

# VI – Atratores Caóticos

Referência Principal: *Chaos*

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Springer (1997)

# 1- Introdução

Movimentos regulares: periódicos ou quase-periódicos

Movimentos irregulares: caóticos

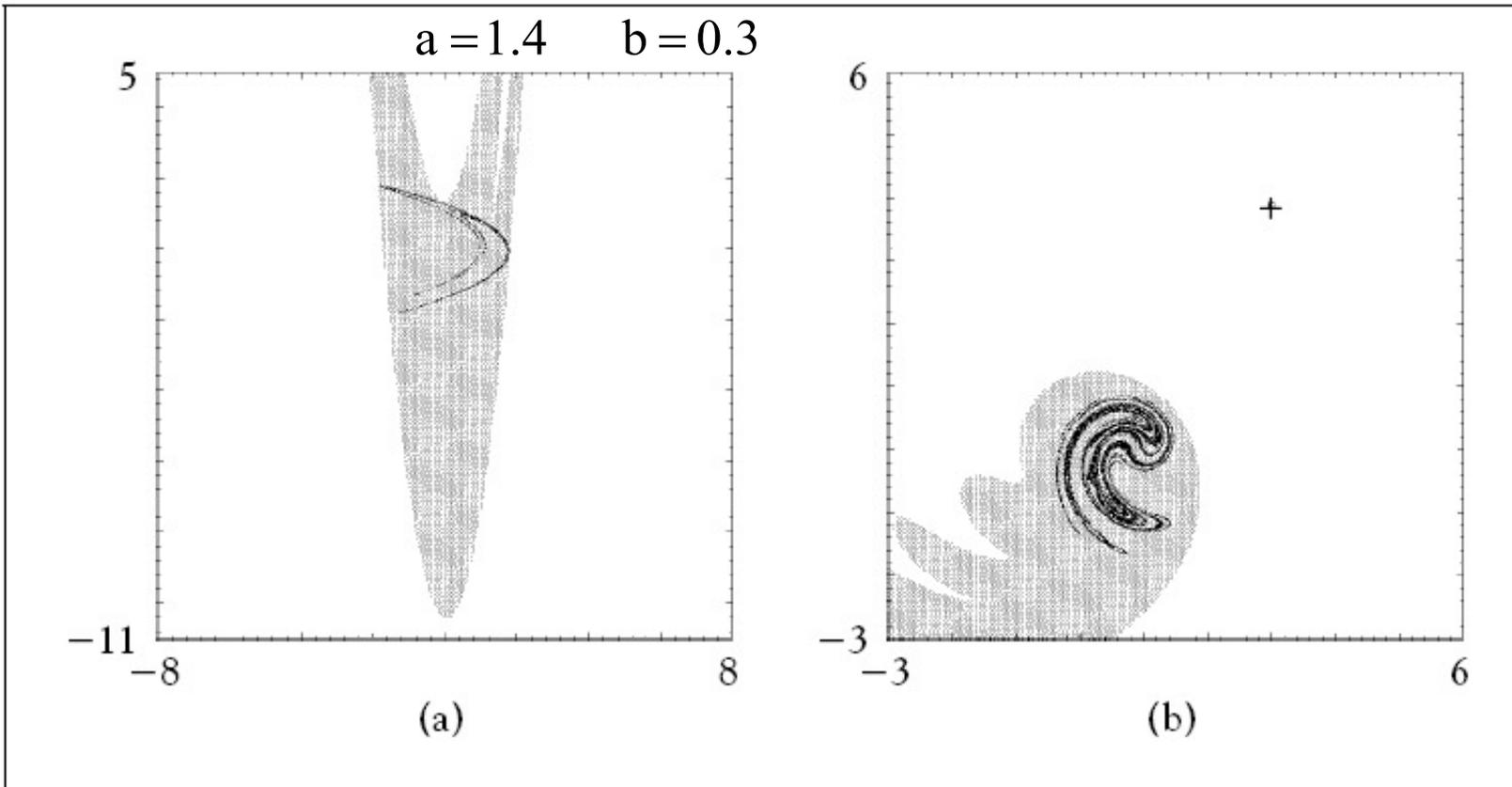
Atratores: atraem condições iniciais próximas à órbita observada

Atratores caóticos: instável localmente (expoente de Lyapunov positivo), mas estável globalmente (bacia de atração)

# Atratores e Bacias

Mapa de Hénon  $f(x, y) = (a - x^2 + by, x)$

Mapa de Ikeda



**Figure 6.1** Chaotic attractors of plane maps.

Each part shows a chaotic attractor in black inside a gray basin. (a) The chaotic attractor of the Hénon map. White points are attracted to infinity. (b) The chaotic attractor of the Ikeda map. Gray points approach the chaotic attractor; white points approach the sink in the upper right corner at  $(3.00, 3.89)$ .

Mapa de Ikeda

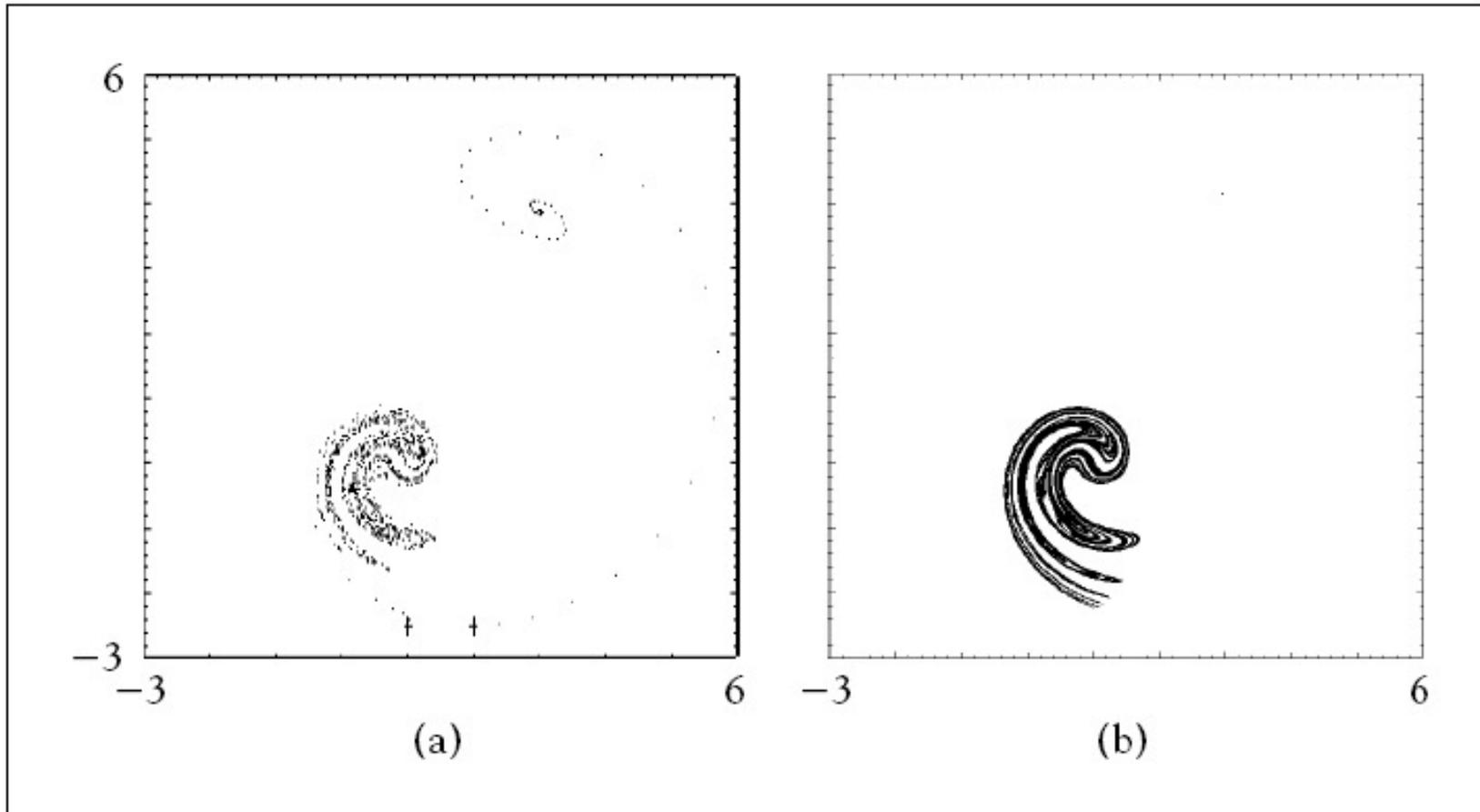
$$F(x, y) = \begin{cases} R + C_2 (x \cos \tau - y \sin \tau) \\ C_2 (x \sin \tau + y \cos \tau) \end{cases}$$

$$\tau = C_1 + \frac{-C_3}{1 + x^2 + y^2}$$

$C_1, C_2, C_3, R$  parâmetros reais

Para  $R = 0.9, C_1 = 0.4, C_2 = 0.9, C_3 = 6 \Rightarrow \begin{cases} \text{atrator caótico} \\ \text{atrator periódico} \end{cases}$

# Atratores do Mapa de Ikeda



**Figure 6.2** Two orbits of the Ikeda map.

(a) Two initial values are denoted by crosses. The next 1000 iterates of each initial value are plotted. One orbit approaches the chaotic attractor, the other approaches a sink. (b) Points 1,000,001–2,000,000 of each orbit are plotted. Part (b) shows some points of the forward limit set of the chaotic orbit of part (a).

## Mapa de Ikeda



Atrator caótico

$$R = 0.9, C_1 = 0.4, C_2 = 0.9, C_3 = 6$$

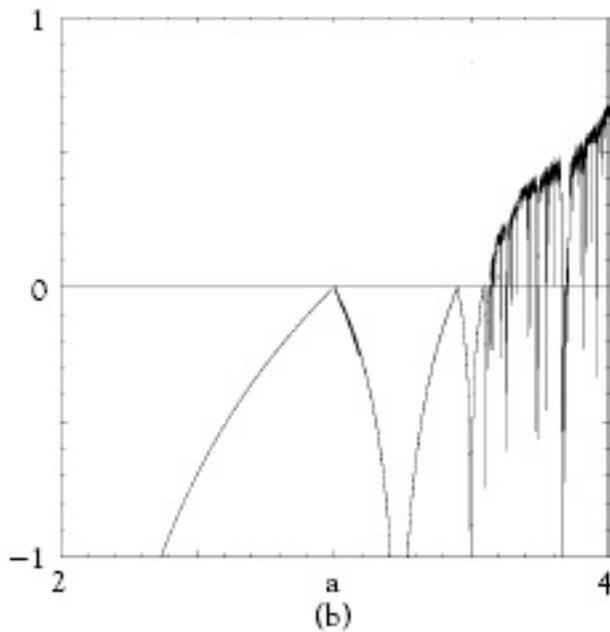
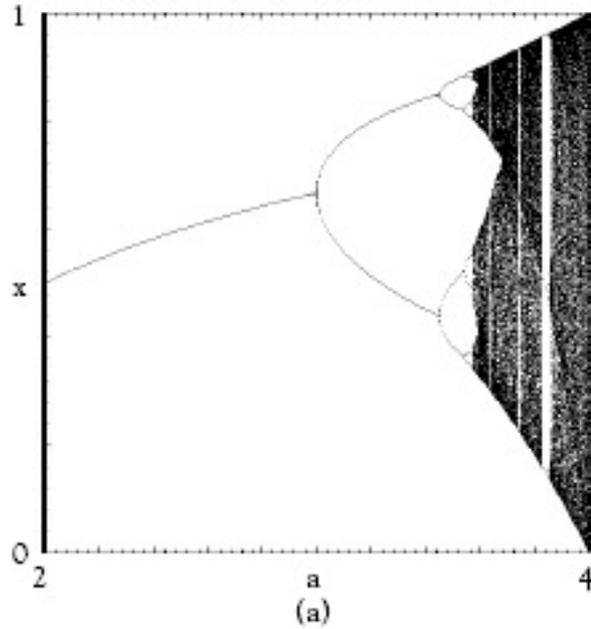
**Figure 5.3** The Ikeda attractor of Example 5.7.

The attractor has fractal structure and a largest Lyapunov exponent of approximately 0.51.

# Atrator e expoente de Lyapunov

Mapa logístico

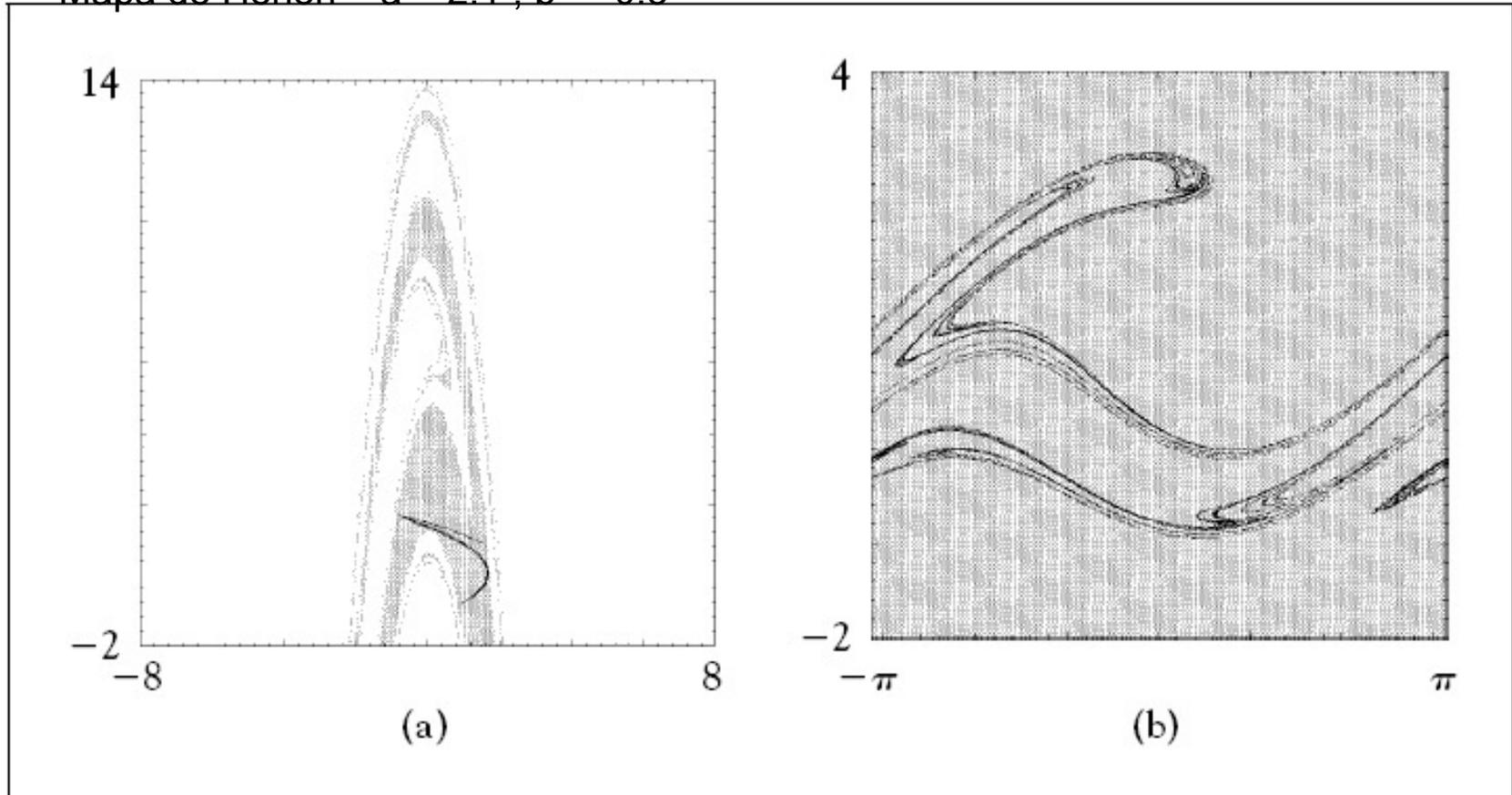
$$f(x) = a x (1 - x)$$



# Atratores Caóticos e suas Bacia

Mapa de Hénon  $a = 2.1$ ,  $b = -0.3$

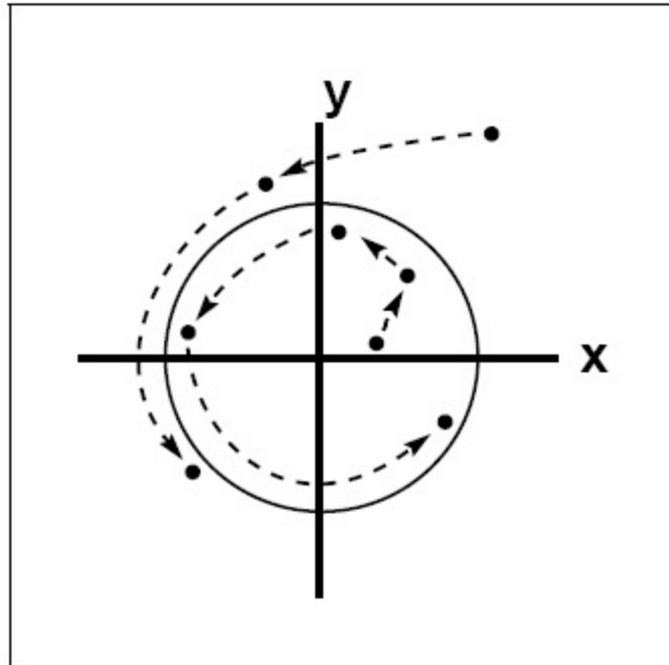
Pêndulo amortecido forçado



**Figure 6.4** More chaotic attractors of plane maps.

The attractors are shown in black inside their basins, which are shown in gray. (a) A chaotic attractor of the Hénon map (6.1) with  $a = 2.1$ ,  $b = -0.3$ . The basin boundary, as well as the attractor, is fractal. White points are attracted to infinity. (b) The chaotic attractor of the time- $2\pi$  map of the forced damped pendulum. The basin of this attractor, shown in gray, consists of virtually all initial conditions.

# Atrator Caótico no Círculo



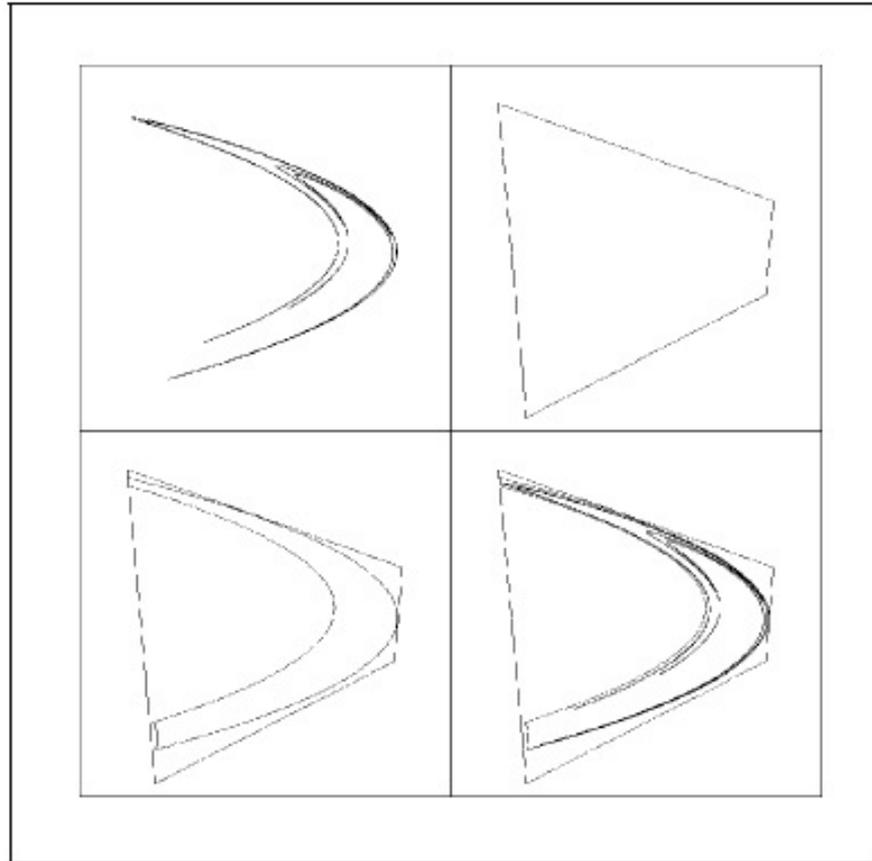
Ponto fixo instável em  $r = 0$

Círculo limite em  $r = 1$  é um atrator caótico ( $h = \ln 2$ )

**Figure 6.5** Map for which the unit circle is a chaotic attractor.

Two orbits are shown for the map  $f(r, \theta) = (r^{1/2}, 2\theta)$  given in polar coordinates. All orbits except the origin converge to the unit circle  $r = 1$ , which has a dense chaotic orbit.

# Atrator Caótico em Região Limitada



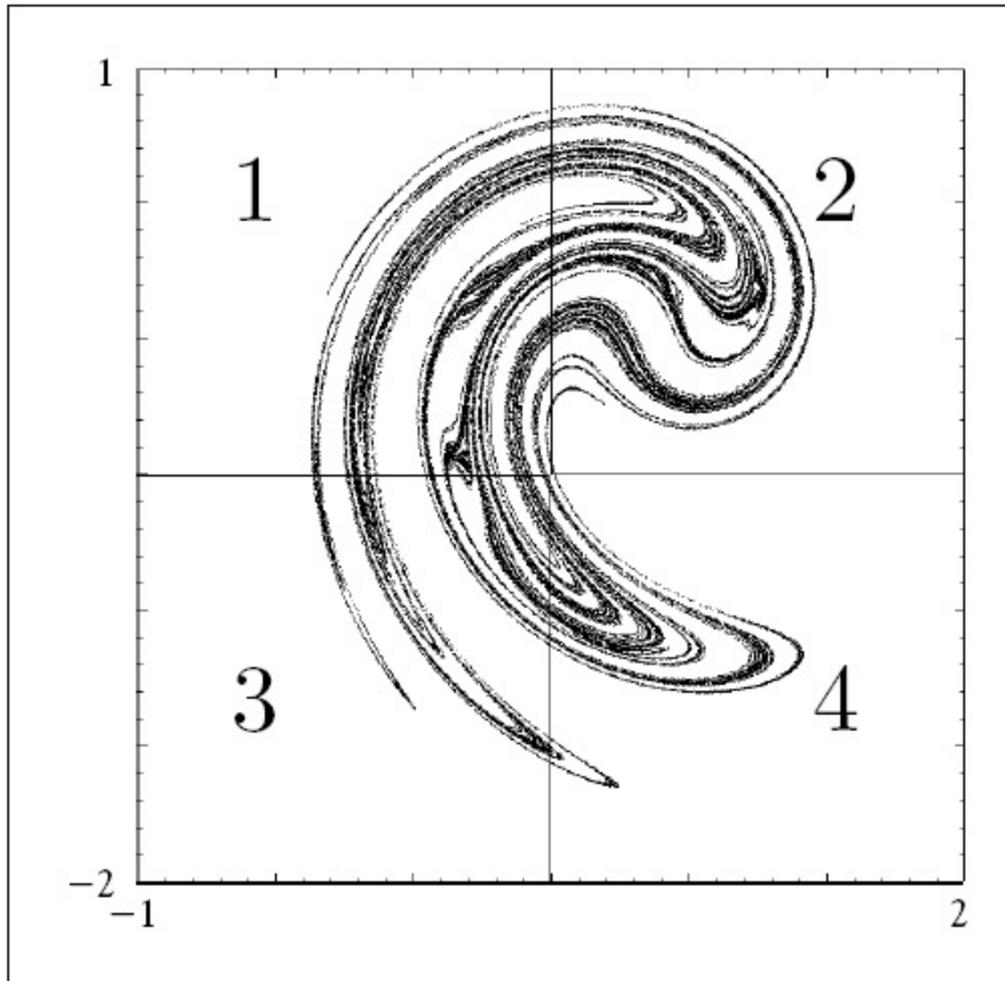
**Figure 6.7** The Hénon attractor and a trapping region.

The attractor of the map  $f(x, y) = (1.4 - x^2 + 0.3y, x)$  shown in (a) is contained in a quadrilateral trapping region shown in (b). The trapping region maps into itself and contracts in area by the factor 0.3. The quadrilateral and its image under  $f$  are shown in (c). Since  $f$  is area-contracting, forward iterates of the trapping region shrink down to a limit set (the attractor) that has zero area. The figures in (a) and (c) are superimposed in (d).

## 4- Medidas

- Avaliação da concentração do atrator em uma dada região.
- Distribuição em determinadas regiões.
- A medida de qualquer conjunto é um número não negativo.
- A medida de um número finito ou infinito de conjuntos mensuráveis é a soma das medidas desses conjuntos.

# Medida no Atrator de Ikeda



Distribuição de pontos nas quatro regiões indicadas.

Um ponto inicial no atrator, 100.000 pontos de uma órbita.

**Figure 6.11** 100,000 points on the Ikeda attractor.

The proportion landing in each of the 4 boxes is an approximation to the natural measure of the Ikeda attractor for that box. Boxes 1, 2, 3, and 4 contain about 30%, 36%, 17%, and 17%, respectively.

$A \subset B \Rightarrow B / A$ : complemento de A em B

A medida no espaço inteiro é 1

Para um conjunto fechado a medida é invariante

$$\mu(f^{-1}(S)) = \mu(S)$$

## Medida Natural

Definição: Fração das iterações da órbita  $\{f^n(x_0)\}$  no conjunto  $S$

$$F(x_0, S) = \lim_{n \rightarrow \infty} \frac{\#\{f^i(x_0) \in S : 1 \leq i \leq n\}}{n}$$

Exemplo de medidas:

Mapa  $f(x) = x/2$ ,  $x = 0$  é ponto atrator

$$S_1 = [-1, 1]$$

$\forall x_0 \neq 0$ , a órbita vai para esse conjunto (se já não começou nele)

$$\Rightarrow \text{medida } F(x_0, S_1) = 1$$

$$S_2 = [-1, -\delta] \cup [-\delta, 1], \quad \delta \cong 0$$

A órbita sai de  $S_2$ , vai para  $[-\delta, \delta]$  e não sai desse conjunto

$$\Rightarrow \text{medida } F(x_0, S_2) = 0$$

$$S_3 = [0, \infty] \begin{cases} x_0 < 0 \Rightarrow F(x_0, S_3) = 0 \\ x_0 > 0 \Rightarrow F(x_0, S_3) = 1 \end{cases}$$

Se  $f^n$  é atraída para o atrator de Hénon mas não o atinge,  
 $F(\vec{v}_0, S) = 0$ .

Mas se considerarmos uma distância  $r$  do atrator,

$$F(\vec{v}_0, N(r, S)) = 1,$$

sendo  $N(r, S) = \{x : \text{distância}(x, S) \leq r\}$  o conjunto de pontos a uma distância de  $S$  menor ou igual a  $r$ .

Definição :

$f$  : mapa

$S$  : conjunto fechado

A medida natural gerada pelo mapa  $f$ , chamada de medida de  $f$ , é

$$\mu_f(S) \equiv \lim_{r \rightarrow \infty} F(x_0, N(r, S))$$

Com essa definição, para o mapa  $f = x/2$ , obtemos

$F(x_0, N(r, S)) = 1$  para  $S = [0, \infty]$ , desde que  $r > 0$

Portanto  $\mu_f(S) = 1$

# 6- Medidas Invariantes para Mapas Unidimensionais

$f$  mapa suave em  $[0, 1]$ , contínuo por partes

$f, \frac{df}{dx}, \frac{d^2f}{dx^2}$  contínuas e limitadas, a não ser em  
um número limitado de pontos

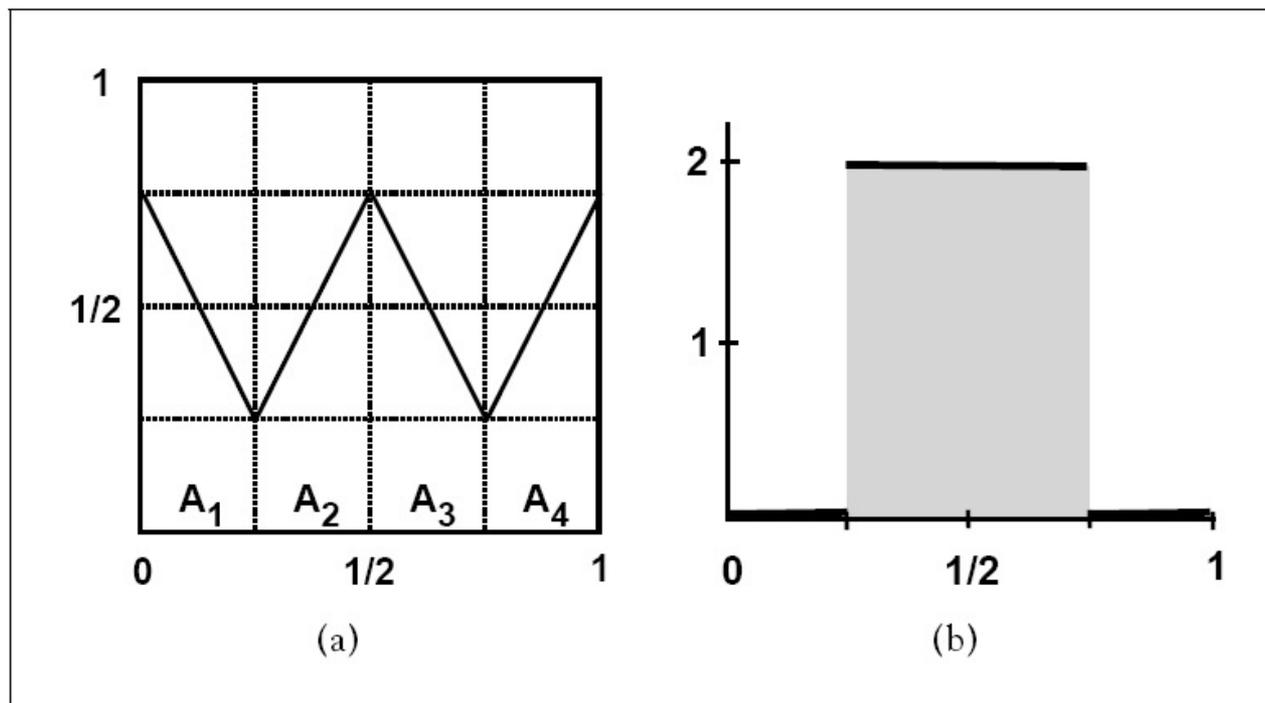
A medida natural é:  $\mu(S) = \int_S p(x) dx$

$f$  mapa suave em  $[0,1]$ , contínuo por partes

$f, \frac{df}{dx}, \frac{d^2f}{dx^2}$  contínuas e limitadas, a não ser em um número limitado de pontos

A medida natural é:  $\mu(S) = \int_S p(x) dx = 2 \times 0.5 = 1$

$$p = \begin{cases} p_1 = 0 \\ p_2 = 2 \end{cases}$$

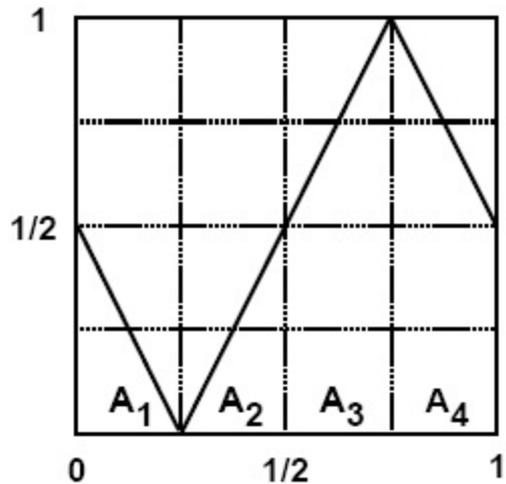


**Figure 6.12 The W-map.**

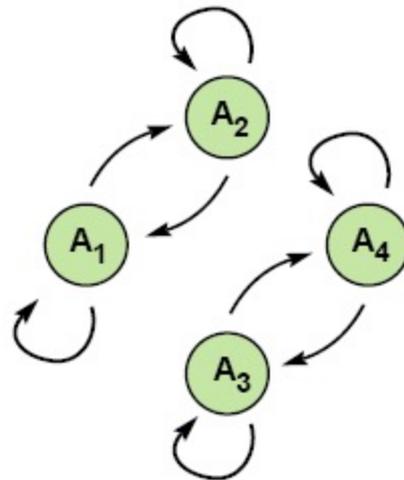
(a) The map is linear on each of four subintervals. (b) The density  $p(x)$  that defines the invariant measure of (a). According to this graph, the measure of an interval inside  $[1/4, 3/4]$  is twice its length.

A medida natural é:  $\mu(S) = \int_S p(x) dx = 1$

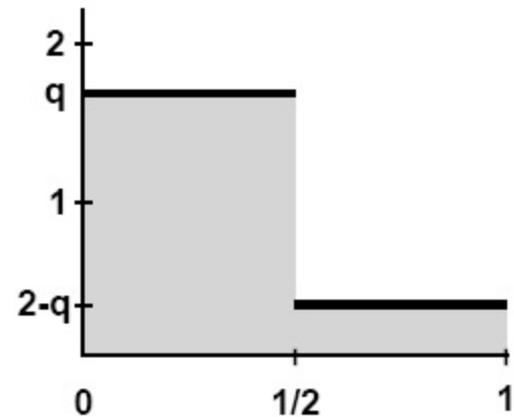
$$p = \begin{cases} p_1 = q \\ p_2 = 2 - q \end{cases}$$



(a)



(b)



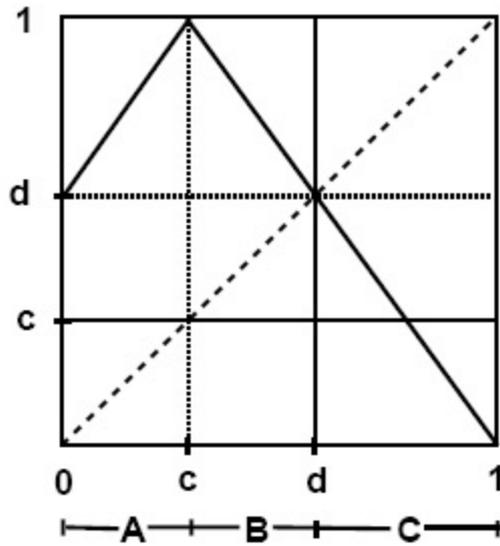
(c)

**Figure 6.13** A piecewise linear map with no dense orbit.

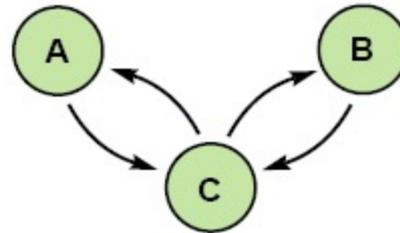
(a) Sketch of map. (b) Transition graph. (c) One invariant measure for (a). In this case, there is no unique invariant measure—there are infinitely many.

A medida natural é:  $\mu(S) = \int_S p(x) dx$

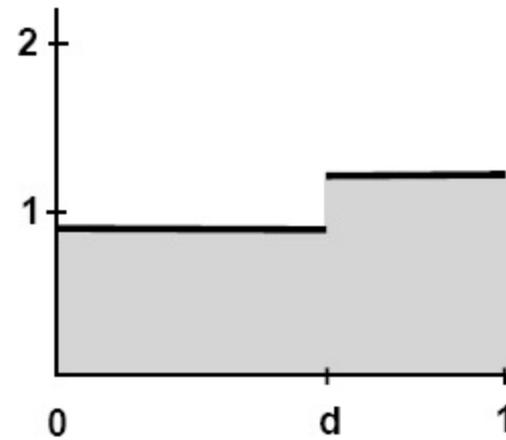
$$p = \begin{cases} p_1 = \frac{1}{2d} \\ p_2 = \frac{1}{\sqrt{2}d} \end{cases}$$



(a)



(b)



(c)

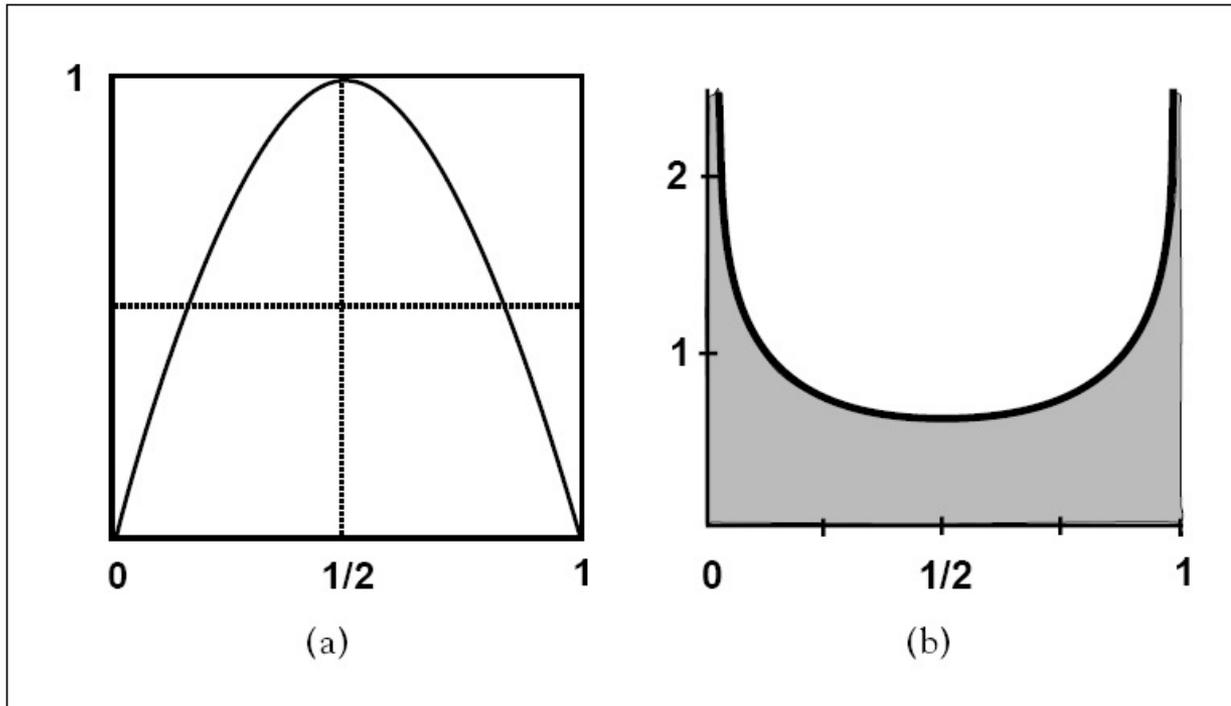
**Figure 6.14** Map in which the critical point is eventually periodic.

(a) Sketch of map. (b) Transition graph. (c) Invariant measure for (a).

# Mapa Logístico (b = 4)

A medida natural é:  $\mu(S) = \int_S p(x) dx$

$$p_{\text{tenda}} = 1 \quad \Rightarrow \quad p_{\text{logístico}} = \frac{1}{\pi \sqrt{x - x^2}}$$



**Figure 6.15** The logistic map  $G$ .

(a) Sketch of map. (b) The invariant measure for  $G$ .

Mapa logístico  $x_{n+1} = G(x_n) = 4x_n(1-x_n)$

Mapa da tenda  $x_{n+1} = T(x_n) = \begin{cases} 2x_n & 0 \leq x_n \leq 1/2 \\ 2(1-x_n) & 1/2 < x_n \leq 1 \end{cases}$

$$G \circ C(x) = C \circ T(x)$$

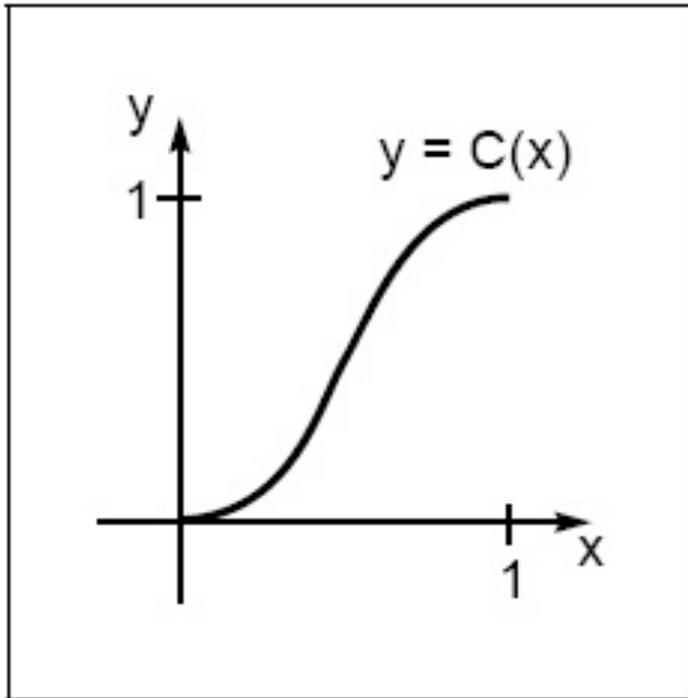
$\{x_i\}$  órbita caótica conjugada do mapa T

$\{C(x_i)\}$  órbita caótica conjugada do mapa G

$$h(\text{mapa T}) = h(\text{mapa G})$$

Portanto, o mapa logístico G (b=4) tem órbitas caóticas

# Conjugação entre os Mapas da Tenda e Logístico

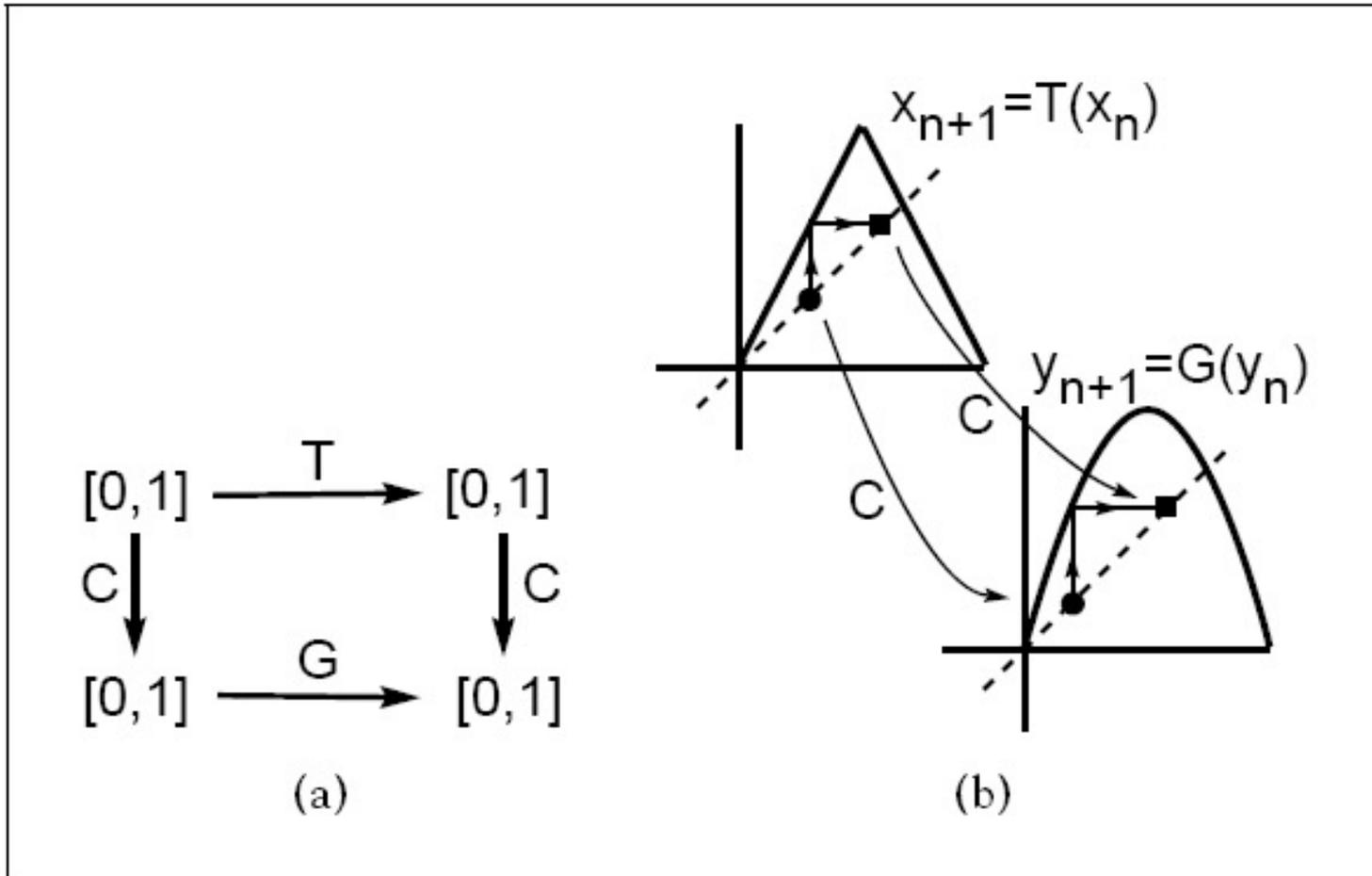


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Chaos

**Figure 3.4 The conjugacy map.**

The graph of  $C(x) = (1 - \cos \pi x)/2$  is continuous and one-to-one. Since  $CT(x) = GC(x)$  for all  $0 \leq x \leq 1$ , it is a conjugacy between the tent map and logistic map.

# Trajetórias Conjugadas



**Figure 3.5** Two illustrations of the fact that the maps  $T$  and  $G$  are conjugate. (a) Both ways of composing maps from upper left to lower right are equal. (b) If  $y_n = C(x_n)$  implies  $y_{n+1} = C(x_{n+1})$ , then  $C$  is a conjugacy between the  $x$ -map and the  $y$ -map.