Contributions to Plasma Physics

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Suprathermal Soliton Solutions to Nonlinear Schrödinger Equation

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ABSTRACT

Maxwell distributions are very difficult to find in the low-pressure environment far away the Earth atmosphere, permeated by high temperatures, various types of radiation, highly energetic particles, space debris, and subjected to microgravity, presenting crucial challenges for spacecraft design and operations, and affecting astronaut's health. In this work, the amplitude and half-width of suprathermal soliton solutions to the nonlinear Schrödinger equation are explored with basis on a deformation parameter of a non-Maxwellian distribution. It is found that the ratio of the square of a normalized phase speed of an ion wave to a normalized electron mass may increase by up to 50% due to intense suprathermal effects. It is also found that the soliton amplitude is generally smaller and that the soliton half-width increases with no limit when suprathermal effects are more intense suggesting a greater propensity for dissipation or instability. Possible applications of our analytical formulation to both laboratory and space plasmas are briefly addressed.

1 | Introduction

Non-Maxwellian solitons and instabilities in systems out of thermal equilibrium have motivated a number of theoretical and numerical investigations, especially in space and astrophysical plasmas [1–6]. The cause for all this interest lies in the fact that Maxwell distributions are quite rare in space. Although the vast majority of space plasmas reside in stationary states, that is, their statistics are, at least temporarily, time-invariant, they are typically not well described by Maxwell distributions, being, therefore, out of thermal equilibrium [7–11].

In 1950, Ginzburg and Landau derived a nonlinear differential equation (the now-called Ginzburg–Landau equation) in order to describe superconductivity [12]. Shortly afterwards, Chiao et al. explored a reduced $(1+1)$-dimensional form of that equation with the purpose of investigating self-trapping of optical beams [13]. They named such an equation the nonlinear Schrödinger equation (NLSE). Later on, the NLSE found several applications from gravity waves on the water surface [14–17] through Langmuir waves in hot plasmas [18–21] to transport of energy along chains of molecules [22–24], nonlinear waves in Bose–Einstein condensates [25–27], ionospheric plasmas [28–30], dispersive media [31–35], optical fibers and waveguides [36–41] to name but a few.

In plasma physics, one may derive the NLSE by attributing a second-order perturbation of the equilibrium concentration to a ponderomotive force in the low-frequency limit of the equation of motion for electrons [42]. The derivation usually assumes a

Maxwellian electron gas, so that the coefficients of the NLSE reflect such an assumption. As a result, properties of solitonic solutions to the NLSE, such as amplitude and half-width, which, in turn, depend on those coefficients, will also reflect that Maxwellian electron gas assumption.

In this work, we derive the NLSE for a plasma whose electron component is assumed to satisfy a non-Maxwellian distribution. The paper is organized as follows. In Section 2, we proceed a linear analysis of the equation of motion for suprathermal electrons. In Section 3, we derive a NLSE whose coefficients depend on a deformation parameter. In Section 4, we explore the behavior of the properties of solitonic solutions to the NLSE in terms of the deformation parameter. In the concluding section, we summarize our work and briefly address possible applications of it to both laboratory and space plasmas.

2 | Linear Analysis

In the absence of a magnetic field, the linearized time evolution of the one-dimensional flow v_e of electrons with mass m and charge $-e < 0$ is governed by the perturbations E_1 in the electric field and $\partial_x P_e$ in the pressure gradient,

$$\frac{\partial v_e}{\partial t} = -\frac{e}{m}E_1 - \frac{1}{n_0 m}\frac{\partial P_e}{\partial x} \tag{1}$$

where n_0 is the equilibrium concentration. In the presence of cold ions (cold in the sense that their temperature is negligible with respect to that of the electrons), the perturbation n_e in the electron concentration is the unique source for the mentioned electric field,

$$\frac{\partial E_1}{\partial x} = -\frac{e}{\epsilon_0}n_e \tag{2}$$

where ϵ_0 is the vacuum electric permittivity. Differentiating both sides of Equation (1) with respect to space, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial v_e}{\partial t}\right) = \frac{e^2}{\epsilon_0 m}n_e - \frac{1}{n_0 m}\frac{\partial}{\partial x}\left(\frac{\partial P_e}{\partial x}\right) \tag{3}$$

where use has been made of Equation (2). In the absence of a source and/or a sink of matter, the conservation of mass in the electron gas establishes that

$$\frac{\partial n_e}{\partial t} = -n_0\frac{\partial v_e}{\partial x} \tag{4}$$

Differentiating both sides of Equation (3) with respect to time, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 v_e}{\partial t^2}\right) = -\frac{n_0 e^2}{\epsilon_0 m}\frac{\partial v_e}{\partial x} - \frac{1}{n_0 m}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\left(\frac{\partial P_e}{\partial x}\right)\right] \tag{5}$$

where use has been made of Equation (4). Using the electron plasma frequency

$$\omega_{pe} = \left(\frac{n_0 e^2}{\epsilon_0 m}\right)^{1/2} \tag{6}$$

we may integrate both sides of Equation (5) in space (the perturbation must vanish at infinity), to obtain

$$\frac{\partial^2 v_e}{\partial t^2} = -\omega_{pe}^2 v_e - \frac{1}{n_0 m}\frac{\partial}{\partial t}\left(\frac{\partial P_e}{\partial x}\right) \tag{7}$$

Now, Equation (7) should be supplied with an expression capable of connecting both gradients of pressure and concentration, essentially, an equation of state. For a Maxwellian gas at temperature T, with f degrees of freedom, such a relation could be adequately given by $\partial_x P_e = (2/f + 1)k_B T \partial_x n_e$, where k_B is the Boltzmann constant. Notwithstanding, in this work, we are interested in a non-Maxwellian gas, specifically, a suprathermal gas, with one degree of freedom. In this case, the appropriate equation of state would be

$$\frac{\partial P_e}{\partial x} = \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{\beta}\right)k_B T\frac{\partial n_e}{\partial x} \tag{8}$$

where the (deformation) parameter β $(0 < \beta < 1)$ is introduced through a mapping from the Kappa distribution [43] to avoid divergences, as detailed in Appendix A. In the limit $\beta \to 1$, Equation (8) recovers the Maxwellian equation of state $\partial_x P_e = 3k_B T \partial_x n_e$, for $f = 1$. Substituting Equation (8) in Equation (7), we get

$$\frac{\partial^2 v_e}{\partial t^2} = -\omega_{pe}^2 v_e - \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{\beta}\right)\frac{k_B T}{n_0 m}\frac{\partial}{\partial x}\left(\frac{\partial n_e}{\partial t}\right) \tag{9}$$

Introducing the thermal speed

$$v_{th} = \left(\frac{k_B T}{m}\right)^{1/2} \tag{10}$$

Equation (9) may be read as

$$\frac{\partial^2 v_e}{\partial t^2} = -\omega_{pe}^2 v_e + \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{\beta}\right)v_{th}^2\frac{\partial^2 v_e}{\partial x^2} \tag{11}$$

where use has been made of Equation (4). It should be noted how suprathermal effects correct the wave-like Equation (11). As β decreases, the reminiscent out-of-equilibrium pressure force term, the second term on the right-hand side of Equation (11) increases.

3 | Nonlinear Schrödinger Equation

As mentioned in Section 1, one may derive the NLSE by introducing a second-order perturbation in Equation (11) [42]. To see how it may be done, start by considering a disturbance in the electron plasma frequency ω_{pe} given by Equation (6). Such a disturbance may be regarded as a perturbation n_1 in the equilibrium concentration n_0,

$$n_0 \to n_0 + n_1 \Rightarrow \omega_{pe}^2 \to \left(1 + \frac{n_1}{n_0}\right)\omega_{pe}^2 \tag{12}$$

where we require quasi-neutrality, that is, $n_1 \approx n_e \approx n_i$, with "e" and "i" standing for "electrons" and "ions," respectively. We note that this is the only second-order perturbation that we consider in this work. As a consequence of such a disturbance, Equation (11) should be replaced with

$$\frac{\partial^2 v_e}{\partial t^2} = -\left(1 + \frac{n_1}{n_0}\right)\omega_{pe}^2 v_e + \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{\beta}\right)v_{th}^2\frac{\partial^2 v_e}{\partial x^2} \tag{13}$$

In order to describe the quasineutral motion of electrons accompanying ions, as the latter move to give rise to n_1, we express the perturbation in the electron flow by

$$v_e(x,t) = v_1(x, t_{sl}) e^{-\imath\omega_{pe}t} \tag{14}$$

where v_1, which includes both fast and slow fluctuations in space, depends on a slow-time scale t_{sl}. Differentiating both sides of Equation (14) twice with respect to time, we have

$$\frac{\partial^2 v_e}{\partial t^2} = \left(\frac{\partial^2 v_1}{\partial t_{sl}^2} - 2\imath\omega_{pe}\frac{\partial v_1}{\partial t_{sl}} - \omega_{pe}^2 v_1\right) e^{-\imath\omega_{pe}t} \tag{15}$$

Equation (15) may be approximated by its terms that oscillate faster in time,

$$\left|\frac{\partial^2 v_1}{\partial t_{sl}^2}\right| \ll \left|\omega_{pe}^2 v_1\right| \Rightarrow \frac{\partial^2 v_e}{\partial t^2} \approx -\left(2\imath\omega_{pe}\frac{\partial v_1}{\partial t_{sl}} + \omega_{pe}^2 v_1\right) e^{-\imath\omega_{pe}t} \tag{16}$$

As a consequence of the approximation in (16), Equation (13) approaches

$$\imath\frac{\partial v_e}{\partial t} = \frac{n_1/n_0}{2}\omega_{pe} v_e - \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{2\beta}\right)\frac{v_{th}^2}{\omega_{pe}}\frac{\partial^2 v_e}{\partial x^2} \tag{17}$$

Equation (17) shows that v_e is given in terms of three independent variables, namely t, x, and n_1. Then, to achieve a dimension free formulation of Equation (17), we take advantage of time, space, concentration, and speed scales, respectively, ω_{pe}^{-1}, λ_{De} (to be soon clarified), n_0, and v_{th} to define the normalized quantities

$$\tau = \frac{t}{\omega_{pe}^{-1}}, \quad \chi = \frac{x}{\lambda_{De}}, \quad \eta = \frac{n_1}{n_0}, \quad \psi = \frac{v_e}{v_{th}} \tag{18}$$

Thus, making use of (18), the dimension free form of Equation (17) may be written as

$$\imath\frac{\partial\psi}{\partial\tau} = \frac{\eta}{2}\psi - \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{2\beta}\right)\frac{v_{th}^2}{\omega_{pe}^2}\frac{1}{\lambda_{De}^2}\frac{\partial^2\psi}{\partial\chi^2} \tag{19}$$

Finally, identifying the electron Debye length

$$\lambda_{De} = \frac{v_{th}}{\omega_{pe}} = \left(\frac{\epsilon_0 k_B T}{n_0 e^2}\right)^{1/2} \tag{20}$$

we conclude that Equation (19) may be read as

$$\imath\frac{\partial\psi}{\partial\tau} = \frac{\eta}{2}\psi - \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{2\beta}\right)\frac{\partial^2\psi}{\partial\chi^2} \tag{21}$$

The reason for the perturbation n_1 in the electron concentration can be well attributed to the conjugate influence of both ponderomotive force and cold ion motion. In fact, as long as those effects are simultaneously taken into account, the normalized disturbance η may be expressed by (combine Equation (B9) in Appendix B with Equation (C5) in Appendix C)

$$\eta = -\left[\frac{|\psi|^2}{2(3-\beta) - 4\phi^2/\mu^2}\right] \tag{22}$$

where ϕ is the phase speed of an ion wave normalized by v_{th}, and μ^2 is the ratio of electron to ion mass. Introducing the coefficients

$$p = \left[\frac{1}{8(3-\beta) - 16\phi^2/\mu^2}\right], \qquad q = \left(\frac{3-\beta}{2}\right)\left(\frac{2+\beta}{2\beta}\right) \tag{23}$$

substitution of Equation (22) in Equation (21) gives the NLSE,

$$\imath\frac{\partial\psi}{\partial\tau} = -2p|\psi|^2\psi - q\frac{\partial^2\psi}{\partial\chi^2} \tag{24}$$

It should be noted that, although we have derived the NLSE by assuming that ψ is the normalized electron flow, at this point, Equation (24) is also satisfied by the normalized electron concentration and electric potential (see these normalizations in (18)). Last, but not least, we emphasize how suprathermal effects correct the NLSE. As β decreases, q increases, and the "Hamiltonian" term, the second term on the right-hand side of Equation (24) gains prominence.

4 | Solitonic Solutions

In order to explore the qualitative features of the suprathermal solitons supported by the NLSE, it suffices to examine the behavior of the amplitude A, and half-width $\Delta/2$ of the basic solution

$$\psi(\chi,\tau) = A\,\mathrm{sech}\left(\frac{\chi}{\Delta/2}\right)e^{\imath\Omega\tau} \tag{25}$$

to Equation (24) in terms of the deformation parameter β, where Ω is some frequency normalized by the electron plasma frequency ω_{pe}. Substituting Equation (25) in Equation (24), we obtain

$$\left(\Omega - \frac{q}{\Delta^2/4}\right) = 2\left(pA^2 - \frac{q}{\Delta^2/4}\right)\mathrm{sech}^2\left(\frac{\chi}{\Delta/2}\right) \tag{26}$$

Equation (26) is satisfied for all possible values of χ, provided that the soliton half-width $\Delta/2$, and amplitude A are expressed by

$$\frac{\Delta}{2} = \left(\frac{q}{\Omega}\right)^{1/2}, \qquad A = \left(\frac{\Omega}{p}\right)^{1/2} \tag{27}$$

in terms of the coefficients q, and p, which, in accordance with (23), are, in turn, expressed in terms of the deformation parameter β. It should be noted that (23) and (27) establish the constraint

$$\frac{\phi^2}{\mu^2} < \frac{3-\beta}{2} \tag{28}$$

on the ratio ϕ^2/μ^2 of normalized quantities. Let us discuss the behavior of the suprathermal soliton properties by exhibiting some selected numerical illustrations of our results.

In Figure 1, we show the upper-limit, $\left(\phi^2/\mu^2\right)_{ul}$, of the ratio ϕ^2/μ^2 as a function of the deformation parameter β, in accordance with the constraint (28). We see that intense suprathermal effects ($\beta \to 0$) may increase the ratio ϕ^2/μ^2 by up to 50%.

In Figure 2, we show the amplitude A of the soliton as a function of the deformation parameter β, in accordance with (27)

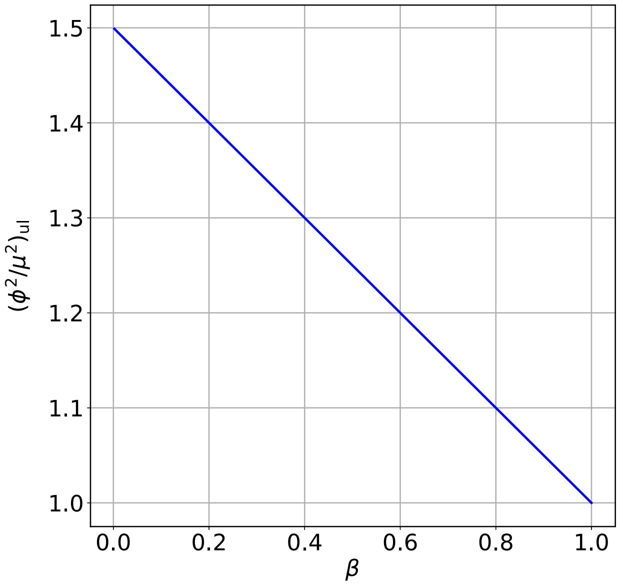

FIGURE 1 | The upper-limit, $\left(\phi^2/\mu^2\right)_{ul}$, of the ratio ϕ^2/μ^2 as a function of the deformation parameter β, according to the constraint (28). Intense suprathermal effects ($\beta \to 0$) may increase such a ratio by up to 50%.

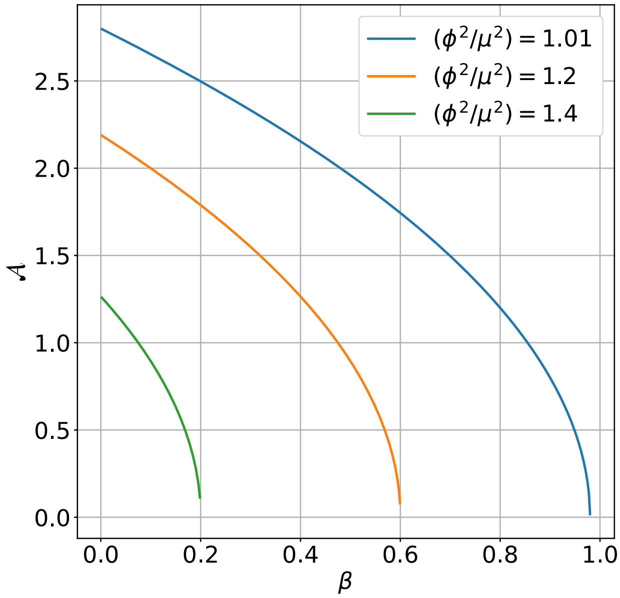

FIGURE 2 | The soliton amplitude A as a function of the deformation parameter β, according to (27) and (23), for selected possible values of the ratio ϕ^2/μ^2 allowed by the constraint (28). The amplitude is generally smaller when suprathermal effects are more intense ($\phi^2/\mu^2 \to 1.5$). The soliton frequency is taken to be equal to the electron plasma frequency ($\Omega = 1$, see Equation (25)), with no loss of generality.

and (23), for selected possible values of the ratio ϕ^2/μ^2 allowed by the constraint (28). We see that the soliton amplitude A is generally smaller when suprathermal effects are more intense ($\phi^2/\mu^2 \to 1.5$).

In Figure 3, it is shown the half-width $\Delta/2$ of the soliton as a function of the deformation parameter β, in accordance with (27) and (23). We see that the soliton half-width $\Delta/2$ increases with no limit as the suprathermal effects intensify ($\beta \to 0$). Such a

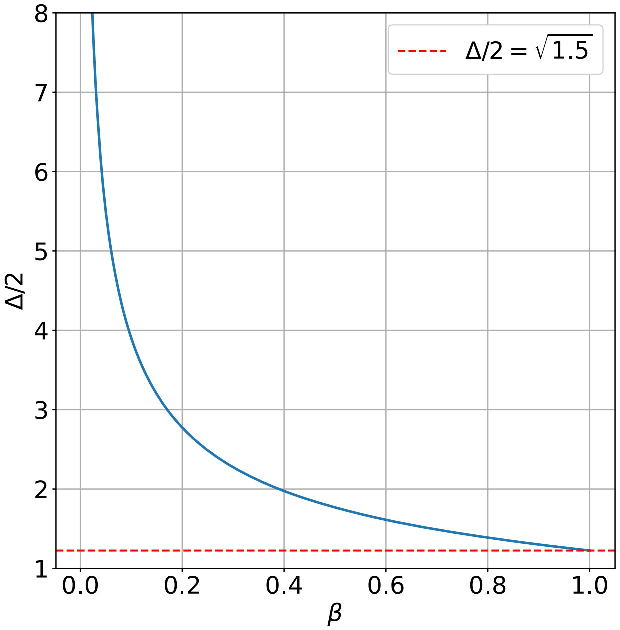

FIGURE 3 | The soliton half-width $\Delta/2$ in terms of β, the deformation parameter, according to (27) and (23). The half-width increases with no limit as the suprathermal effects intensify ($\beta \to 0$), suggesting a greater propensity for dissipation, or instability. The soliton frequency is taken to be equal to the electron plasma frequency ($\Omega = 1$ in Equation (25)), with no loss of generality.

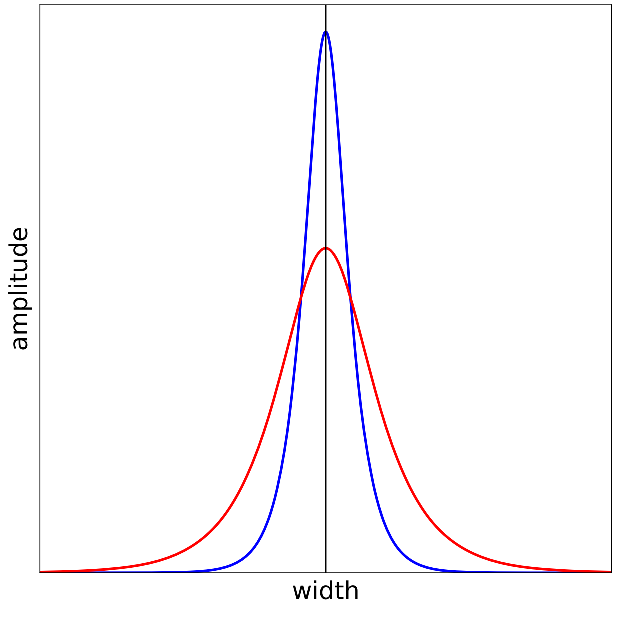

FIGURE 4 | Suprathermal effects are capable to decrease the amplitude and increase the half-width of solitonic solutions to the NLSE. Scales on axes are arbitrary.

result strongly suggests that suprathermal solitons can be more prone to dissipative effects, or influence from instabilities, than Maxwellian solitons are.

In Figure 4, it is shown a phase diagram, illustrating how a solitonic solution to the NLSE can change shape with increasing suprathermal effects. As we have found, the soliton amplitude decreases, and its half-width, increases.

5 | Conclusion

We have explored the behavior of the amplitude and half-width of suprathermal solitonic solutions to the NLSE with basis on a deformation parameter describing a non-Maxwellian distribution. It has been found that: (1) intense suprathermal effects may increase the ratio of the square of a normalized phase speed of an ion wave to a normalized electron mass by up to 50%, (2) when suprathermal effects are more intense, the soliton amplitude is generally smaller, and (3) as suprathermal effects intensify, the soliton half-width increases with no limit, suggesting that such solutions can be more prone to dissipation or instability.

Observations, measurements, numerical simulations, and theoretical models for the scaling of solitonic properties with suprathermal parameters in both laboratory and space plasmas have attracted the attention of several researchers [44–48]. In this sense, given that our formulation provides a rigorous analytical treatment of the dynamics of suprathermal solitons, we may advocate that it can actually contribute to more feasible and reliable comparisons and contrasts of theory and numerical simulations with experiments and observations. Let us illustrate our point through three examples.

In Reference [49], soliton solutions and chaotic dynamics have been explored in the (3 + 1)-dimensional Wazwaz–Benjamin–Bona–Mahony equation, a partial differential system that can include the NLSE, by a generalized rational exponential function approach, providing the time, and frequency dependence of the Lyapunov, and power spectra, respectively. In Reference [50], soliton solutions, bifurcation, and chaos analysis have been examined in a one-dimensional wave model that can include the Korteweg–De Vries equation, or NLSE, through inverse scattering, or Bäcklund, or Darboux transformations. In Reference [51], the so-called modulation instability, the exponential time growth of the modulus of the field due to a perturbation, and optical solitary wave solutions have been analyzed in the high-order dispersive parabolic Schrödinger–Hirota equation.

In all three examples mentioned above, the employment of our analysis can serve as a kind of *theoretical benchmark* for the simplest limiting situations. Such applications shall be discussed in future communications.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Appendix A

Equation of State

The Kappa distribution of the electron concentration n_e in terms of the electric potential Φ is given by [43]

$$n_e = n_0 \left[1 - \frac{1}{(\kappa - 3/2)} \left(\frac{e\Phi}{k_B \Theta}\right)\right]^{-(\kappa-1/2)} \tag{A1}$$

where the Kappa temperature Θ of the electron gas is related to its Maxwellian temperature T through

$$\Theta = T\left(\frac{\kappa}{\kappa - 3/2}\right) \tag{A2}$$

with κ denoting a spectral index, $3/2 < \kappa < \infty$. Equation (A2) shows that $\Theta > T$ for all possible values of κ. This is the generic reason why particles whose concentration satisfies Equation (A1) are said to be *suprathermal particles*. But, to be more precise, it should be emphasized that the key suprathermal feature of the Kappa distribution lies on its non-Maxwellian shape, that is, its *enhanced high-energy tail* in contrast with the Maxwellian Gaussian profile, not just on the temperature enhancement. In the limit $\kappa \to \infty$, Equation (A2) shows that $\Theta \to T$, and Equation (A1) recovers the Boltzmann relation $n_e = n_0 \exp\left[(e\Phi)/(k_B T)\right]$. However, in the limit $\kappa \to 3/2$, Equation (A1) approaches $n_e = n_0\left[1 - (2e\Phi)/(3k_B T)\right]^{-1}$, reproducing the behavior ubiquitously observed in space plasmas [52–54], despite the divergence of Θ in the same approximation, as shown by Equation (A2).

In a series of previous works, we have derived equations of state for the Thomas–Fermi distribution [55], Kappa distribution [56], Tsallis distribution [57], the Korteweg–De Vries equation [58], Bohm–Gross dispersion [59], mapping of Kappa onto Tsallis distributions [60], and diffusion equation [61]. For a suprathermal electron gas, we have shown that the Kappa distribution can be mapped onto the Beta distribution

$$n_e = n_0 \left[1 - (1-\beta)\left(\frac{e\Phi}{k_B T_1}\right)\right]^{-1/(1-\beta)} \tag{A3}$$

where the Beta temperature T_1 of the electron gas is related to the Maxwellian temperature T through

$$T_1 = T\left(\frac{3-\beta}{2}\right) \tag{A4}$$

with β denoting a deformation parameter, $0 < \beta < 1$, provided that

$$\beta = \frac{\kappa - 3/2}{\kappa - 1/2} \tag{A5}$$

In the limit $\beta \to 1$ ($\kappa \to \infty$), the Beta distribution recovers the Maxwellian results, as much as the Kappa distribution does. But, the Beta distribution offers a great advantage over the Kappa one: in the limit $\beta \to 0$ ($\kappa \to 3/2$), Equation (A4) shows that $T_1 \to 3T/2$, that is, the Beta temperature acquires a finite upper-bound value for Equation (A3) reproducing the aforementioned ubiquitous behavior exhibited by space plasmas. In this sense, we can say that our suprathermal deformation parameter β plays a role similar to that played by the Tsallis non-extensive index [62].

In one dimension, Equation (A3) shows that the pressure gradient is given by

$$\frac{\partial P_e}{\partial x} = \frac{(3-\beta)}{2\beta} n_0 k_B T \left\{\frac{2}{f}\frac{\partial}{\partial x}\left(\frac{n_e}{n_0}\right) + \frac{\partial}{\partial x}\left[\left(\frac{n_e}{n_0}\right)^{\beta}\right]\right\} \tag{A6}$$

where f is the number of degrees of freedom of a suprathermal gas of electrons. In the limit $\beta \to 1$ ($\kappa \to \infty$), Equation (A6) recovers $\partial_x P_e = (2/f + 1) k_B T \partial_x n_e$, as it should be for a Maxwellian distribution.

Appendix B

Ponderomotive Force

On the one hand, the low-frequency limit of the equation of motion (1) is attained by neglecting the inertial term, changing the thermodynamic regime of the electron gas from adiabatic ($f = 1$) to isothermal ($f \to \infty$) (in the limit $n \approx n_0 + n_1$ of Equation (A3)), and including the ponderomotive force (the gradient of the time average of the field energy),

$$0 = -n_0 e E_1 - \frac{(3-\beta)}{2} k_B T \frac{\partial n_1}{\partial x} - \frac{\partial}{\partial x}\left\langle \frac{\epsilon_0 E_1^2}{2}\right\rangle \tag{B1}$$

On the other hand, the high-frequency limit of Equation (1) is obtained by keeping the inertial term and neglecting the thermal one,

$$\frac{\partial v_e}{\partial t} = -\frac{e}{m} E_1 \tag{B2}$$

Taking the time average of the square of Equation (B2), we get

$$\omega_{pe}^2 \left\langle v_e^2 \right\rangle = \frac{e^2}{m^2}\left\langle E_1^2 \right\rangle \tag{B3}$$

Substituting Equation (6) in Equation (B3), the time average of the field energy equals that of the kinetic one,

$$\left\langle \frac{\epsilon_0 E_1^2}{2}\right\rangle = \left\langle \frac{n_0 m v_e^2}{2}\right\rangle \tag{B4}$$

Expressing the electric field E_1 in terms of the usual gradient of a scalar potential Φ_1, that is, $E_1 = -\partial_x \Phi_1$, and substituting Equation (B4) in Equation (B1), we have

$$0 = n_0 e \frac{\partial \Phi_1}{\partial x} - \frac{(3-\beta)}{2} k_B T \frac{\partial n_1}{\partial x} - \frac{n_0 m}{2}\frac{\partial}{\partial x}\left\langle v_e^2\right\rangle \tag{B5}$$

Introducing the potential energy normalized by the thermal energy,

$$u = \frac{e\Phi_1}{k_B T} \tag{B6}$$

and making use of (18), Equation (B5) may be read as

$$0 = \frac{\partial u}{\partial \chi} - \frac{(3-\beta)}{2}\frac{\partial \eta}{\partial \chi} - \frac{1}{2}\frac{\partial}{\partial \chi}\left\langle \psi^2 \right\rangle \tag{B7}$$

Noting that

$$\left\langle \psi^2 \right\rangle = \frac{|\psi|^2}{2} \tag{B8}$$

we may integrate Equation (B7) in space, to obtain

$$u = \frac{(3-\beta)}{2}\eta + \frac{|\psi|^2}{4} \tag{B9}$$

(the perturbation vanishes at infinity).

Appendix C

Cold Ion Motion

The motion of the cold ions (their temperature is negligible with respect to that of electrons) is described by the linearized one-dimensional equations

$$\frac{\partial v_i}{\partial t} = -\frac{e}{M}\frac{\partial \Phi_1}{\partial x}, \quad \frac{\partial n_1}{\partial t} = -n_0 \frac{\partial v_i}{\partial x} \tag{C1}$$

where $M, e > 0$, and υ_i are the mass, charge, and flow of the ions, respectively. We may eliminate υ_i from (C1), to get

$$\frac{\partial^2 n_1}{\partial t^2} = \frac{n_0 e}{M} \frac{\partial^2 \Phi_1}{\partial x^2} \tag{C2}$$

Equation (C2) is a wave equation for ions, provided that the condition

$$\frac{n_0 e}{M} \Phi_1 = \upsilon_{\text{ph}}^2 n_1 \tag{C3}$$

be satisfied by the perturbation, where υ_{ph} is the phase speed of the wave. In terms of the normalized quantities

$$\phi^2 = \frac{\upsilon_{\text{ph}}^2}{\upsilon_{\text{th}}^2}, \qquad \mu^2 = \frac{m}{M} \tag{C4}$$

Equation (C3) may be read as

$$u = \frac{\phi^2}{\mu^2} \eta \tag{C5}$$

where use has been made of Equation (B6), and (18).