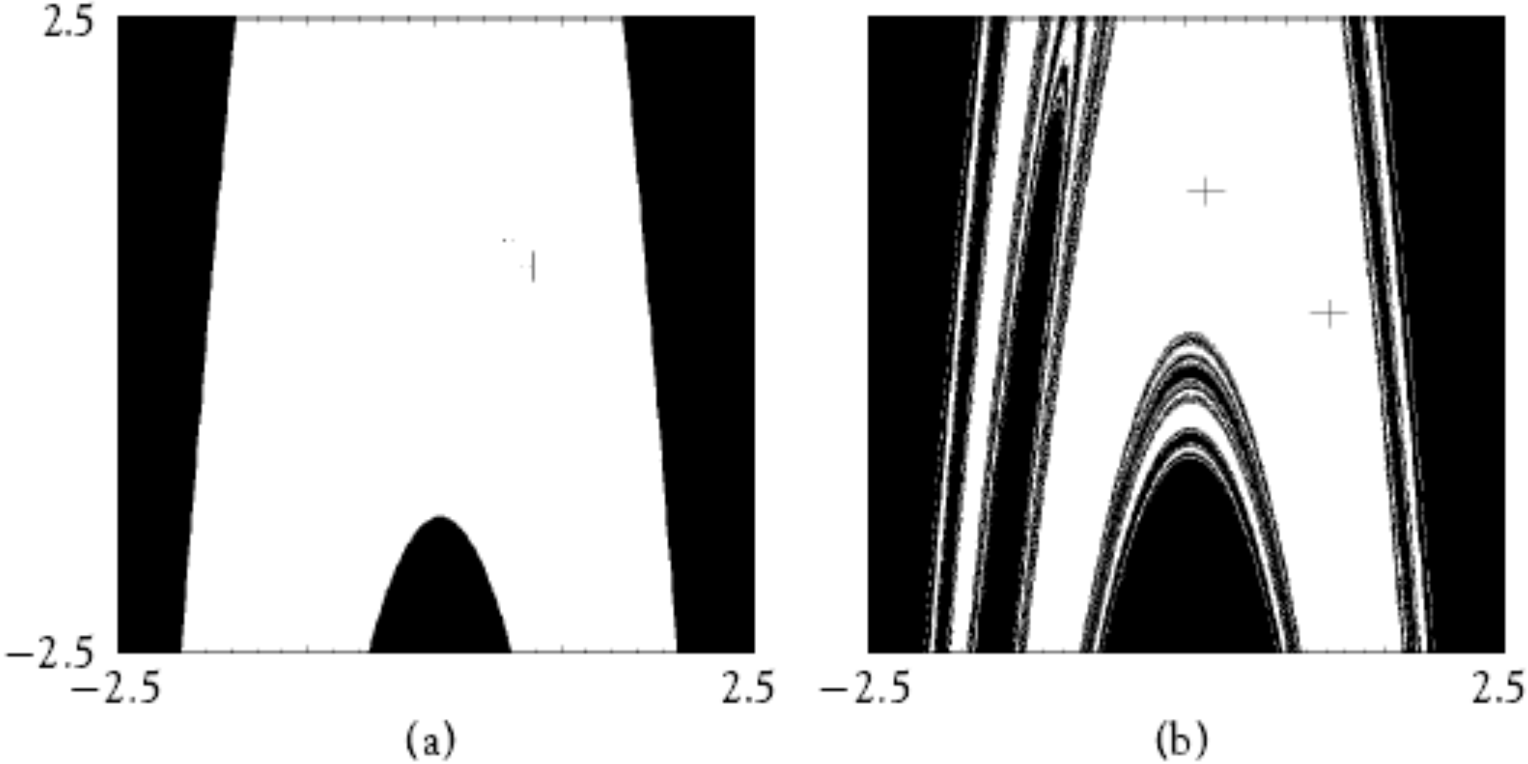



**Figure 2.3 A square of initial conditions for the Hénon map with $b = -0.3$.** Initial values whose trajectories diverge to infinity upon repeated iteration are colored black. The crosses show the location of a period-two sink, which attracts the white initial values. (a) For $a = 1.28$, the boundary of the basin is a smooth curve. (b) For $a = 1.4$, the boundary is "fractal".

(Alligood et al. Chaos...)

# 4 - Definições de Atratores e Repulsores

Definição: O comprimento (euclidiano) de um vetor

$\vec{u} = (x,\ y)$ em $R^2$ é $|\vec{u}| = \sqrt{x^2 + y^2}$.

A vizinhançã é $N_\varepsilon(\vec{p}) : \{\vec{v} \in R^2 : |\vec{v} - \vec{p}| < \varepsilon\}$

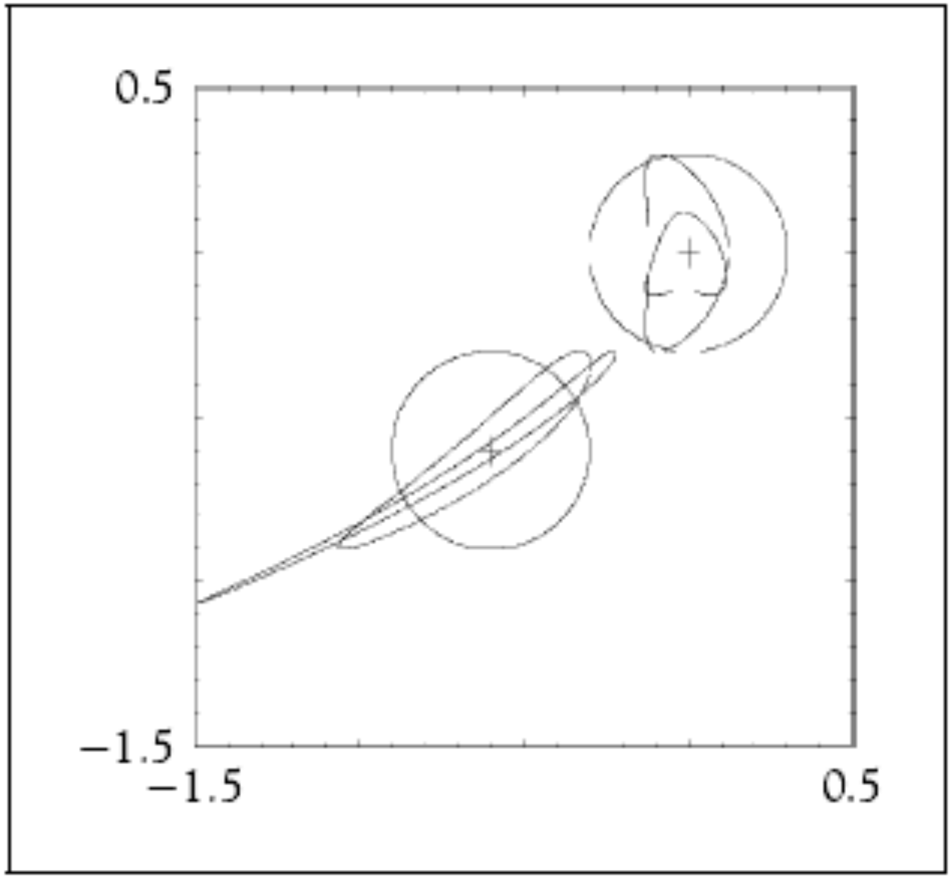
Seja $\vec{f}$ um mapa em $R^2$, $\vec{p}$ um ponto fixo, $\vec{f}(\vec{p}) = \vec{p}$

Definições:

$\vec{p}$ é um atrator se $\exists\ \varepsilon > 0$ tal que $\vec{v} \subset N_\varepsilon(\vec{p}) \Rightarrow \lim\limits_{k \to \infty} \vec{f}^{\,k}(\vec{v}) = \vec{p}$

$\vec{p}$ é um repulsor se $\exists\ \varepsilon > 0$ tal que $\vec{v} \subset N_\varepsilon(\vec{p}) \Rightarrow \lim\limits_{k \to \infty} \vec{f}^{\,k}(\vec{v}) \not\subset N_\varepsilon(\vec{p})$

# Atrator e Ponto de Sela no Mapa de Hénon




a = 0
b = 0.4

**Figure 2.9 Local dynamics near fixed points of the Hénon map.** The crosses mark two fixed points of the Hénon map f with $a = 0$, $b = 0.4$, in the square $[-1.5, 0.5] \times [-1.5, 0.5]$. Around each fixed point a circle is drawn along with its two forward images under f. On the left is a saddle: the images of the disk are becoming increasingly long and thin. On the right the images are shrinking, signifying a sink.

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# Bacias de atração para o Mapa de Hénon

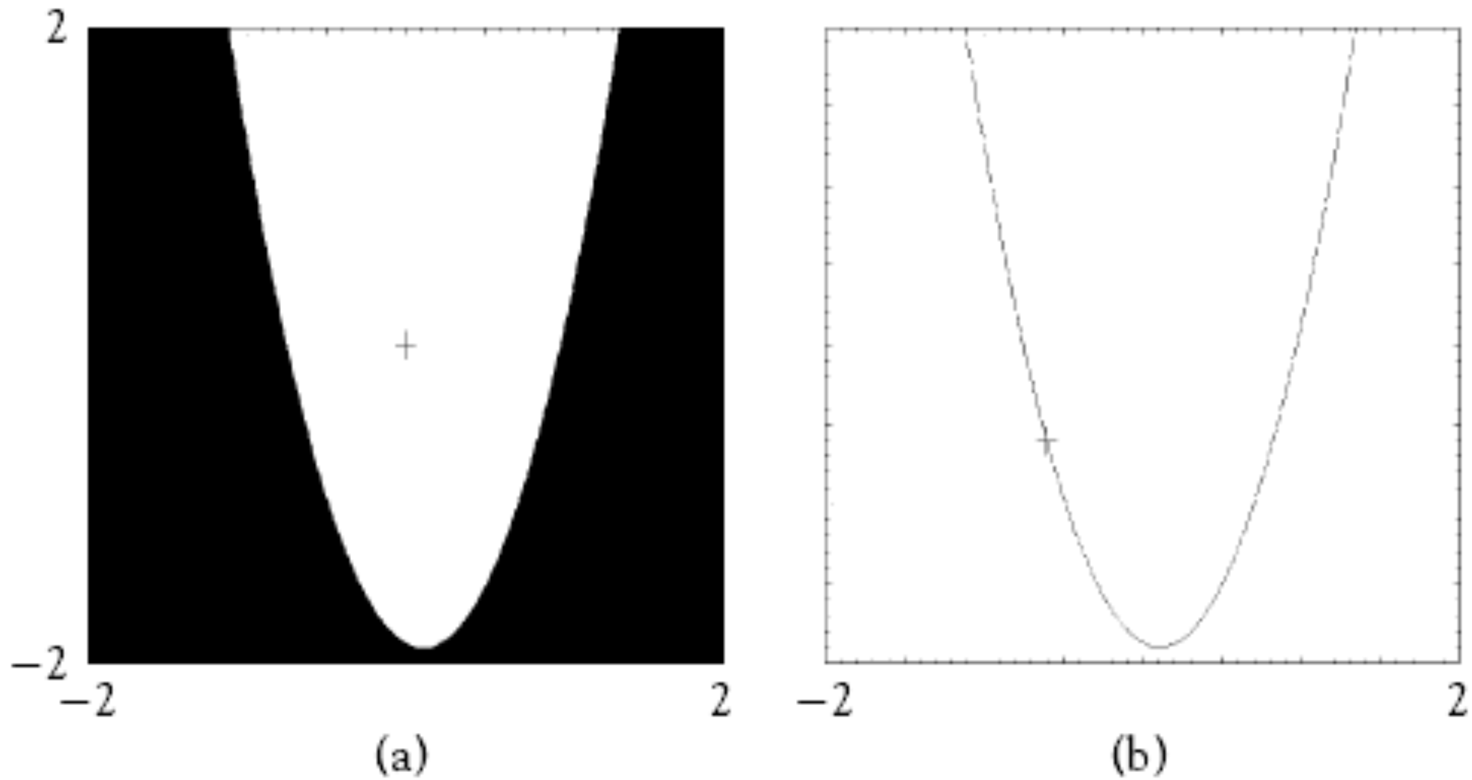


**Figure 2.10 Basins of attraction for the Hénon map with $a = 0, b = 0.4$.**
(a) The cross marks the fixed point $(0, 0)$. The basin of the fixed point $(0, 0)$ is shown in white; the points in black diverge to infinity. (b) The initial conditions that are on the boundary between the white and black don't converge to $(0, 0)$ or infinity; instead they converge to the saddle $(-0.6, -0.6)$, marked with a cross. This set of boundary points is the stable manifold of the saddle (to be discussed in Section 2.6).

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# Trator Caótico para o Mapa de Hénon

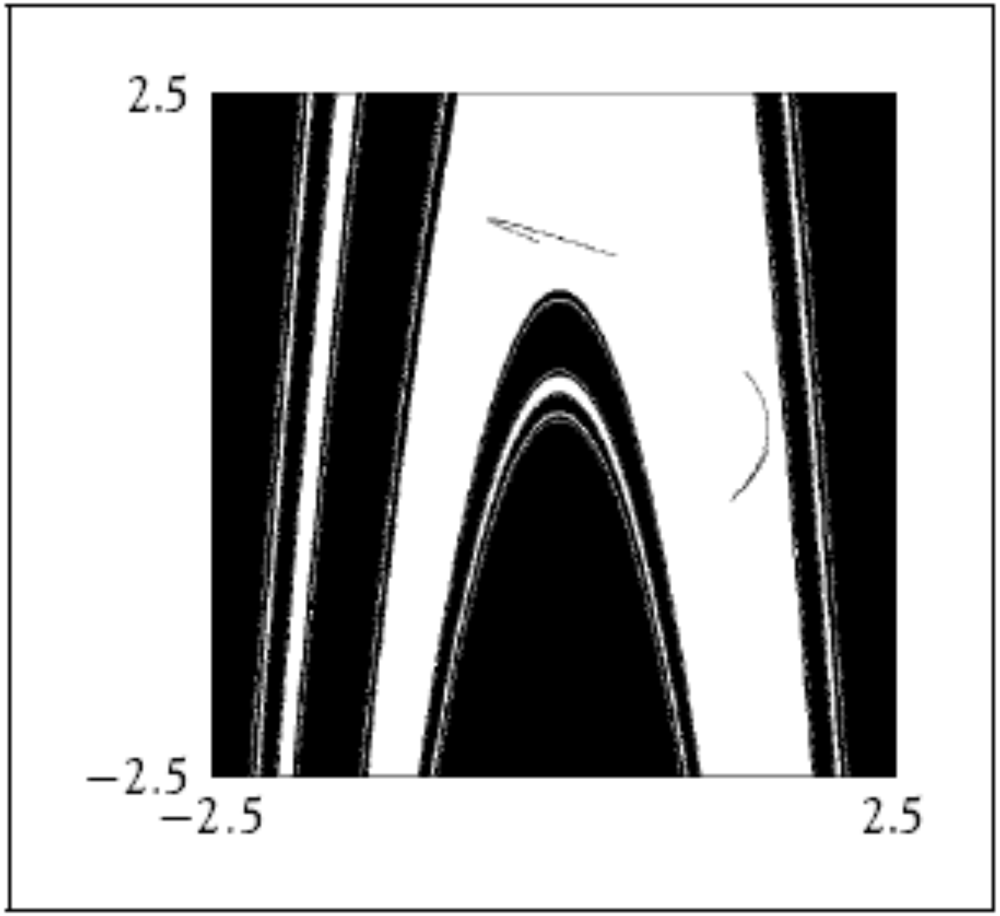


**Figure 2.11 Attractors for the Hénon map with $a = 2$, $b = -0.3$.**
Initial values in the white region are attracted to the hairpin attractor inside the white region. On each iteration, the points on one piece of the attractor map to the other piece. Orbits from initial values in the black region diverge to infinity.

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# 5 - Mapas Lineares

$$A\begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}\, x + a_{12}\, y \\ a_{21}\, x + a_{22}\, y \end{pmatrix} \qquad \vec{\mathrm{V}} \equiv \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix}$$

Definição: A é linear se $A(a\vec{v} + b\vec{w}) = a\,A\,(\vec{v}) + b\,A\left(\vec{w}\right)$

$\lambda$ é um auto-valor da matriz A se (para $\vec{v} \neq \vec{0}$)

$$A\,\vec{v} = \lambda\,\vec{v}$$

$$\vec{v}_{n+1} = A\,\vec{v}_{n} \Rightarrow \vec{v}_{n+1} = \lambda^{n+1}\,\vec{v}_{0}$$

Exemplo

Para a matriz $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \text{a é auto-valor e } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ auto-vetor}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} = b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \text{b é auto-valor e } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ auto-vetor}$$

# Expansão e Contração no Mapeamento de Um Disco de Pontos Iniciais

Para n iterações $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$$\begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a^n \\ 0 \end{pmatrix} = a^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ b^n \end{pmatrix} = b^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pontos iniciais em um disco de raio unitário são mapeados em uma elípse com semi-eixos $|a^n|$ na direção x e $|b^n|$ em y.

Um disco de raio $\varepsilon$, i.e.,

na vizinhança $N_{\varepsilon}(0,0)$ do ponto $\vec{P}(0,0)$,

torná-se uma elípse com semi-eixos $\varepsilon|a^{n}|$ e $\varepsilon|b^{n}|$

$|a| > 1 \Rightarrow$ repulsão na direção x

$|b| < 1 \Rightarrow$ atração na direção y

$$\text{Exemplo}: A = \begin{pmatrix} 2 & 0 \\ 0 & 0{,}5 \end{pmatrix} \quad \text{e} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2\,x_0 \\ 0{,}5\,y_0 \end{pmatrix} \;; A^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 4\,x_0 \\ 0{,}25\,y_0 \end{pmatrix}$$

$$\text{Em geral, } A(\vec{v}) = |\vec{v}|\, A\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

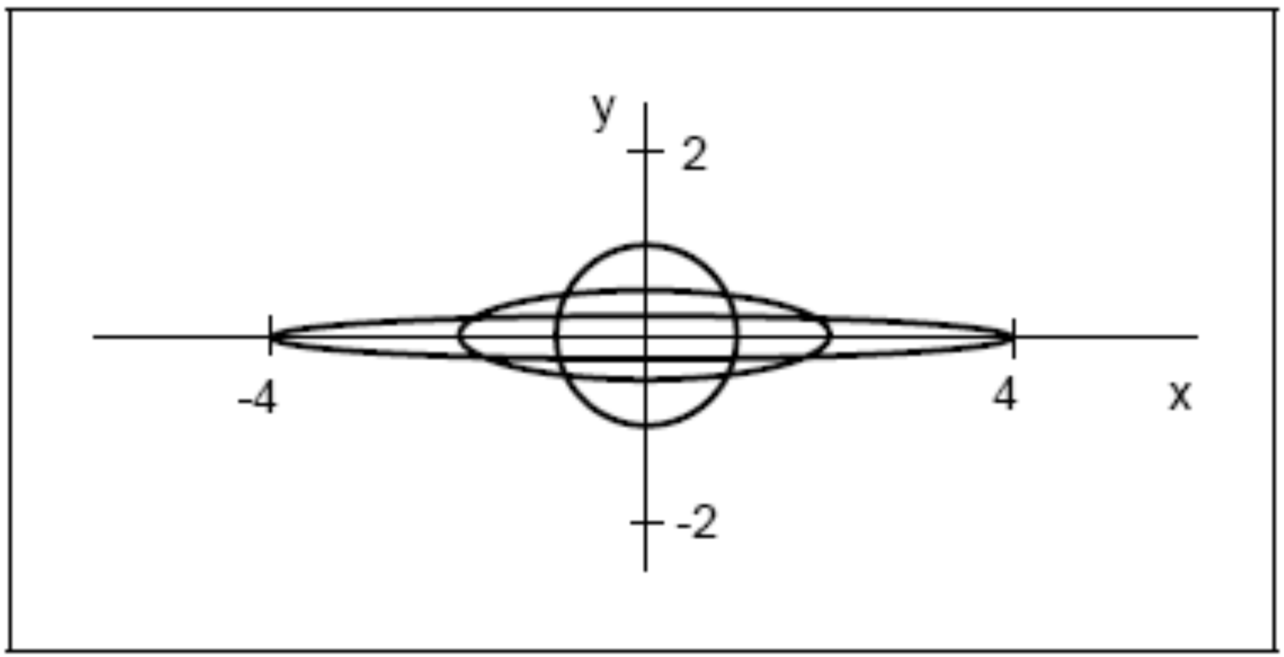


**Figure 2.12 The unit disk and two images of the unit disk under a linear map.** The origin is a saddle fixed point.

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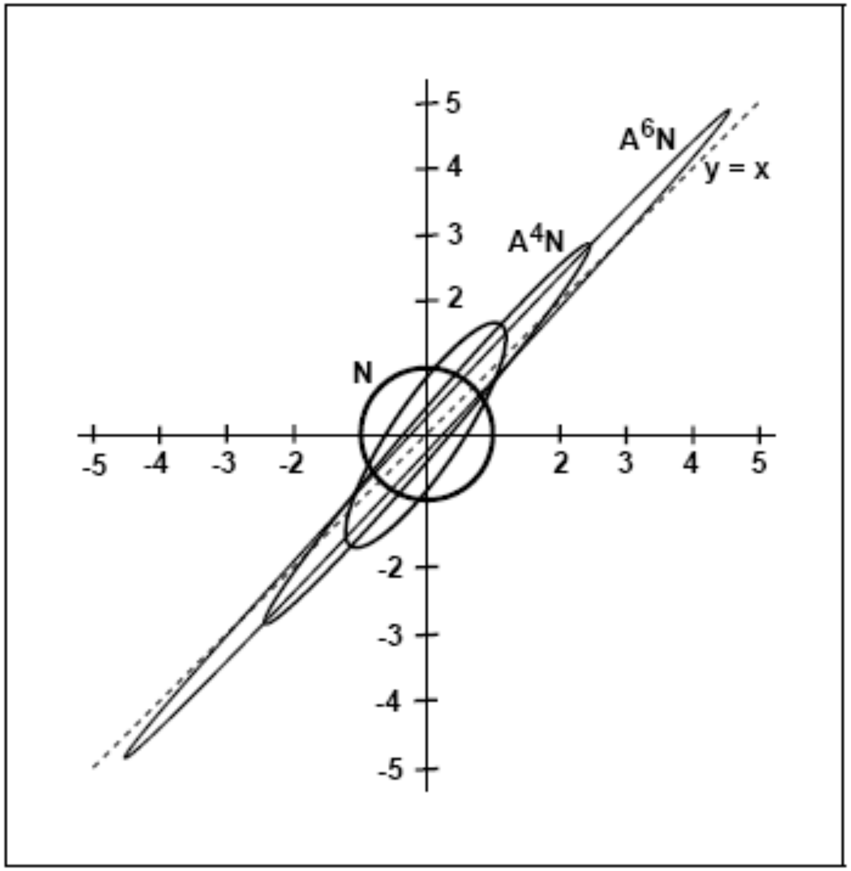


**Figure 2.25 Successive images of the unit circle *N* for a saddle fixed point.** The image of a circle under a linear map is an ellipse. Successive images are therefore also ellipses, which in this example line up along the expanding eigenspace.

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$$\text{Exemplo: } A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \Rightarrow \text{ponto iterado gira de } \theta \text{ e dilata de } r$$

$$\text{Auto-valores } a \pm i\,b \qquad \text{auto-vetores } \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

$$\text{Transformação } a = r\,\text{sen}\,\theta, \; b = r\cos\theta$$

$$r^2 = a^2 + b^2 \;, \quad \theta = \text{arctg}\, a/b$$

$$A = r\begin{pmatrix} a/r & -b/r \\ b/r & a/r \end{pmatrix} = \sqrt{a^2 + b^2}\begin{pmatrix} \text{sen}\,\theta & -\cos\theta \\ \cos\theta & \text{sen}\,\theta \end{pmatrix}$$

$$A\begin{pmatrix} x_0 \\ 0 \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_0 \\ 0 \end{pmatrix} = r\begin{pmatrix} x_0\cos\theta \\ x_0\,\mathrm{sen}\,\theta \end{pmatrix}$$

Rotação de $\theta$ e dilatação (contração) de r

# 7 – Matriz Jacobiana

Em uma dimensão, com um ponto fixo $p = f(p)$,

$$f(p + h) \cong f(p) + h\, f'(p)$$

$$|f'(p)| < 1 \Rightarrow p \text{ é um atrator}$$

$$|f'(p)| > 1 \Rightarrow p \text{ é um repulsor}$$

Em duas dimensões, com um ponto fixo $\vec{p} = \vec{f}(\vec{p})$

$$\vec{f}(\vec{p}+\vec{h}) \cong \vec{f}(\vec{p}) + \vec{D}\vec{f}(\vec{p})\cdot\vec{h} = \vec{p} + \vec{D}\vec{f}(\vec{p})\cdot\vec{h}$$

Teorema:

1 - Se os módulos dos auto-valores da matriz $\vec{D}\vec{f}(\vec{p})$ forem menores que 1, $\vec{p}$ é um atrator.

2 - Se os módulos dos auto-valores da matriz $\vec{D}\vec{f}(\vec{p})$ forem maiores que 1, $\vec{p}$ é um repulsor.

Definição: Se um auto-valor for maior que 1 e o outro menor que 1, $\vec{p}$ é um ponto de sela.

Em uma dimensão, $f^2(p_0)' = f(p_0)' \, f(p_1)'$

Em duas dimensões, $D\vec{f}^{\,2}(\vec{p}_0) = D\vec{f}(\vec{p}_0)\, D\vec{f}(\vec{p}_1)$

Exemplo: Mapa de Hénon $\quad f(x, y) = (a - x^2 + b\,y,\ x) \quad a = 0$ e $b = -0{,}4$

Ponto fixo $(0, 0)$

$$J = \begin{pmatrix} \partial_x (a - x^2 + b\,y) & \partial_y (a - x^2 + b\,y) \\ \partial_x x & \partial_y x \end{pmatrix} = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0{,}4 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 - \lambda & b \\ 1 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow |\lambda| = \left|\pm\sqrt{0{,}4}\right| < 1 \Rightarrow \text{atrator}$$

Ponto fixo $(-0{,}6,\ -0{,}6)$

$$J = \begin{pmatrix} \partial_x (a - x^2 + b\,y) & \partial_y (a - x^2 + b\,y) \\ \partial_x x & \partial_y x \end{pmatrix} = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1{,}2 & 0{,}4 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1{,}2 - \lambda & b \\ 1 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow |\lambda_1| = |1{,}472| > 1\ ;\ |\lambda_2| = |-0{,}272| < 1 \Rightarrow \text{ponto de sela}$$

# Esqema Dinâmico de um Ponto de Sela

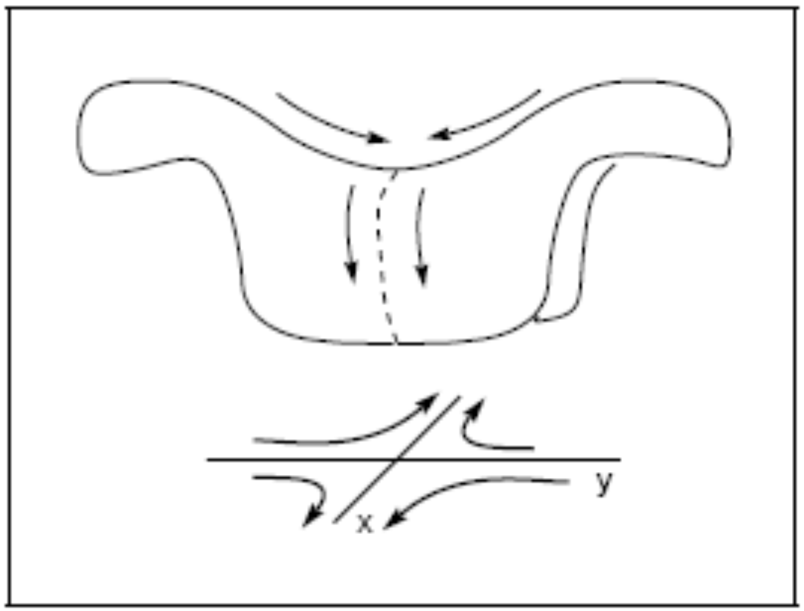


**Figure 2.13 Dynamics near a saddle point.**
Points in the vicinity of a saddle fixed point (here the origin in the xy-plane) move as if responding to the influence of gravity on a saddle.

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# Órbitas Próximas de Um Ponto de Sela

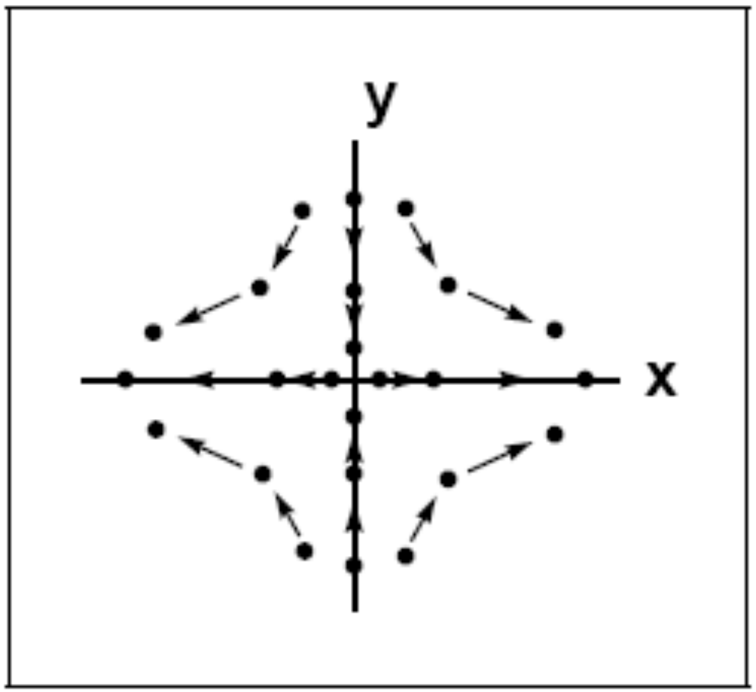


**Figure 2.14 Saddle dynamics.**
Successive images of points near a saddle fixed point are shown.

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Exemplo : Mapa de Hénon $f(x, y) = (a - x^2 + b\,y,\ x) \quad a = 0,43 \quad b = 0,4$

Órbita de período 2: $(0.7,\ -0,1) \Leftrightarrow (-0.1, 0.7)$

$$J^2 = J\,J = \begin{pmatrix} -2(-0.1) & 0.4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2(0.7) & 0.4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0.12 & 0.008 \\ -1.4 & 0.4 \end{pmatrix}$$

$$\begin{vmatrix} 0.12-\lambda & 0.008 \\ -1.4 & 0.4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0.26 \pm i\,0.30 \Rightarrow |\lambda| = 0.40 < 1 \ \text{ atrator}$$

# Mudançã de  Atratores do Mapa de Hénon

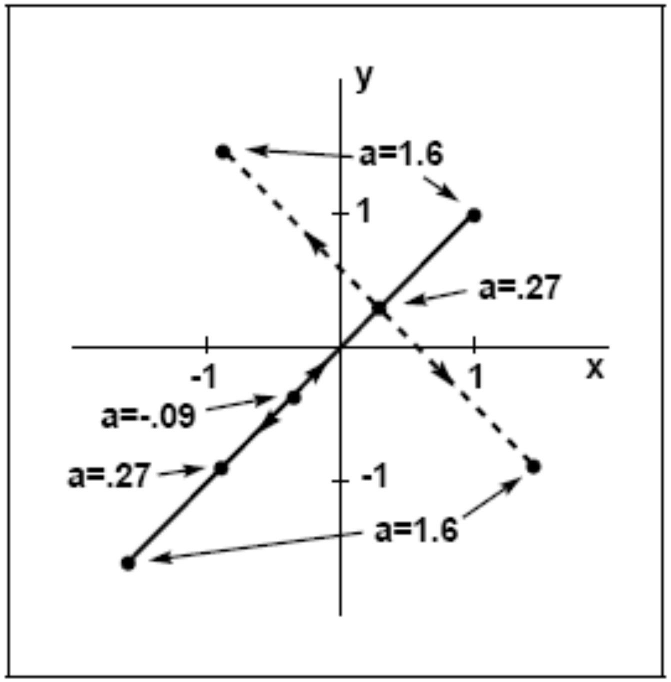


**Figure 2.15 Fixed points and period-two points for the Hénon map with *b* fixed at 0.4.**
The solid line denotes the trails of the two fixed points as $a$ moves from $-0.09$, where the two fixed points are created together, to 1.6 where they have moved quite far apart. The fixed point that moves diagonally upward is attracting for $-0.09 < a < 0.27$; the other is a saddle. The dashed line follows the period-two orbit from its creation when $a = 0.27$, at the site of the (previously) attracting fixed point, to $a = 1.6$.

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# Diagrama de Bifurcação do Mapa de Hénon

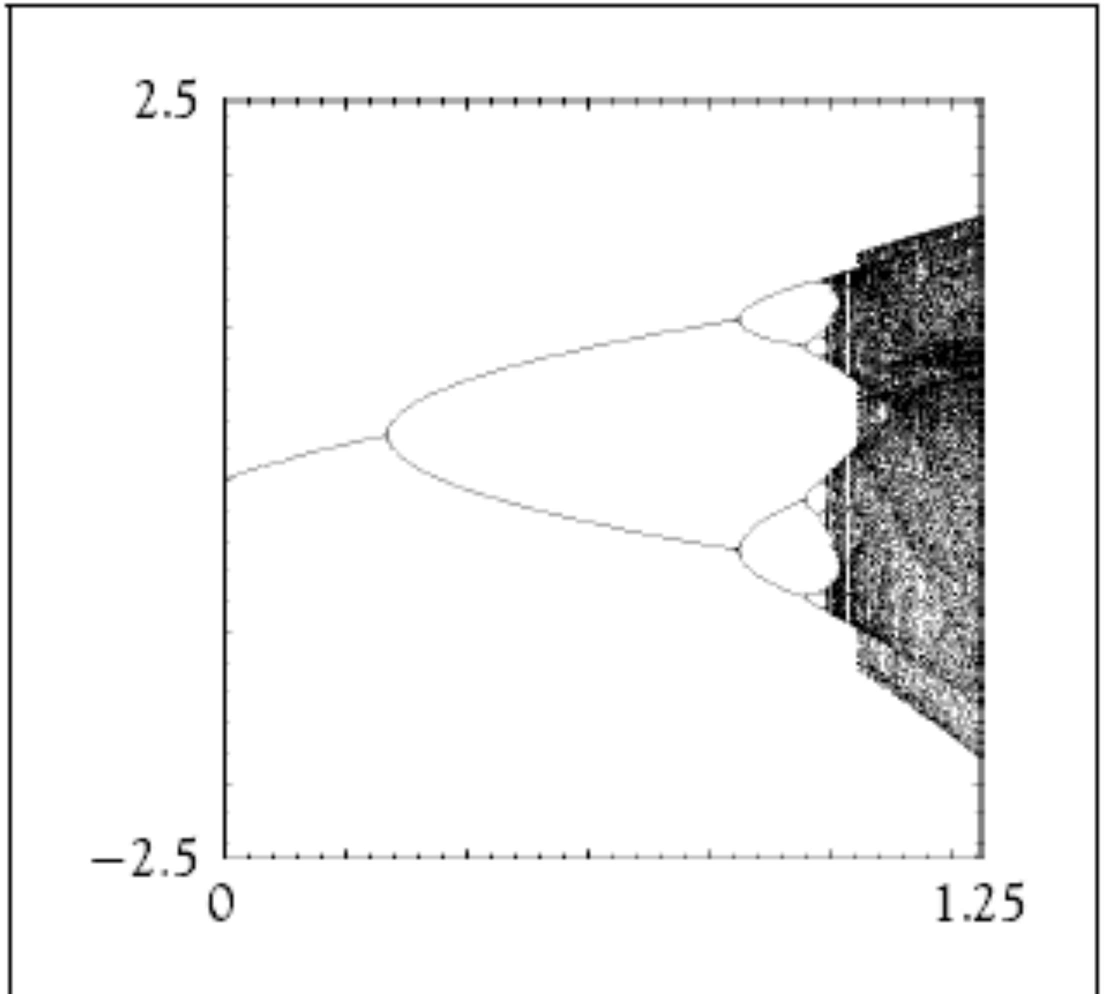


**Figure 2.16 Bifurcation diagram for the Hénon map, $b = 0.4$.**
Each vertical slice shows the projection onto the $x$-axis of an attractor for the map for a fixed value of the parameter $a$.

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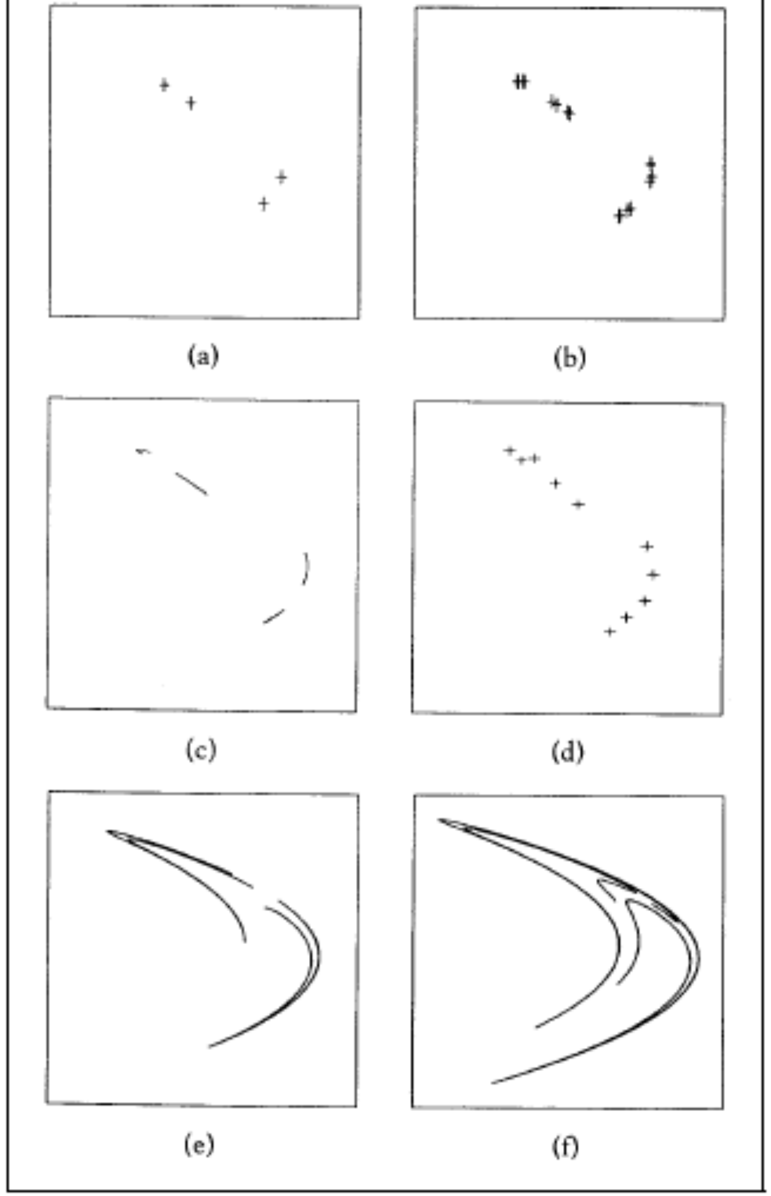


**Figure 2.17 Attractors for the Hénon map with $b = 0.4$.**
Each panel displays a single attracting orbit for a particular value of the parameter $a$. (a) $a = 0.9$, period 4 sink. (b) $a = 0.988$, period 16 sink. (c) $a = 1.0$, four-piece attractor. (d) $a = 1.0293$, period-ten sink. (e) $a = 1.045$, two-piece attractor. The points of an orbit alternate between the pieces. (f) $a = 1.2$, two pieces have merged to form one-piece attractor.

Mudança de atratores com a, para b=0.4

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# Variedades Estáveis e Instáveis

Variedade estável: conjunto dos pontos iniciais que convergem para um ponto de sela.

Variedade instável: variedade estável da transformação inversa

# Definições das Variedades

A variedade estável de P, S(P), é o conjunto de pontos $\vec{v}$ tal que

$$\lim_{n \to \infty} \left| f^{n}(\vec{v}) - f^{n}(P) \right| \to 0$$

A variedade instável de P, U(P), é o conjunto de pontos $\vec{v}$ tal que

$$\lim_{n \to \infty} \left| f^{-n}(\vec{v}) - f^{-n}(P) \right| \to 0$$

Exemplo: O mapa linear $f(x, y) = (-2x + \frac{5}{2}y, -5x + \frac{11}{2}y)$

tem um ponto de sela em $(0, 0)$

com auto-valores $0.5$ e $3$ e auto-vetores $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ e $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} -2 & \frac{5}{2} \\ -5 & \frac{11}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.5 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & \frac{5}{2} \\ -5 & \frac{11}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Coordenadas dos pontos satisfazendo a condição $y = x$,

linha na direção do auto-vetor $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, sofrem um decréscimo

de um fator 0.5 a cada iteração.

Esses pontos constituem a variedade estável.

Coordenadas dos pontos satisfazendo a condição $y = 2x$,

linha na direção do auto-vetor $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, sofrem um acréscimo

de um fator 3 a cada iteração.

Esses pontos constituem a variedade instável.

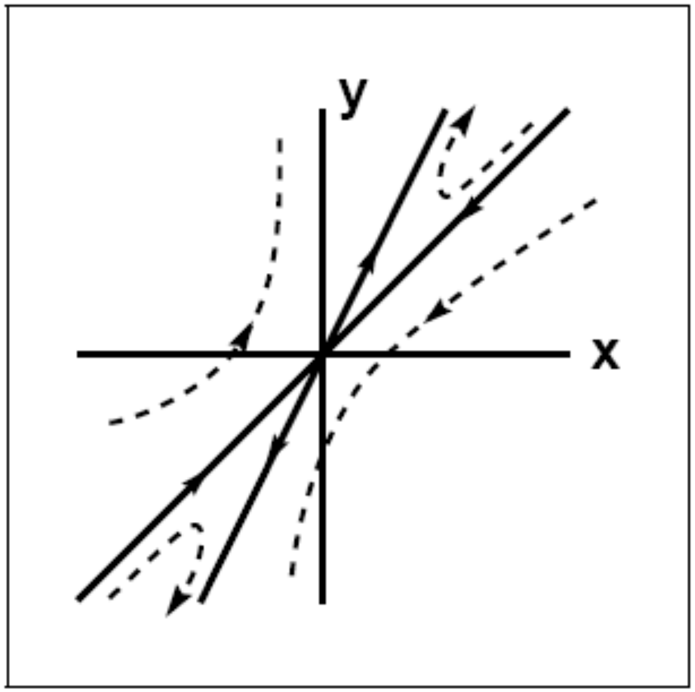


**Figure 2.18 Stable and unstable manifolds for regular saddle.**
The stable manifold is the solid inward-directed line; the unstable manifold is the solid outward-directed line. Every initial condition leads to an orbit diverging to infinity except for the stable manifold of the origin.

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$$\text{Exemplo: Mapa com } f(x, y): (2x + 5y, \ -0.5y)$$

Ponto de sela alternado (*flip saddle point*)

$$\text{Auto-valores: } 2, \ -0{,}5$$

$$\text{Auto-vetores: } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ e } \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

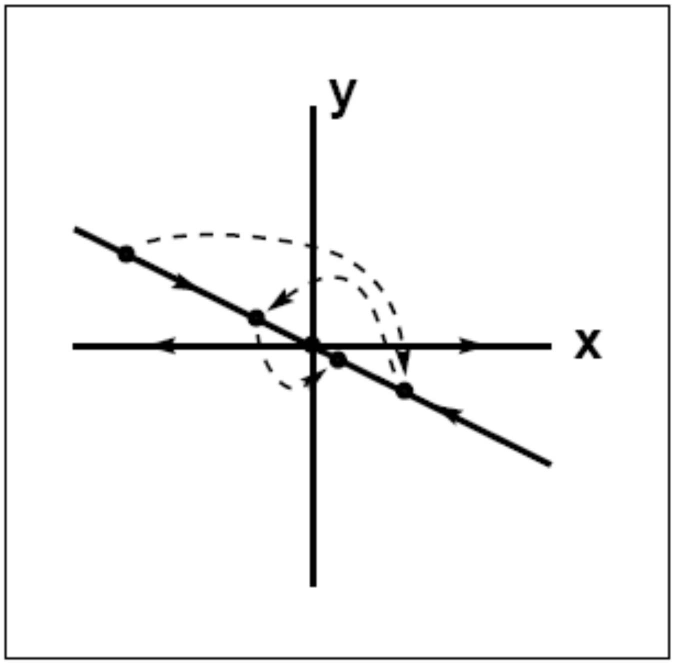


**Figure 2.19 Stable and unstable manifolds for flip saddle.**
Flipping occurs along the stable manifold (inward-directed line). The unstable manifold is the $x$-axis.

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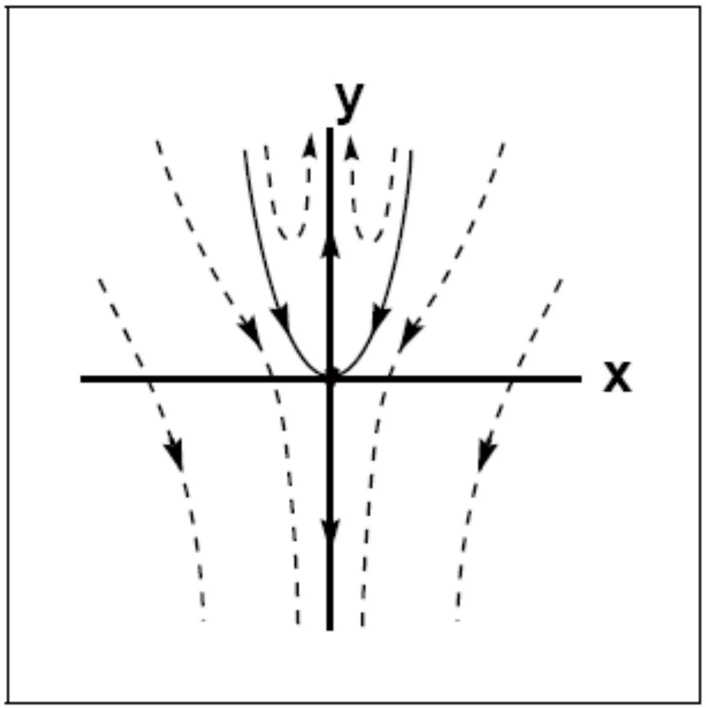


**Figure 2.20 Stable and unstable manifolds for the nonlinear map of Example 2.21.**
The stable manifold is a parabola tangent to the $x$-axis at 0; the unstable manifold is the $y$-axis.

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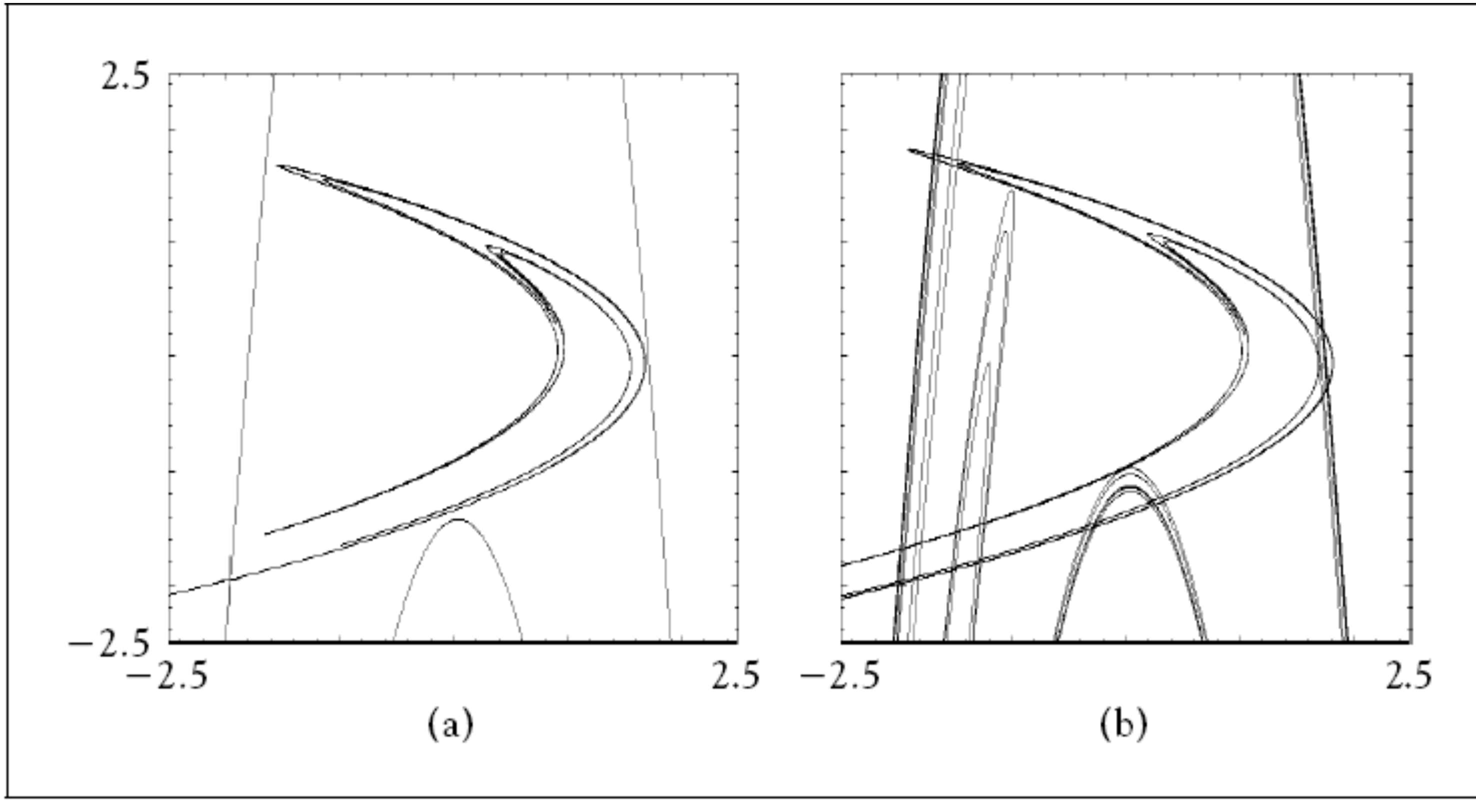


**Figure 2.21 Stable and unstable manifolds for a saddle point.**
The stable manifolds (mainly vertical) and unstable manifolds (more horizontal) are shown for the saddle fixed point (marked with a cross in the lower left corner) of the Hénon map with $b = -0.3$. Note the similarity of the unstable manifold with earlier figures showing the Hénon attractor. (a) For $a = 1.28$, the leftward piece of the unstable manifold moves off to infinity, and the rightward piece initially curves toward the sink, but oscillates around it in an erratic way. The rightward piece is contained in the region bounded by the two components of the stable manifold. (b) For $a = 1.4$, the manifolds have crossed one another.

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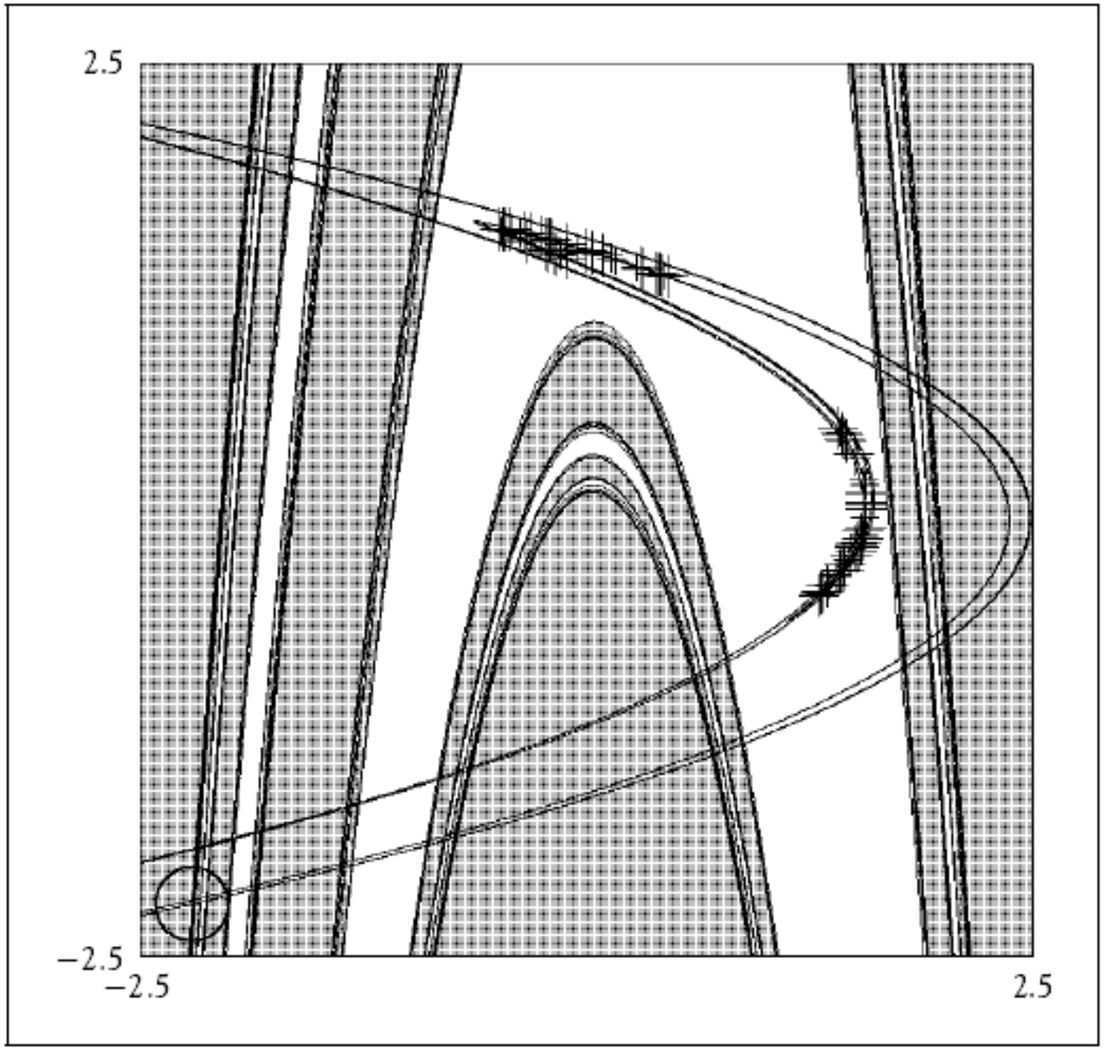


**Figure 2.22 A two-piece attractor of the Hénon map.**
The crosses mark 100 points of a trajectory lying on a two-piece attractor. The basin of attraction of this attractor is white; the shaded points are initial conditions whose orbits diverge to ∞. The saddle fixed point circled at the lower left is closely related to the dynamics of the attractor. The stable manifold of the saddle, shown in black, forms the boundary of the basin of the attractor. The attractor lies along the unstable manifold of the saddle, which is also in black.

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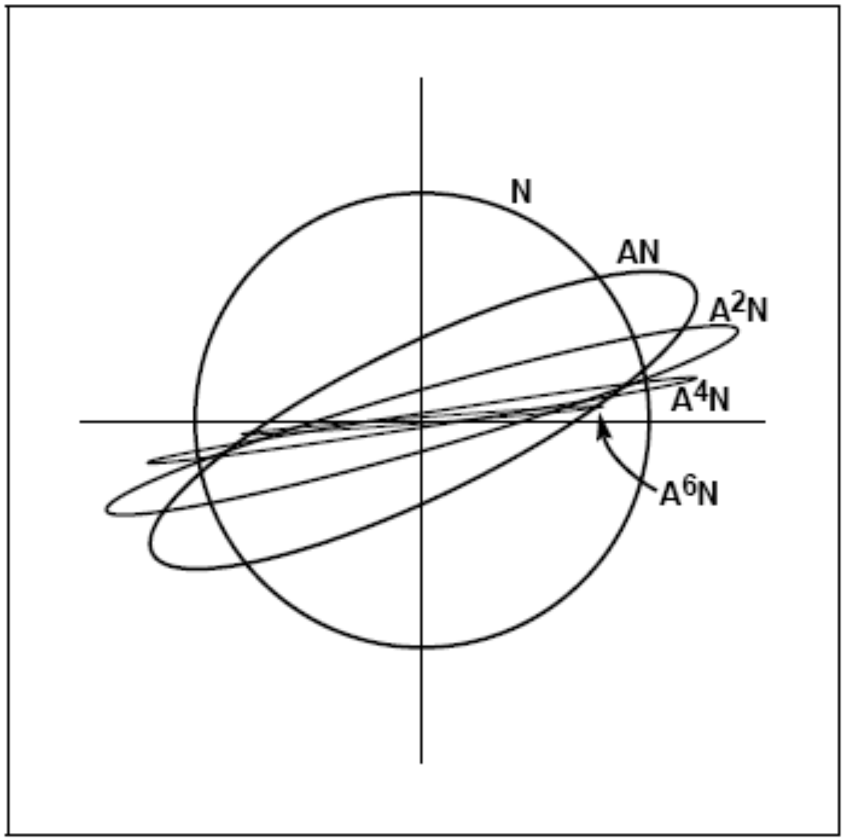


**Figure 2.26 Successive images of the unit circle *N* under a linear map *A* in the case of repeated eigenvalues.**

The $n$th iterate $A^nN$ of the circle lies wholly inside the circle for $n \geq 6$. In this case, the origin is a sink.

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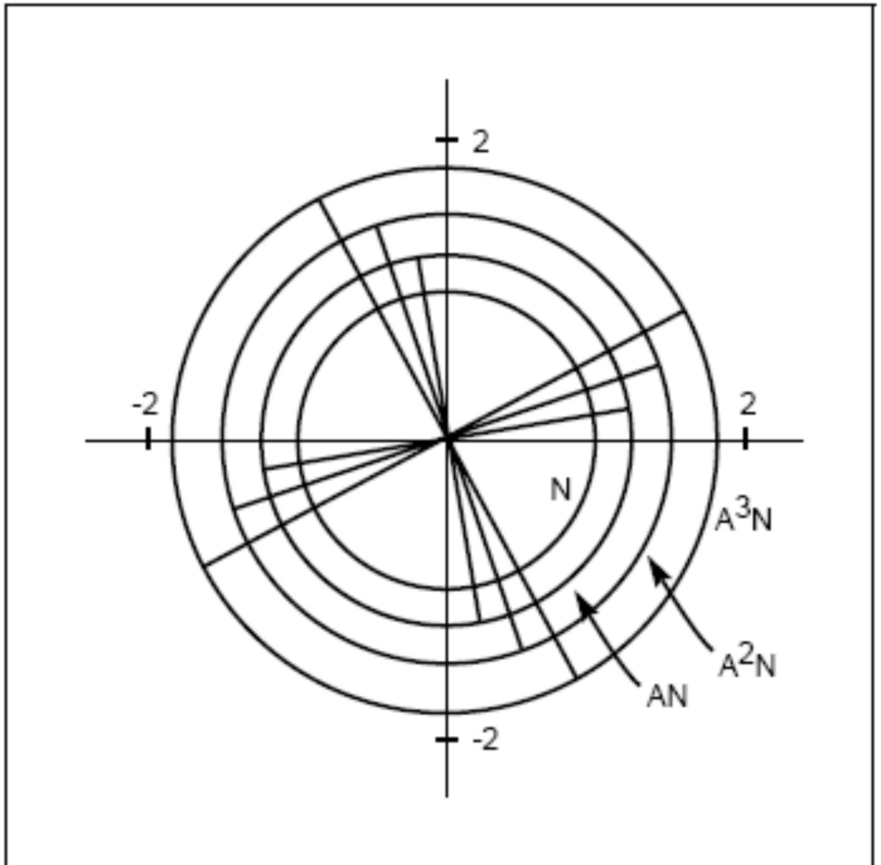


**Figure 2.27 Successive images of the unit circle *N*.**
The origin is a source with complex eigenvalues.

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# Bacias de Atração
# e Atrator Caótico no Pêndulo Forçado Periódicamente

# Pêndulo Forçado

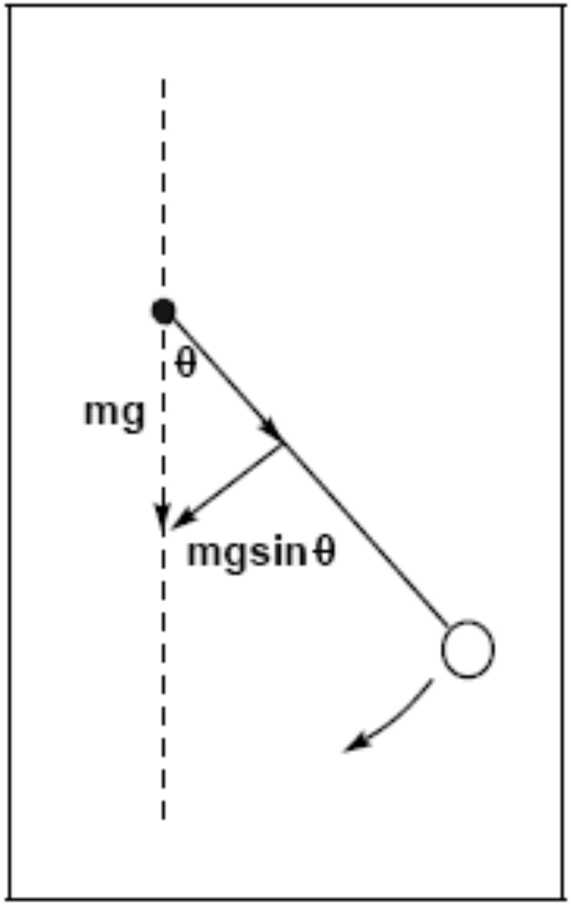



**Figure 2.4 The pendulum under gravitational acceleration.**
The force of gravity causes the pendulum bob to accelerate in a direction perpendicular to the rod. Here $\theta$ denotes the angle of displacement from the downward position.

(Alligood et al. Chaos...)

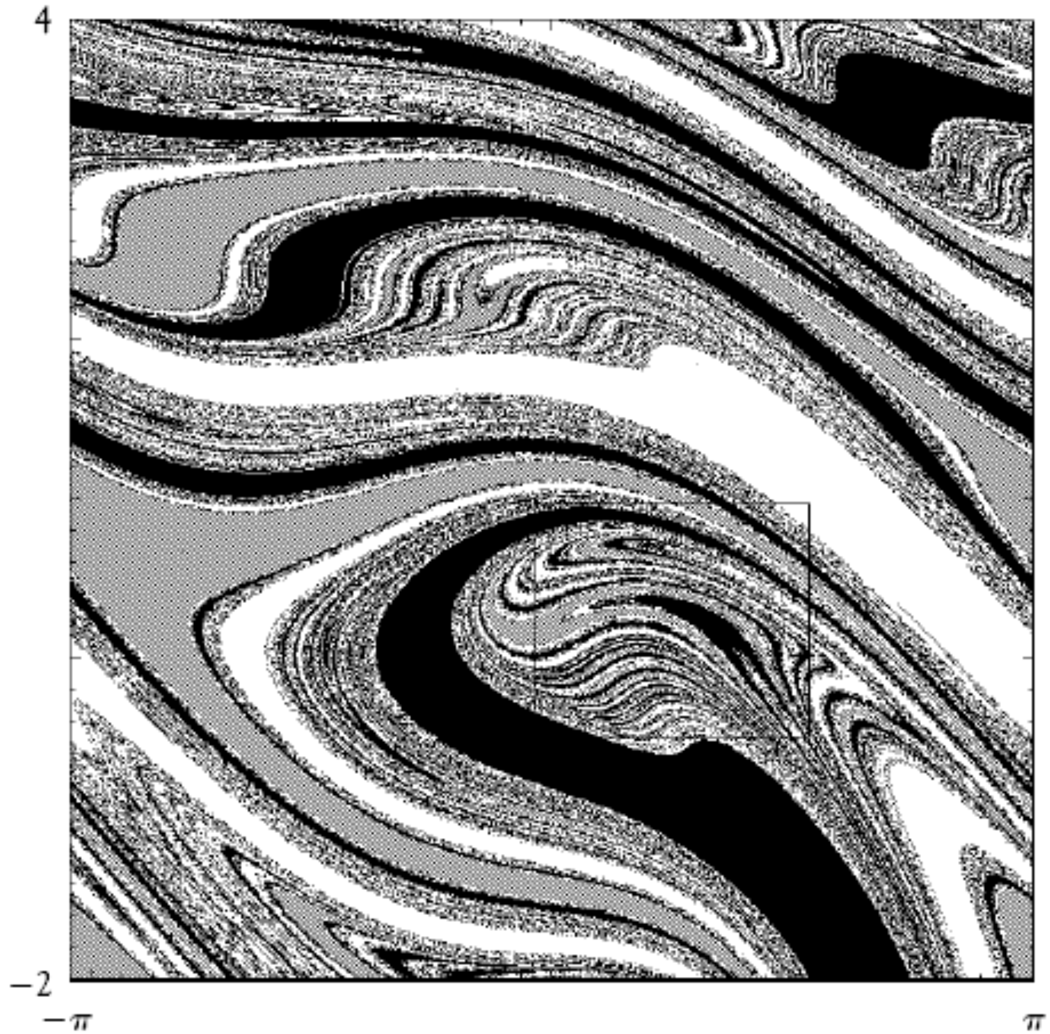
Equação de movimento: $\ddot{\theta} = -c\,\dot{\theta} - \operatorname{sen}\theta + \rho \operatorname{sen} t$

$\theta(t)$ e $\theta(t + 2\pi N)$ são soluções, $N = 0, 1, 2, 3, \ldots$

Portanto, vamos integrar a equação de movimento e anotar os valores das variáveis nos instantes $t = 2\pi N$ .

Para $c = 0{,}2$ ; $\rho = 1{,}66$ , há 1 ponto fixo e duas órbitas de período 2. As fronteiras das bacias dessas três órbitas são fractais.

Há ainda 5 pontos fixos instáveis



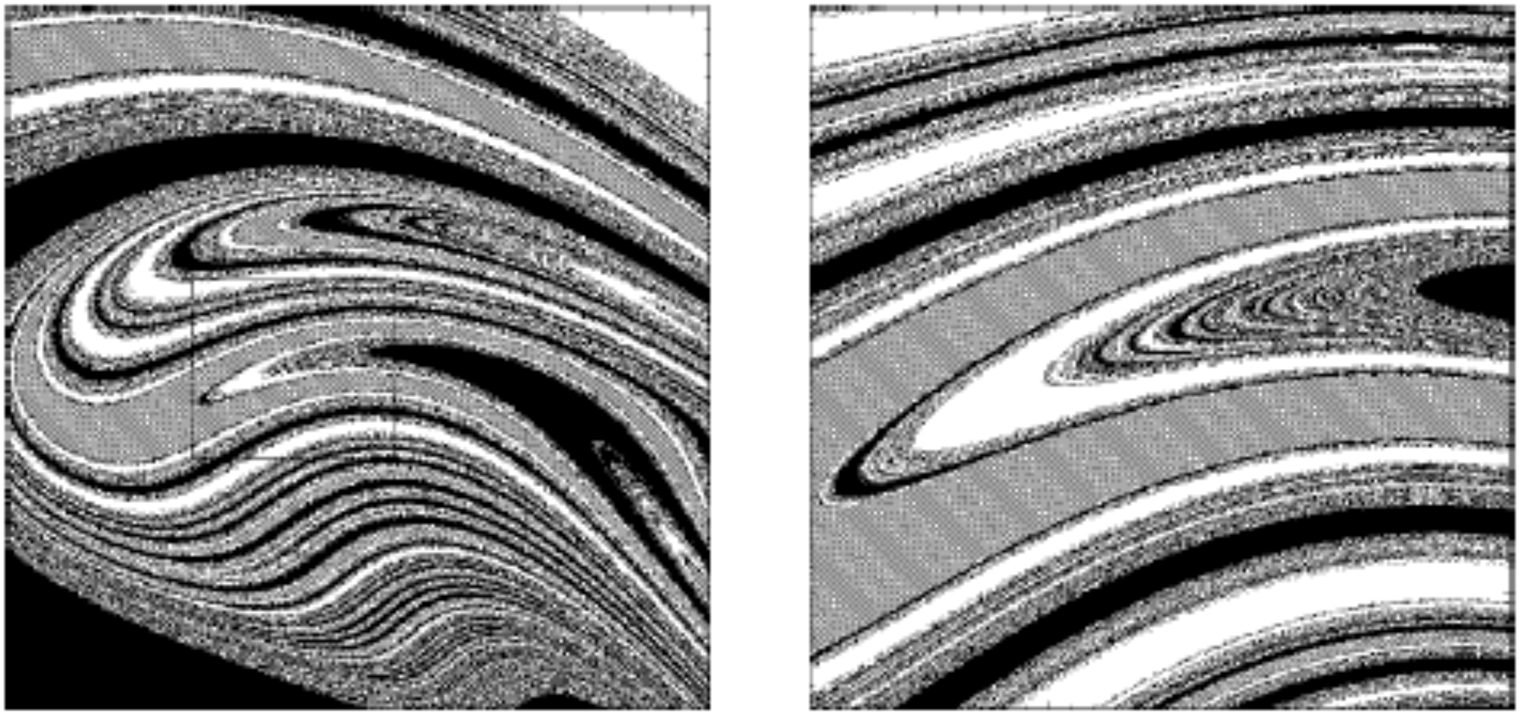

# Bacias de Atração de 1 Ponto Fixo e 2 Órbitas Periódicas

c = 0.2
ρ = 1.66

**Figure 2.5 Basins of three coexisting attracting fixed points.** Parameters for the forced damped pendulum are $c = 0.2$, $\rho = 1.66$. The basins are shown in black, gray, and white. Each initial value $(\theta, \dot{\theta})$ is plotted according to the sink to which it is attracted. Since the horizontal axis denotes angle in radians, the right and left edge of the picture could be glued together, creating a cylinder. The rectangle shown is magnified in figure 2.6.

Alligood et al. Chaos...)

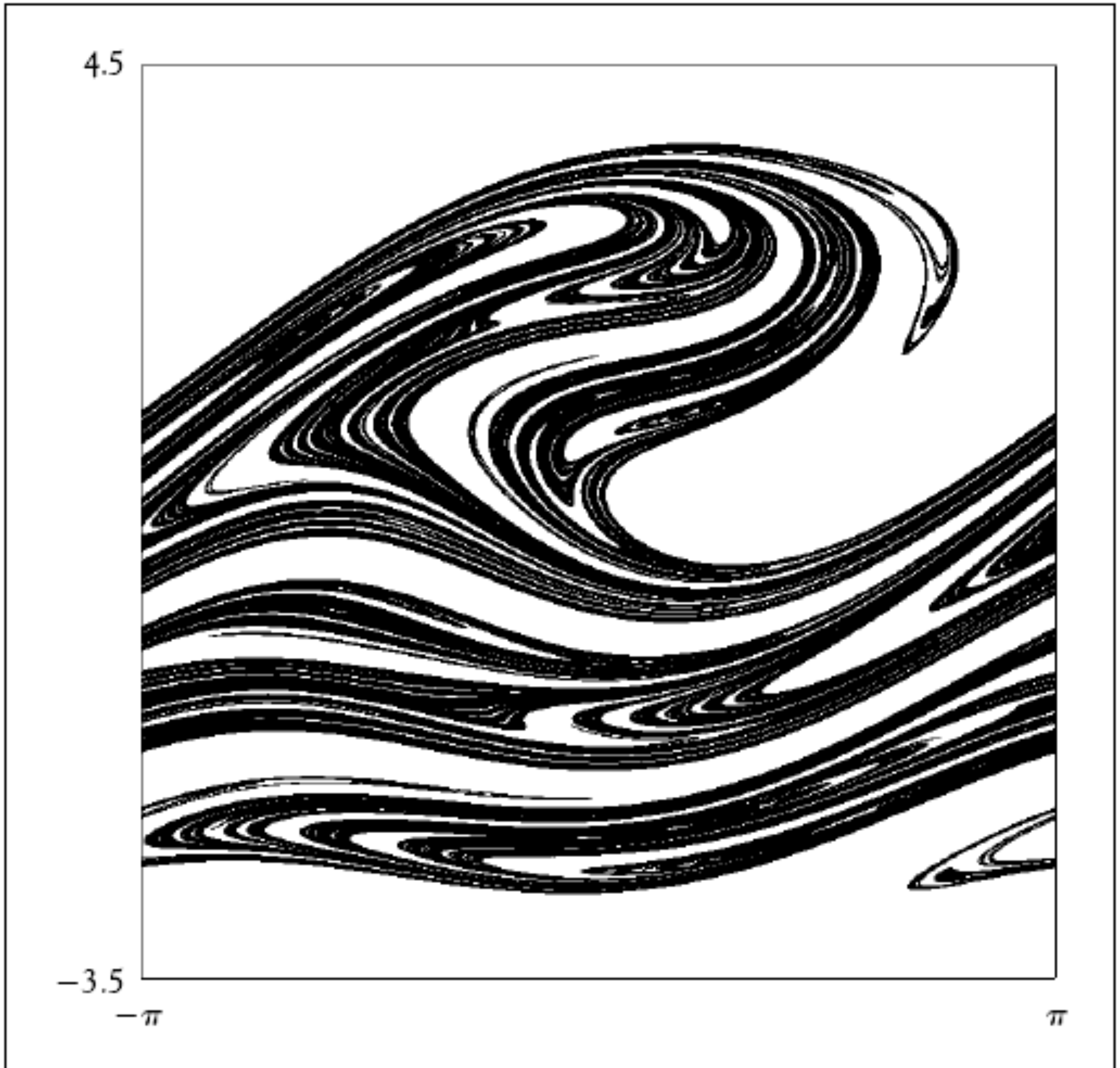
# Ampliações das Três Bacias



(a) (b)

**Figure 2.6 Detail views of the pendulum basin.**
(a) Magnification of the rectangle shown in Figure 2.5. (b) Magnification of the rectangle shown in (a).

(Alligood et al. Chaos...)




# Órbita Caótica do Pêndulo Forçado

c = 0.05
ρ = 2.5

**Figure 2.7 A single orbit of the forced damped pendulum with $c = 0.05$, $\rho = 2.5$.**
Different initial values yield essentially the same pattern, unless the initial value is an unstable periodic orbit, of which there are several (see Figure 2.23).

(Alligood et al. Chaos...)

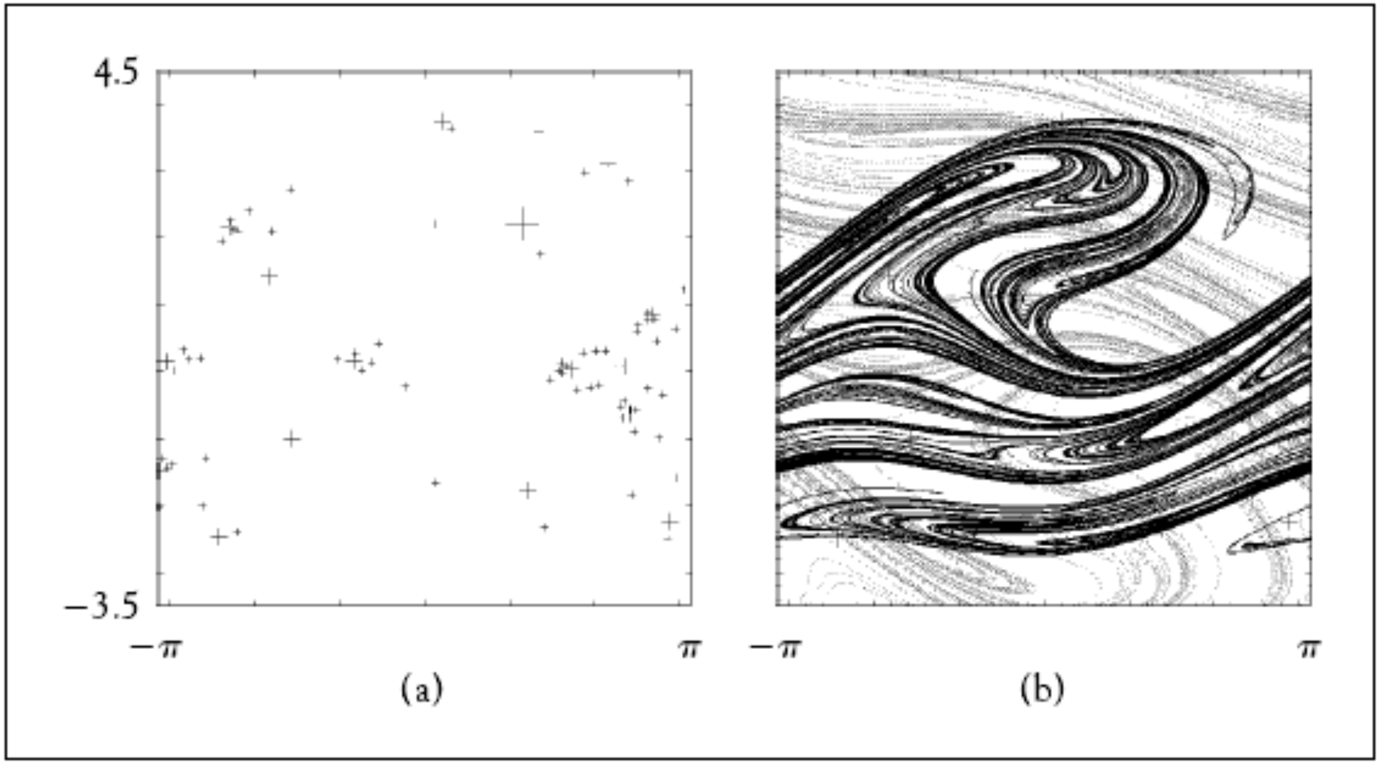



**Figure 2.23 The forced damped pendulum.**
(a) Periodic orbits for the time-$2\pi$ map of the pendulum with parameters $c = 0.05, \rho = 2.5$. The large crosses denote 18 fixed points, and the small crosses, 38 period-two orbits. (b) The stable and unstable manifolds of the largest cross in (a). The unstable manifold is drawn in black; compare to Figure 2.7. The stable manifold is drawn in gray dashed curves. The manifolds overlay the periodic orbits from (a)—note that without exception these orbits lie close to the unstable manifold.

Alligood
Chaos

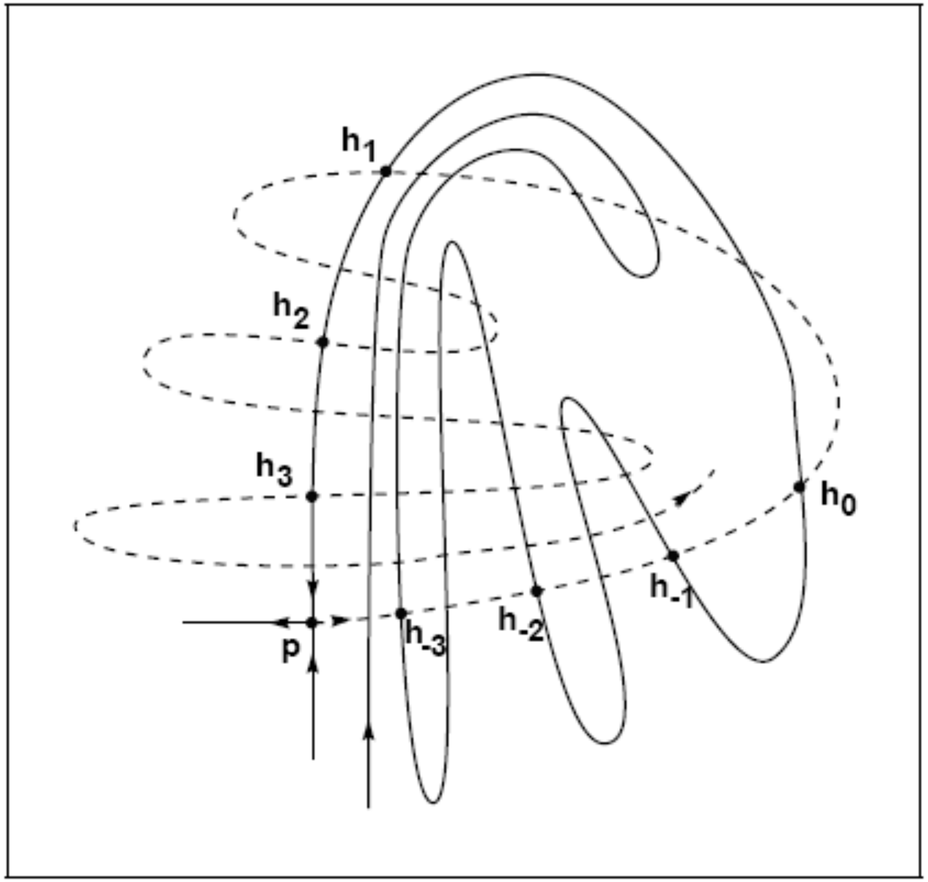



**Figure 2.24 A schematic view of a homoclinic point $h_0$.**
The stable manifold (solid curve) and unstable manifold (dashed curve) of the saddle fixed point $\mathbf{p}$ intersect at $\mathbf{h}_0$, and therefore also at infinitely many other points. This figure only hints at the complexity. Poincaré showed that if a circle was drawn around any homoclinic point, there would be infinitely many homoclinic points inside the circle, no matter how small its radius.

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Chaos

# Órbitas Planetárias Caóticas

# Movimento de Três Corpos Restrito (Num Plano)

Dois corpos pesados descrevem círculos ao redor do centro de massa.
Uma partícula leve de massa m descreve a trajetória mostrada na figura (sem influenciar o movimento dos dois corpos pesados).

Método de análise introduzido por Poincaré

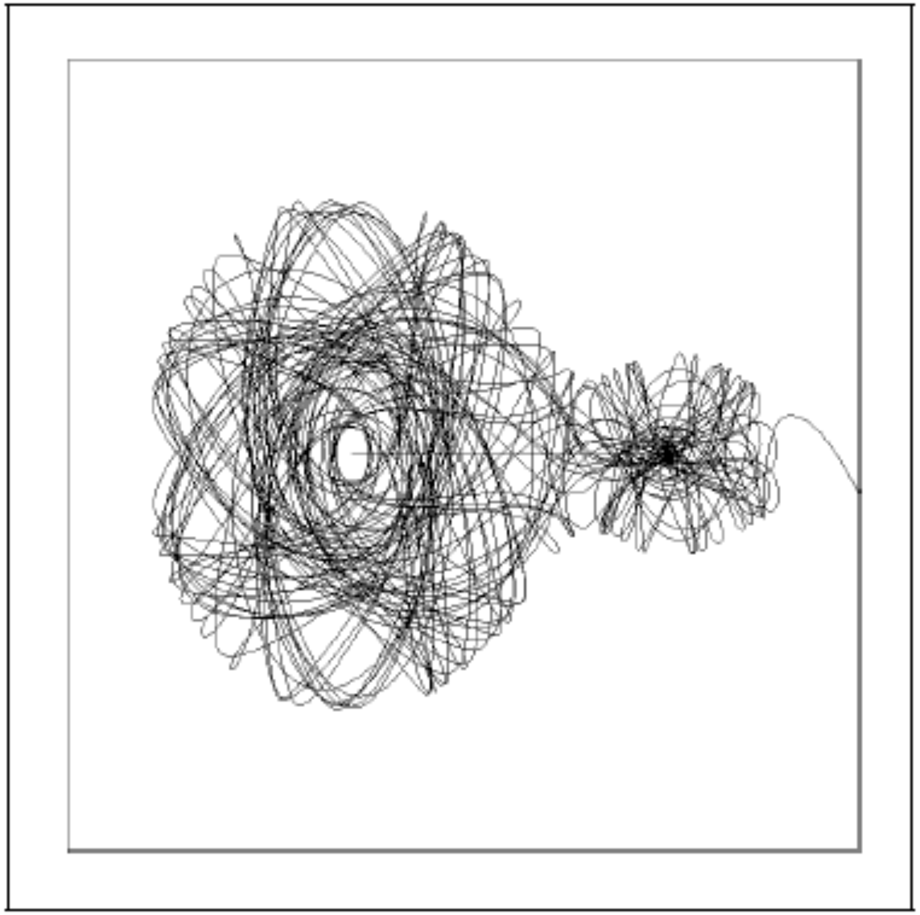
$$\text{Órbita no espaço de fase} : (x, \dot{x}, y, \dot{y})$$

$$\text{As intersecções da órbita no plano } y = o, \text{ com } \dot{y} > 0 \text{ e } H = cte,$$

$$\text{são os pontos } (x, \dot{x}) \text{ no mapa de Poincaré.}$$

$$\text{Este mapa é bidimensional.}$$

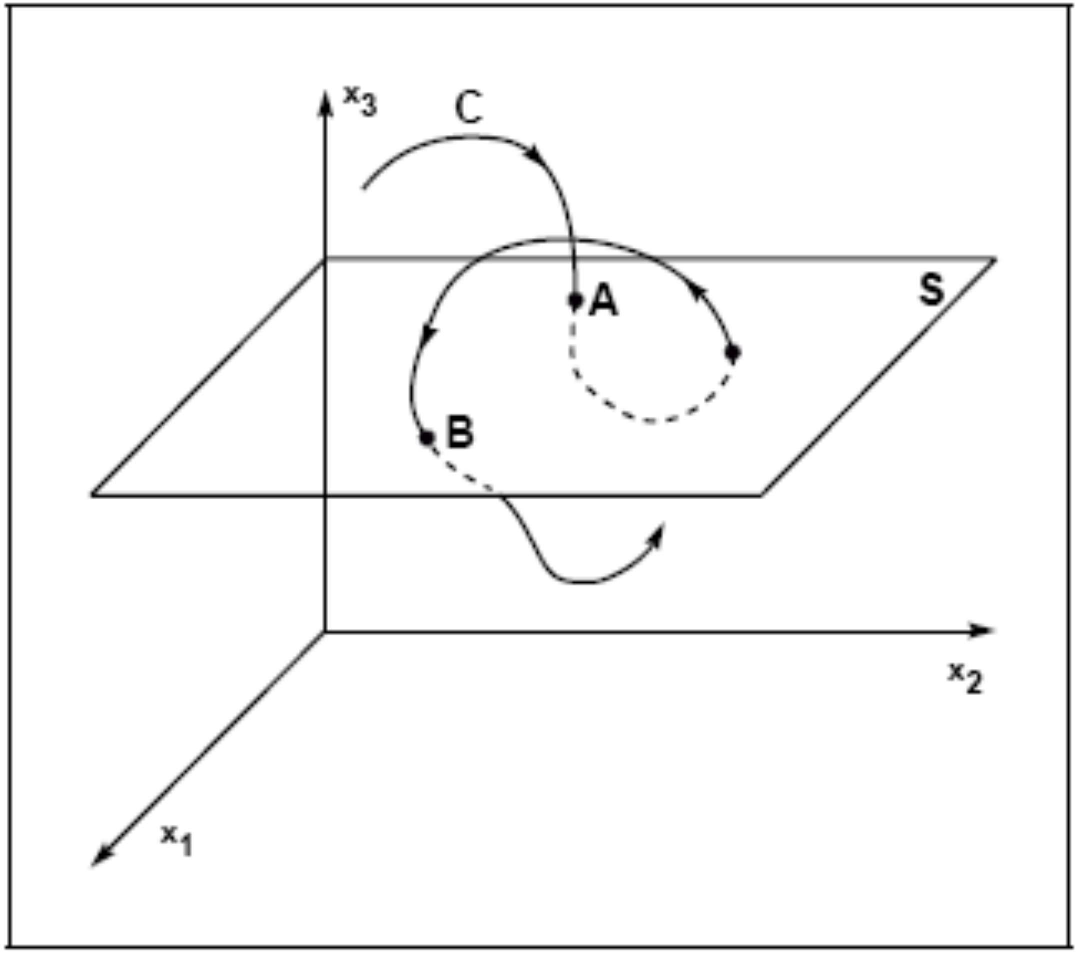
# Trajetória de Uma Massa Leve no Sistema de Três Corpos



(Alligood et al. Chaos...)

**Figure 2.1 A trajectory of a tiny mass in the three-body problem.**
Two larger bodies are in circular motion around one another. This view is of a rotating coordinate system in which the two larger bodies lie at the left and right ends of the horizontal line segment. The tiny mass is eventually ejected toward the right. Other trajectories starting close to one of the bodies can be forever trapped.

# Mapa de Poincaré Bidimensional




Órbita tridimensional

(Alligood et al.
Chaos...)

**Figure 2.2 A Poincaré map derived from a differential equation.**
The map G is defined to encode the downward piercings of the plane S by the solution curve C of the differential equation so that G(A) = B, and so on.

# Caos no Sistema Solar

Prêmio do rei Oscar da Suécia em 1889 para trabalho sobre a estabilidade do sistema solar.
Poincaré mostrou que as trajetórias de 3 corpos que se atraem (Terra, Sol e Jupiter)  são sensíveis às condições iniciais se ocorrerem cruzamentos homoclínicos.

Sussman, Wisdom, *Numerical evidence that the motion of Pluto is chaotic*, Science 241, 433 (1988)

# Sussman, Wisdom, *Numerical evidence that the motion of Pluto is chaotic*, Science 241, 433 (1988)

Integração das equações do movimento de Plutão para um intervalo de 845 milhões de anos.

Computador construido para essa investigação: Digital Orrey.
Em exposição no Smithsonin Institution em Washington, D.C.

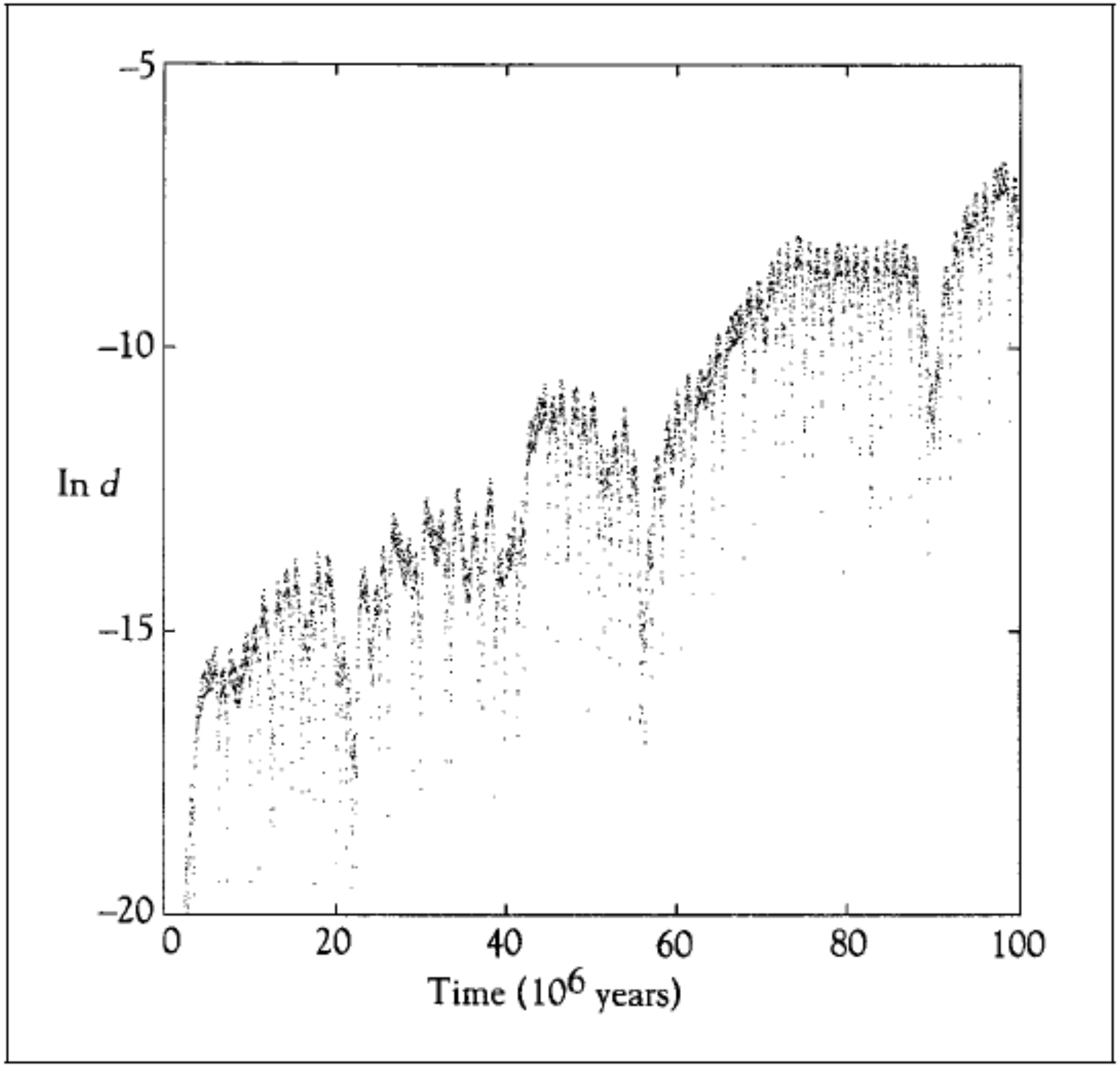



**Figure 2.32 Comparison of two computer simulations.**
The difference in the position of Pluto in two simulations with slightly different initial conditions is plotted as a function of time. The vertical scale is ln of the difference, in units of AU (astronomical units).

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Chaos

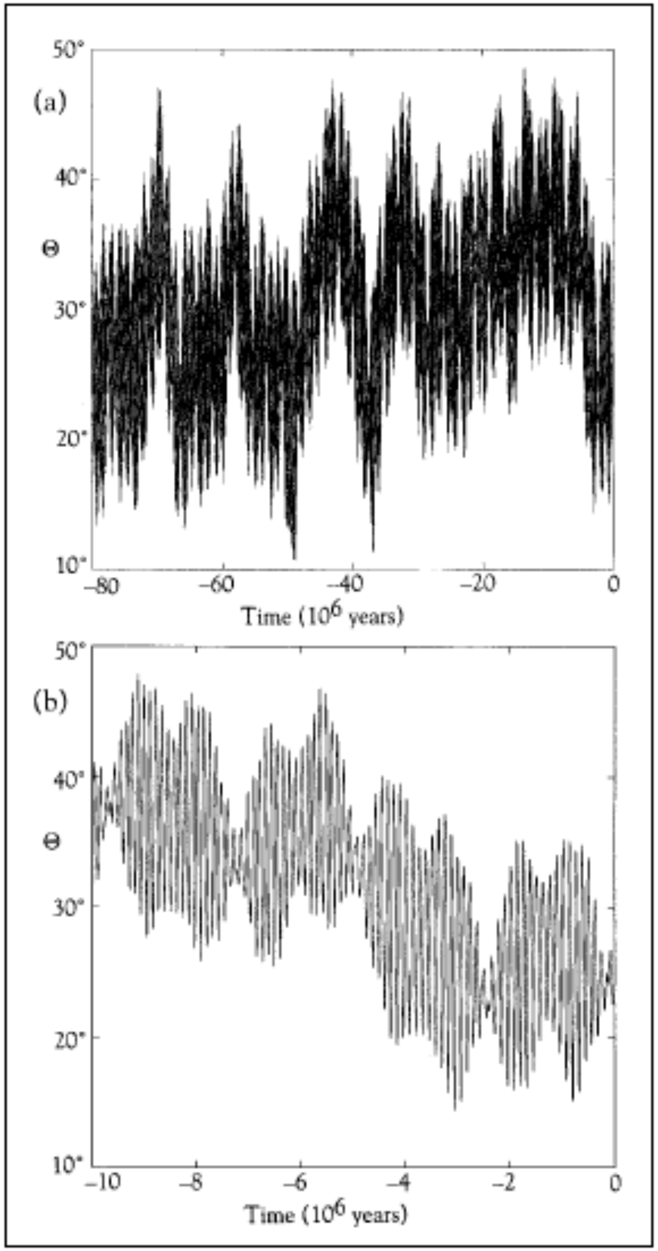


**Figure 2.33 The obliquity of Mars.**
(a) The result of a computer simulation of the solar system shows that the obliquity of Mars undergoes erratic variation as a function of time. (b) Detail from (a), showing only the last 10 million years. There is an abrupt transition about 4 million years ago.

Inclinação do eixo de rotação de Marte
(em relação ao plano do sistema solar)

Transições abruptas

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