VII – Equações Diferenciais

Referência Principal: *Chaos*

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Springer (1997)

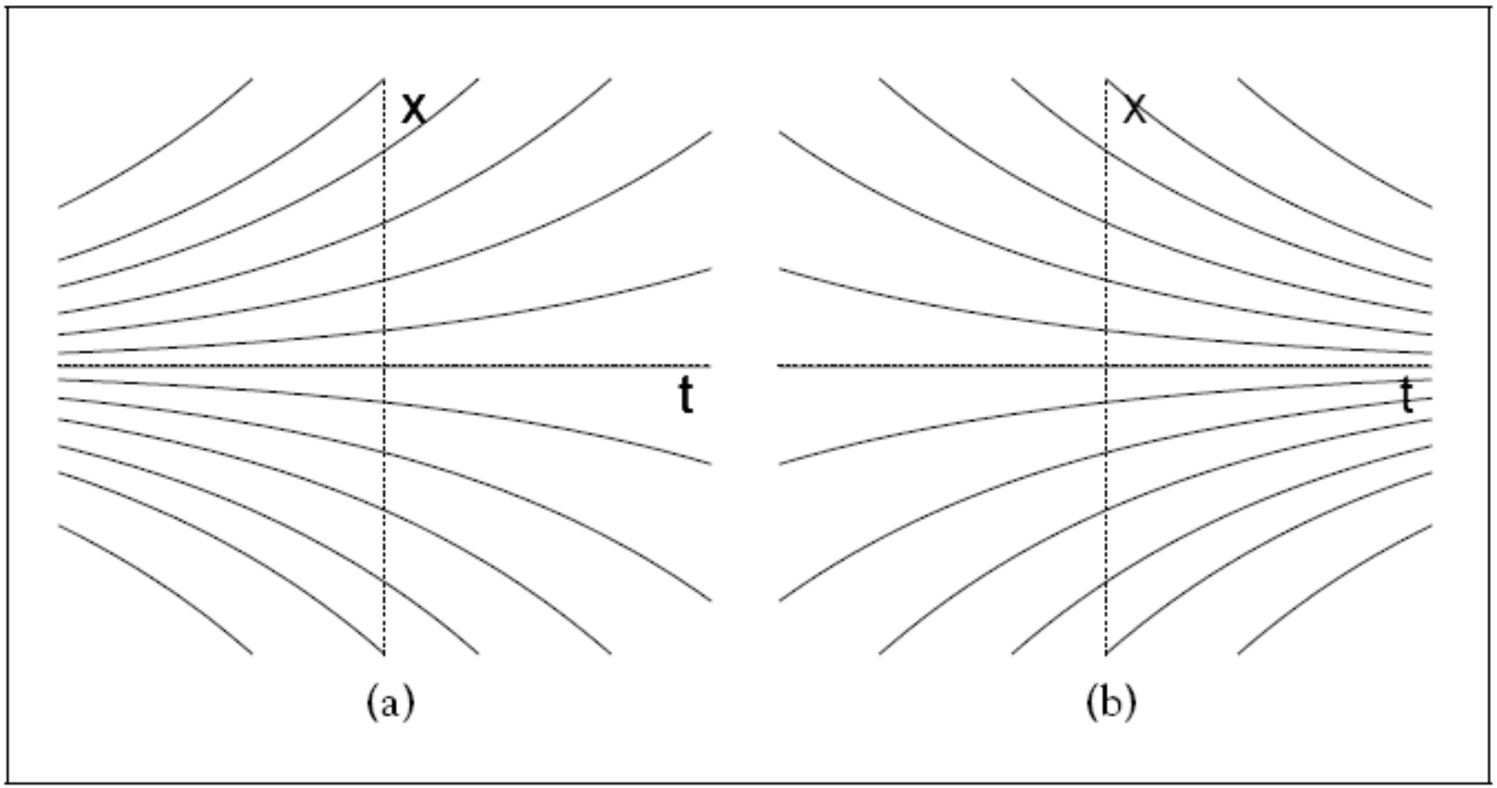


Figure 7.1 The family of solutions of $\dot{x} = ax$.
(a) $a > 0$: exponential growth (b) $a < 0$: exponential decay.

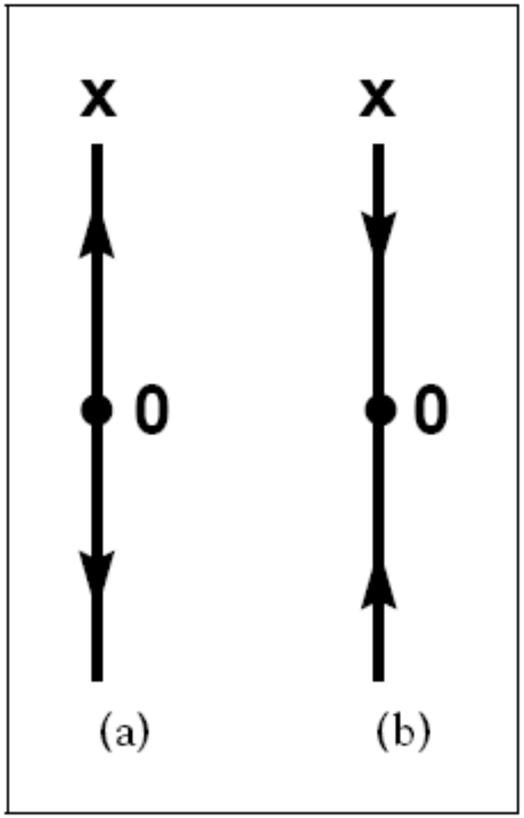


Figure 7.2 Phase portraits of $\dot{x} = ax$.
Since x is a scalar function, the phase space is the real line $\mathbb{R}$. (a) The direction of solutions is away from the equilibrium for $a > 0$. (b) The direction of solutions is toward the equilibrium for $a < 0$.

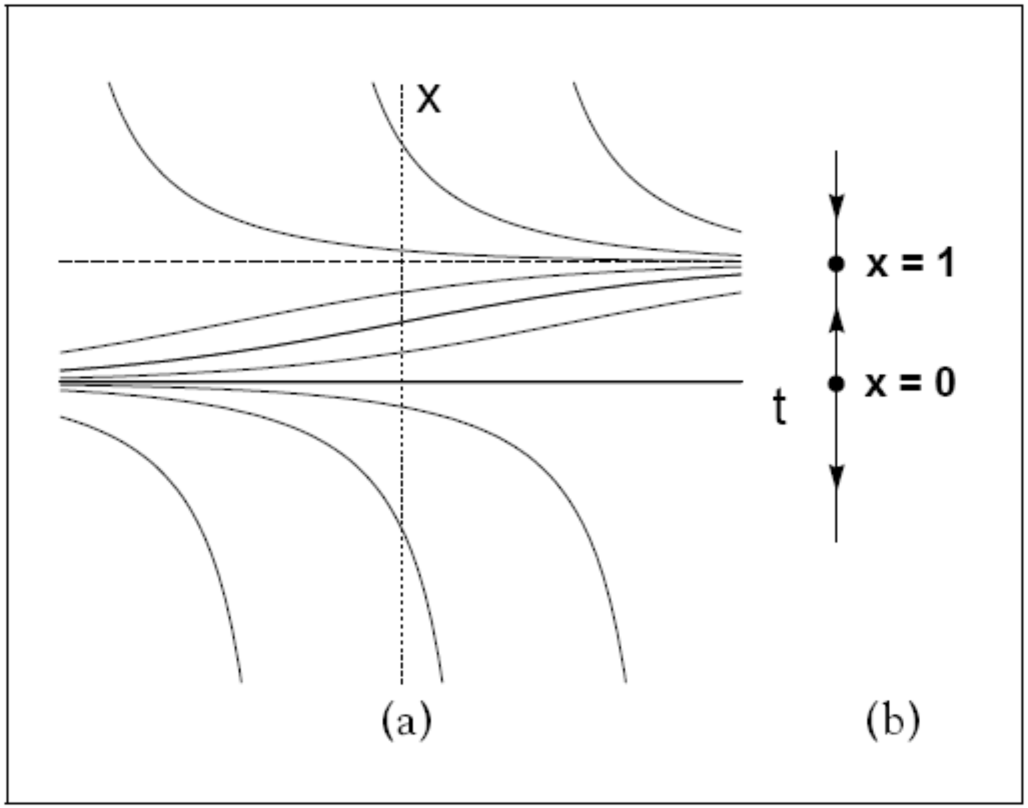


Figure 7.3 Solutions of the logistic differential equation.
(a) Solutions of the equation $\dot{x} = x(1 - x)$. Solution curves with positive initial conditions tend toward $x = 1$ as t increases. Curves with negative initial conditions diverge to $-\infty$. (b) The phase portrait provides a qualitative summary of the information contained in (a).

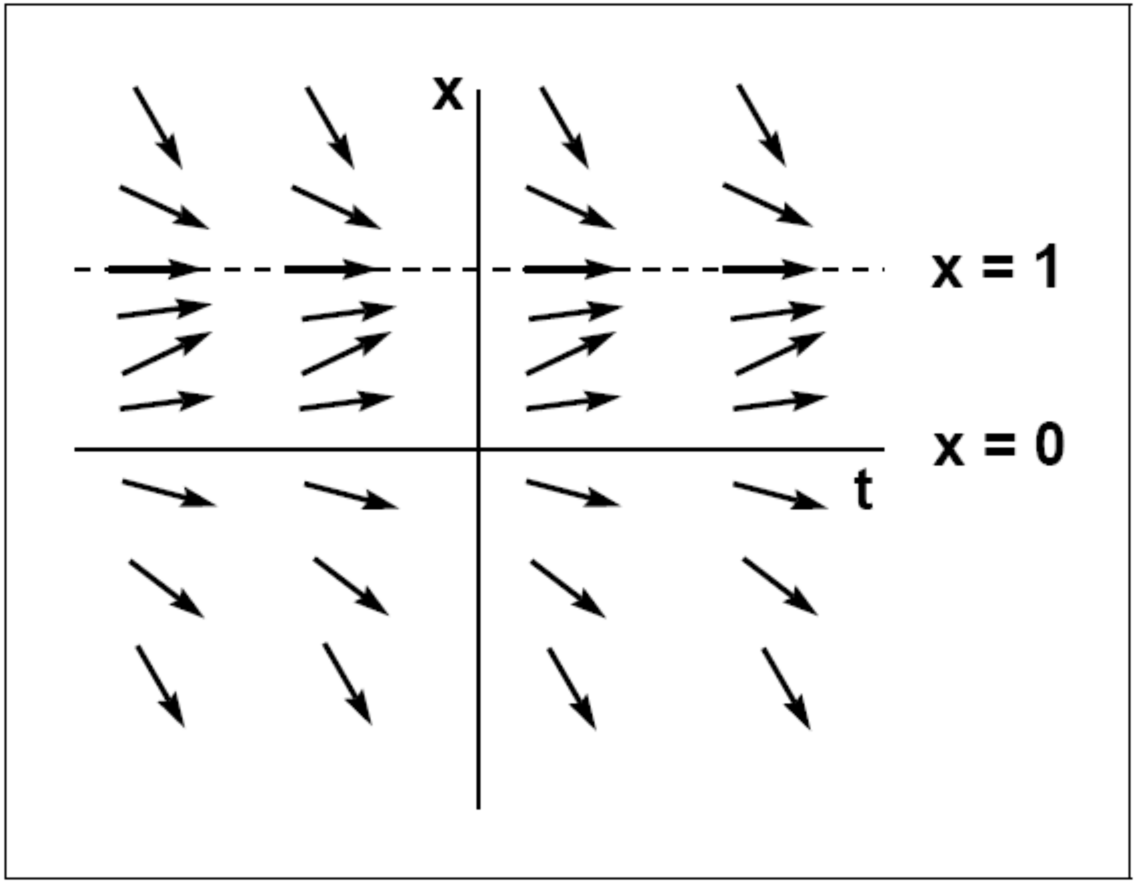


Figure 7.4 Slope field of the logistic differential equation.
At each point (t, x), a small arrow with slope $ax(1 - x)$ is plotted. Any solution must follow the arrows at all times. Compare the solutions in Figure 7.3.

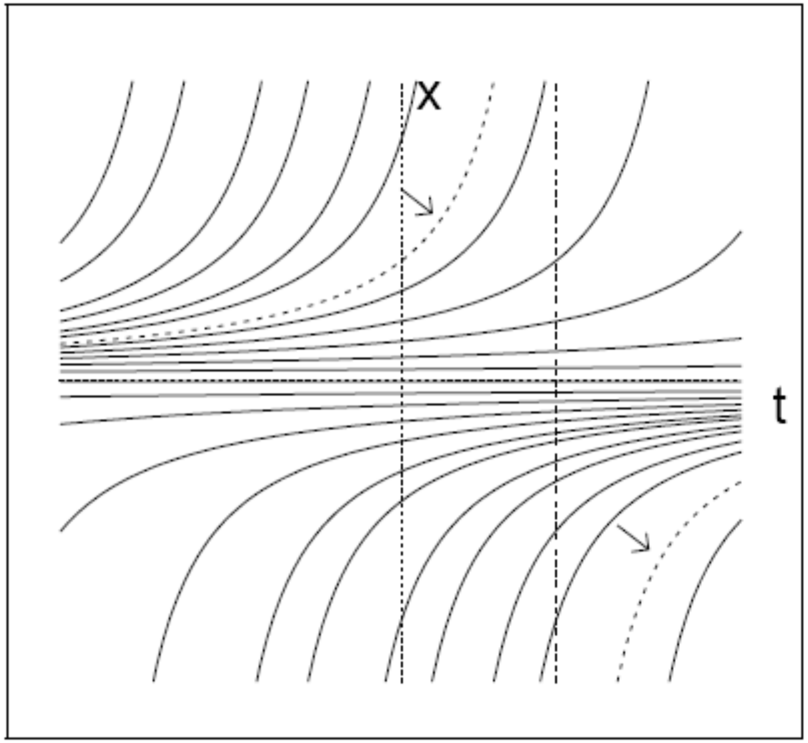


Figure 7.5 Solutions that blow up in finite time.
Curves shown are solutions of the equation $\dot{x} = x^2$. The dashed curve in the upper left is the solution with initial value $x(0) = 1$. This solution is $x(t) = 1/(1 - t)$, which has a vertical asymptote at $x = 1$, shown as a dashed vertical line on the right. The dashed curve at lower right is also a branch of $x(t) = 1/(1 - t)$, one that cannot be reached from initial condition $x(0) = 1$.

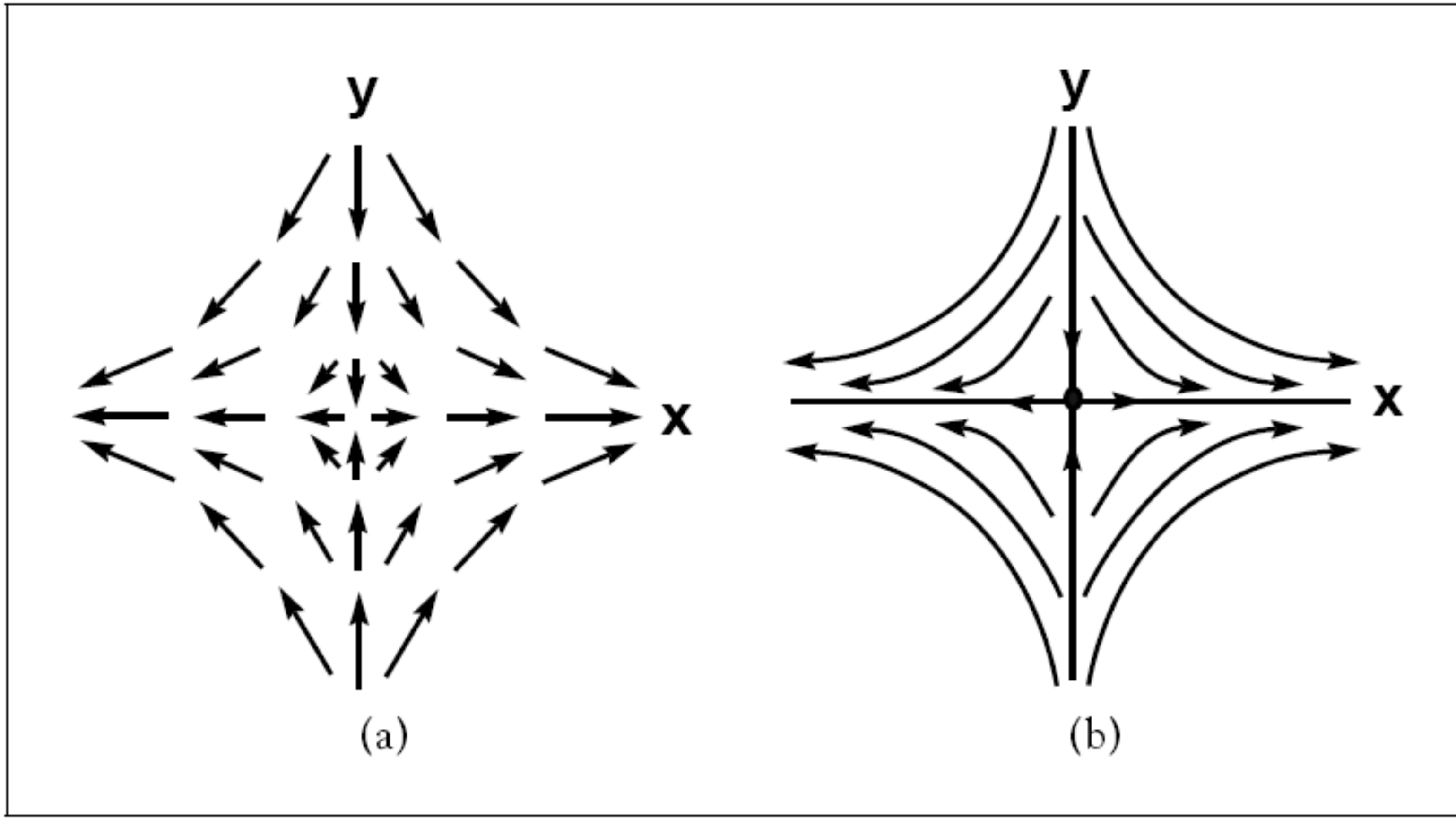


Figure 7.6 Vector field and phase plane for a saddle equilibrium.
(a) The vector field shows the vector $(\dot{x}, \dot{y})$ at each point (x, y) for (7.14). (b) The phase portrait, or phase plane, shows the behavior of solutions. The equilibrium $(x, y) \equiv (0, 0)$ is a saddle. The time coordinate is suppressed in a phase portrait.

$$\dot{x} = -4x - 3y$$

$$\dot{y} = 2x + 3y$$

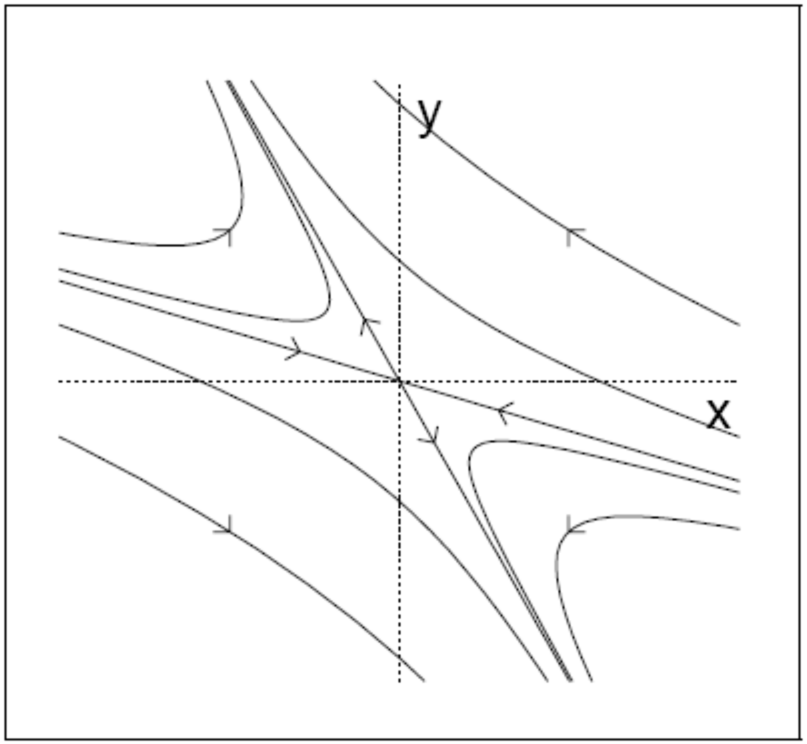


Figure 7.7 Phase plane for a saddle equilibrium.
For (7.16), the origin is an equilibrium. Except for two solutions that approach the origin along the direction of the vector $(3, -1)$, solutions diverge toward infinity, although not in finite time.

$$\dot{x} = -4x - 3y$$

$$\dot{y} = 2x + 3y$$

$$\mathbf{v}(0) = (1, 1)$$

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{A}\mathbf{u} = \lambda\,\mathbf{u}$$

Auto valores 2 e -3
Auto vetores (1, -2) e (3, -1)

$$\mathbf{v}(t) = e^{\lambda t}\mathbf{u}$$

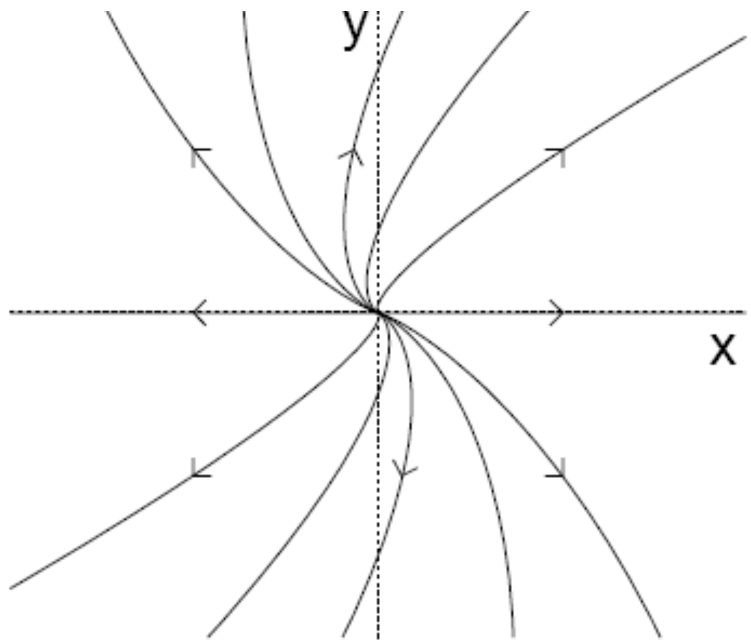
Soluções $$\mathbf{v}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2$$

$(1, 1)$ at $t = 0$ $$\mathbf{v}(t) = -4/5e^{2t}\mathbf{u}_1 + 3/5e^{-3t}\mathbf{u}_2$$

The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

has only one eigenvalue $\lambda = 3$, and $(1, 0)$ is the only eigenvector up to scalar multiple. The phase plane for this system is shown in Figure 7.8. The x-axis is the eigenspace; it contains all positive and negative scalar multiples of $(1, 0)$.



$$\dot{x} = 3x + y$$

$$\dot{y} = 3y$$

Figure 7.8 Phase plane for Equation (7.17).
The coefficient matrix $\mathbf{A}$ for this system has only one eigenvector, which lies along the x-axis. All solutions except for the equilibrium diverge to infinity.

$$\dot{x} = 3x + y \qquad \mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

$$\dot{y} = 3y$$

only one eigenvalue $\lambda = 3$, and $(1, 0)$ is the only eigenvector

$$\dot{x} = y$$

$$\dot{y} = -x$$

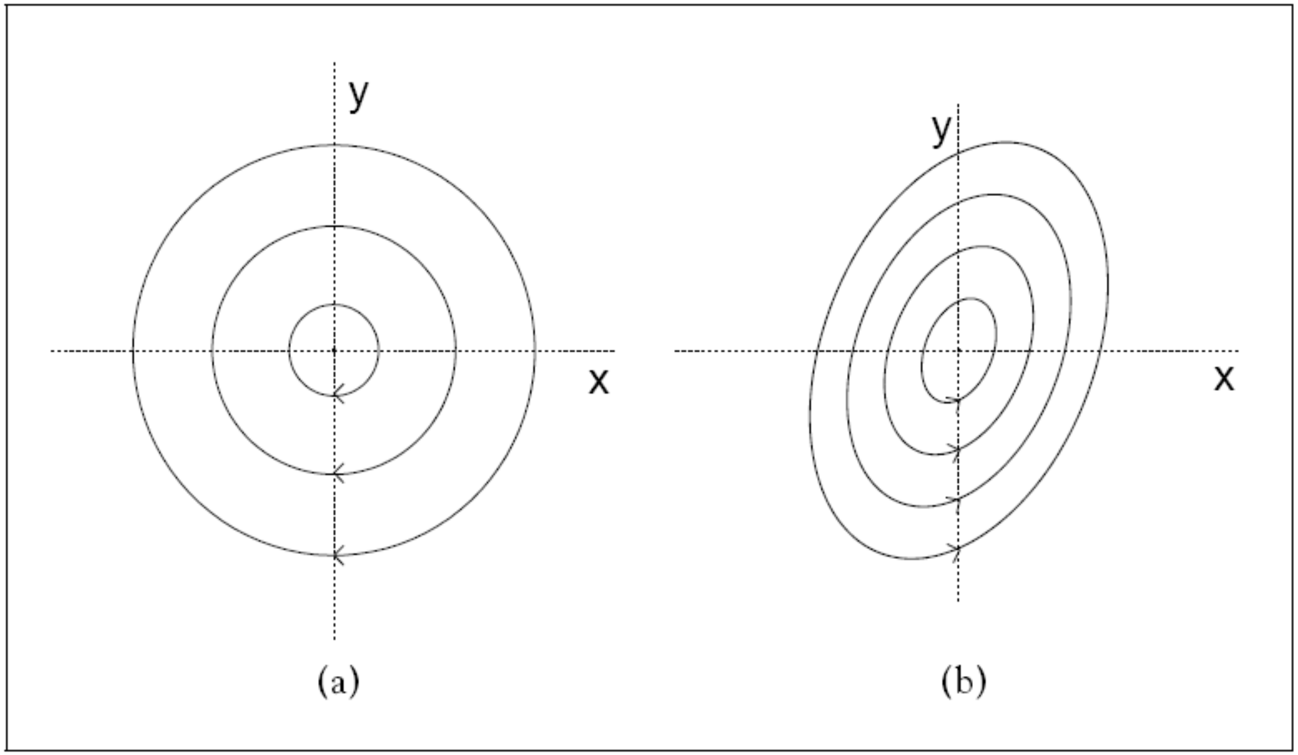


Figure 7.9 Phase planes for pure imaginary eigenvalues.
(a) In (7.18), the eigenvalues are $\pm i$. All solutions are circles around the origin, which is an equilibrium. (b) In (7.23), the eigenvalues are again pure imaginary. Solutions are elliptical. Note that for this equilibrium, some points initially move farther away, but not too far away. The origin is (Lyapunov) stable but not attracting.

$$\dot{x} = y$$

$$\dot{y} = -x$$

Verify that the eigenvalues are $\pm i$. Solutions of this system are $x(t) = c_1 \cos t + c_2 \sin t$ and $y(t) = c_2 \cos t - c_1 \sin t$, where c_1 and c_2 are any real constants. The

the distance squared is $|\mathbf{v}(t)|^2 = x^2 + y^2$

$$\dot{x} = -x - 10y$$
$$\dot{y} = 10x - y$$

Auto valores: $-1 \pm 10i$

$$\dot{x} = x - 10y$$
$$\dot{y} = 10x + y$$

Auto valores: $1 \pm 10i$

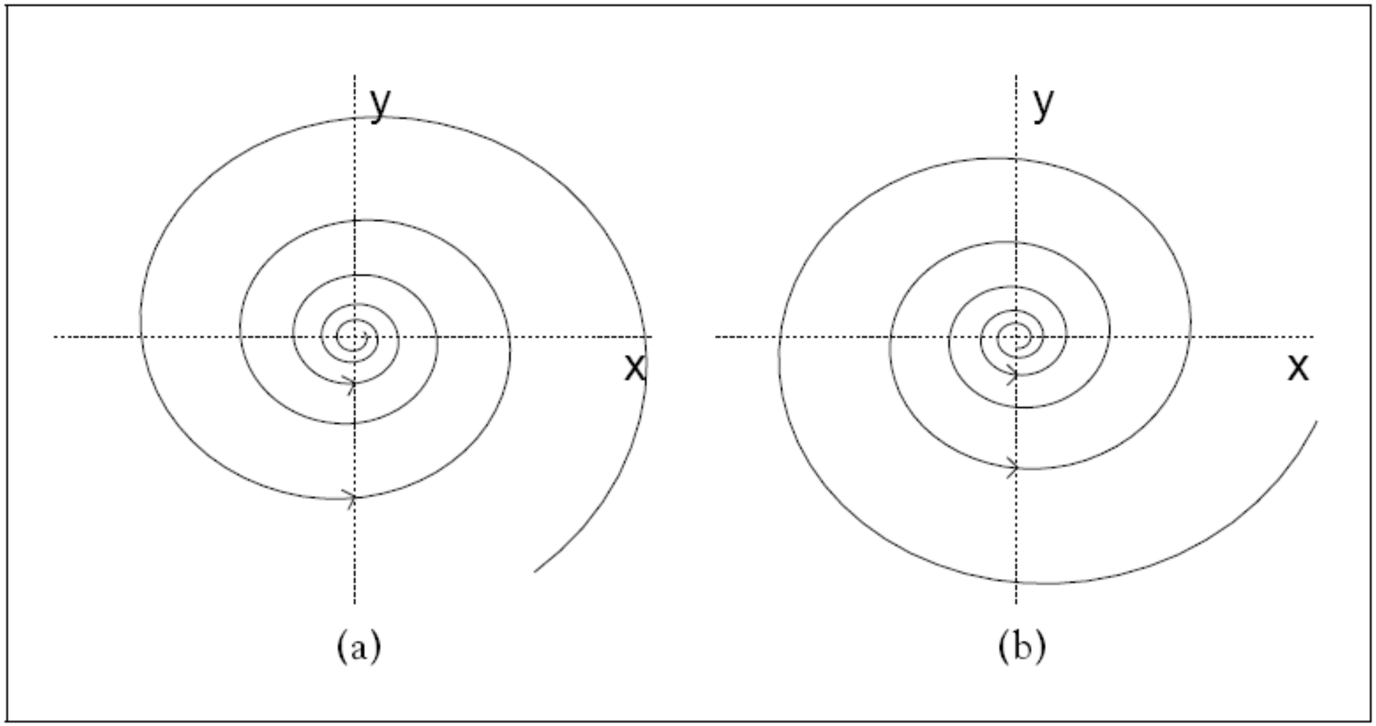


Figure 7.10 Phase planes for complex eigenvalues with nonzero real part. (a) Under (7.21), trajectories spiral in to a sink at the origin. The eigenvalues of the coefficient matrix A have negative real part. (b) For (7.22), the trajectories spiral out from a source at the origin.

$$\dot{x} = 5x + y + z$$

$$\dot{y} = -2y - 3z$$

$$\dot{z} = 3y - 2z$$

Auto valores: 5 and $-2 \pm 3i$

$|x(t)| \to \infty$ $z(t) \to 0$

Variedade instavel no eixo x
Variedade estável no plano x, y

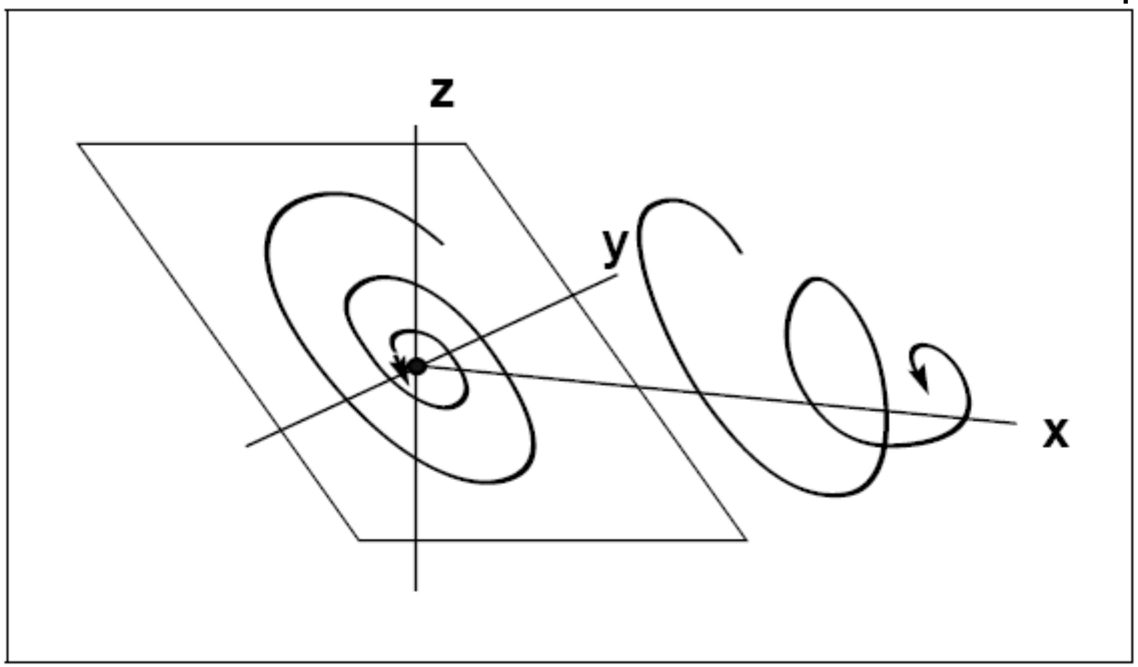


Figure 7.11 A three-dimensional phase portrait.
In Example 7.13, the origin $(0, 0, 0)$ is a saddle equilibrium. Trajectories whose initial values lie in the plane move toward the origin, and all others spiral away along the x-axis.

Equações Não Lineares

Soluções

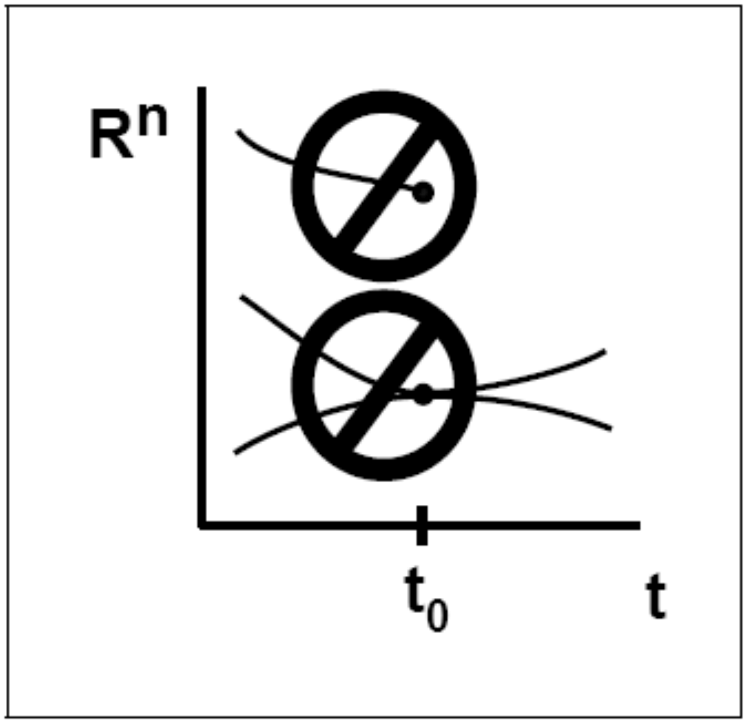


Figure 7.12 Solutions that are outlawed by the existence and uniqueness theorem.
Solutions cannot suddenly stop at t_0, and there cannot be two solutions through a single initial condition.

Definition 7.15 Let U be an open set in $\mathbb{R}^n$. A function $\mathbf{f}$ on $\mathbb{R}^n$ is said to be **Lipschitz** on U if there exists a constant L such that

$$|\mathbf{f}(\mathbf{v}) - \mathbf{f}(\mathbf{w})| \le L|\mathbf{v} - \mathbf{w}|,$$

for all $\mathbf{v}, \mathbf{w}$ in U. The constant L is called a **Lipschitz constant** for $\mathbf{f}$.

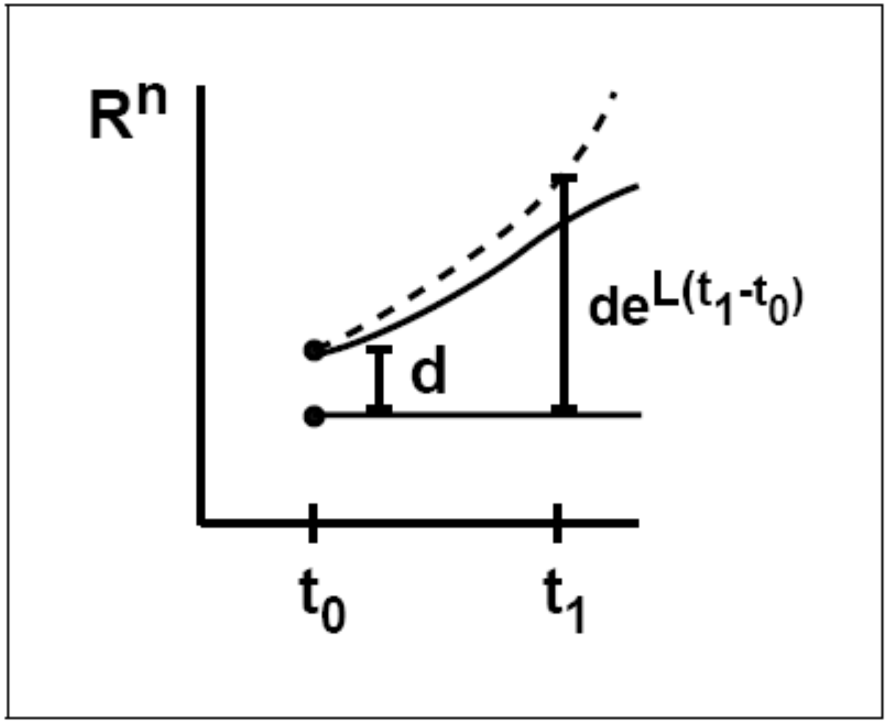


Figure 7.13 The Gronwall inequality.
Nearby solutions can diverge no faster than an exponential rate determined by the Lipschitz constant of the differential equation.

Equações de Primeira Ordem

$$\dot{x}_1 = f_1(x_1, \ldots, x_n)$$

$$\vdots$$

$$\dot{x}_n = f_n(x_1, \ldots, x_n).$$

$$\dot{\mathbf{v}} = \mathbf{f}(\mathbf{v})$$

Equações de Primeira Ordem

$$x^{(n)} = f(x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)})$$

Derivadas em relação à t

Definir novas n variáveis

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \\ &\vdots \\ x_n &= x^{(n-1)} \end{aligned}$$

n equações de primeira ordem

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(x_1, x_2, \ldots, x_n) \end{aligned}$$

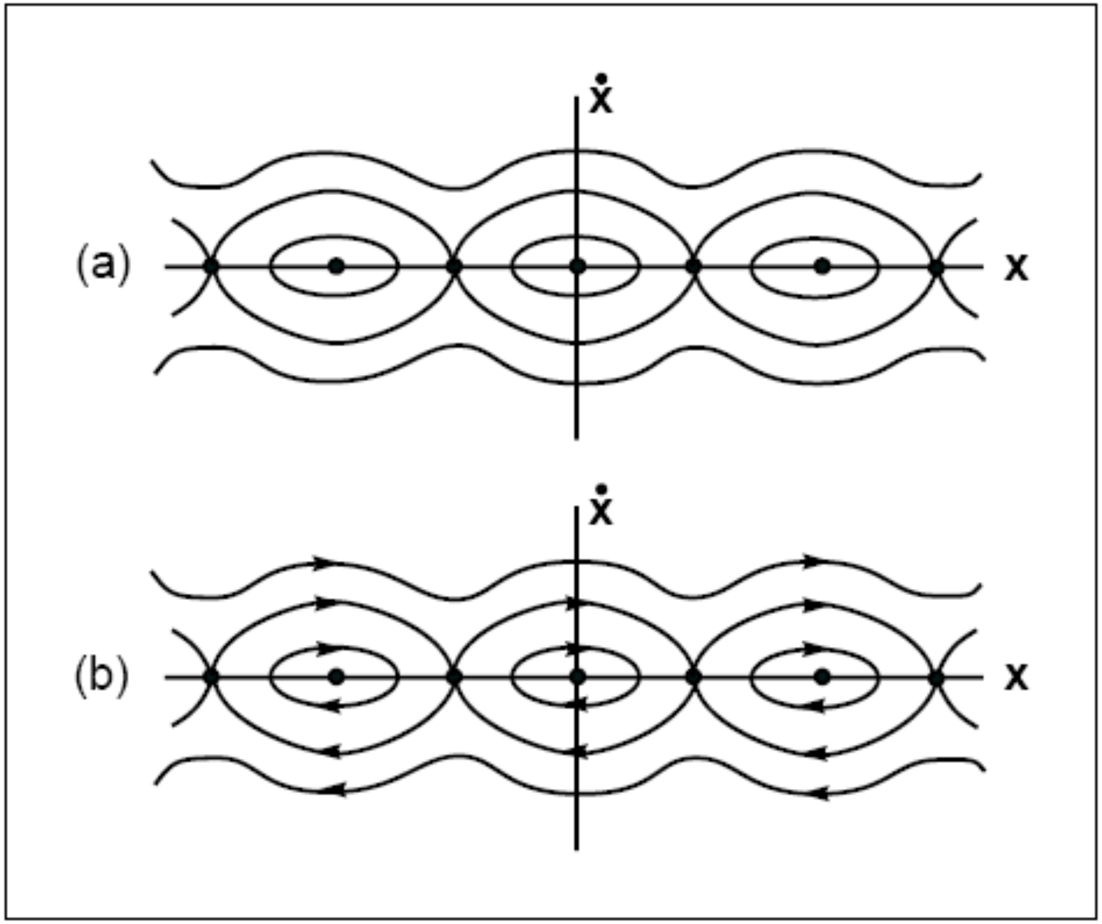


Figure 7.14 Solution curves of the undamped pendulum.
(a) Level curves of the energy function. (b) The phase plane of the pendulum. The solutions move along level curves; equilibria are denoted by dots. The variable x is an angle, so what happens at x also happens at $x + 2\pi$. As a results, (a) and (b) are periodic in x with period 2π.

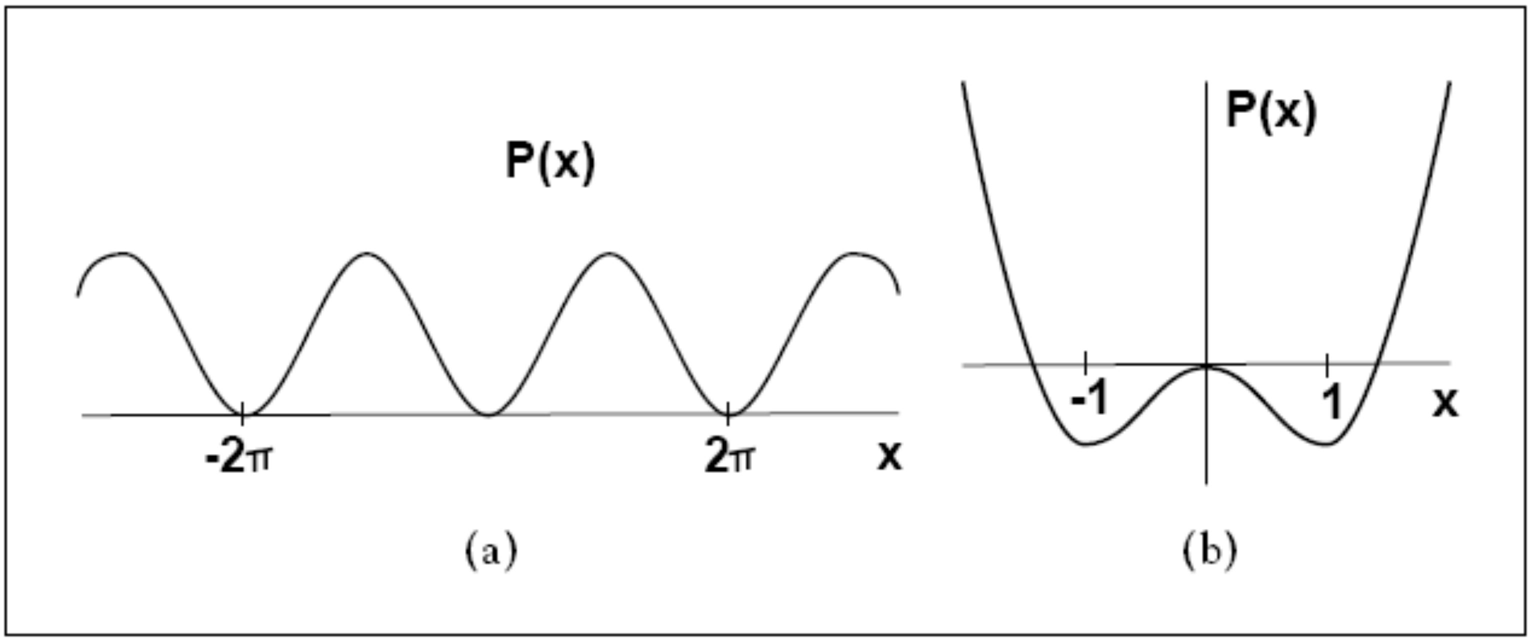


Figure 7.15 Potential energy functions.
(a) The potential function for the pendulum is $P(x) = 1 - \cos x$. There are infinitely many wells. (b) The double-well potential $P(x) = x^4/4 - x^2/2$.

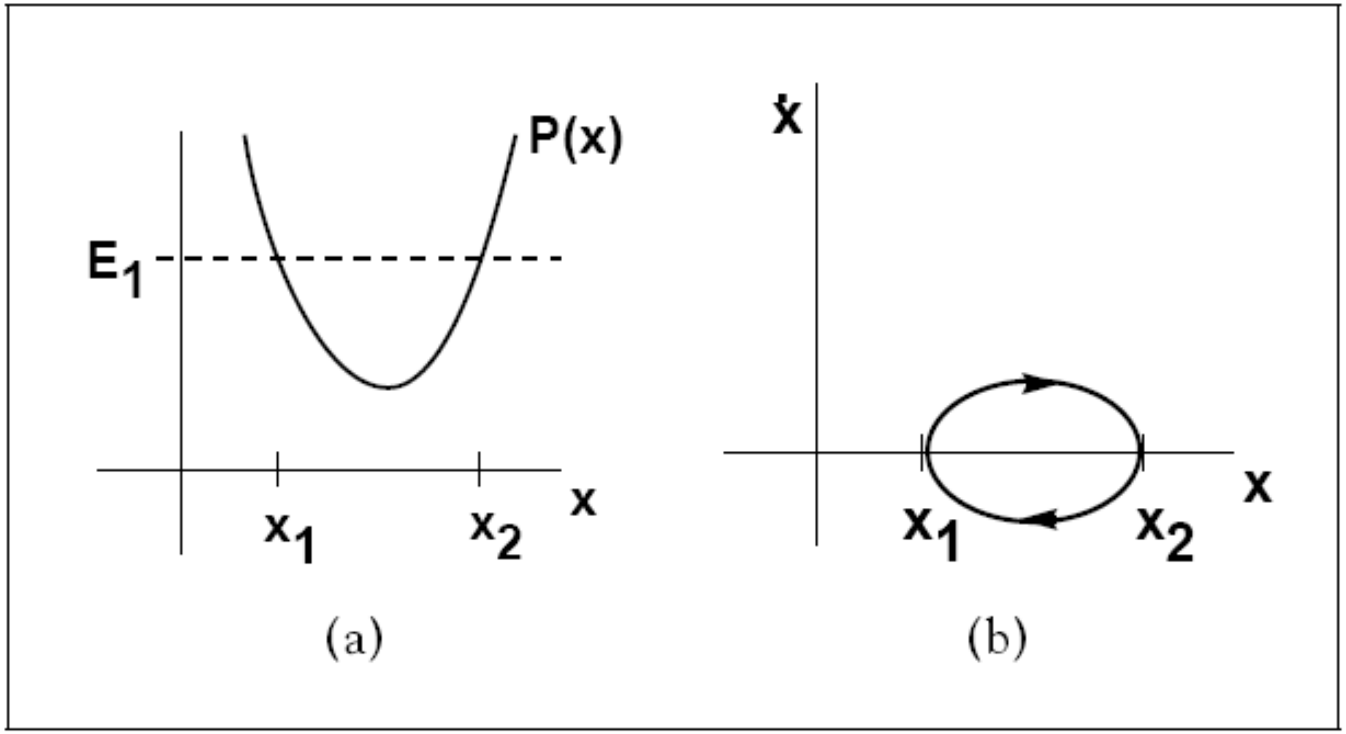


Figure 7.16 Drawing phase plane curves from the potential.
(a) Graph of the potential energy function $P(x)$. Each trajectory of the system is trapped in a potential energy well. The total energy $\dot{x}^2/2 + P(x)$ is constant for trajectories. As a trajectory with fixed total energy E_1 tries to climb out near x_1 or x_2, the kinetic energy $\dot{x}^2/2 = E_1 - P(x)$ goes to zero, as the energy E converts completely into potential energy. (b) A periodic orbit results: The system oscillates between positions x_1 and x_2.

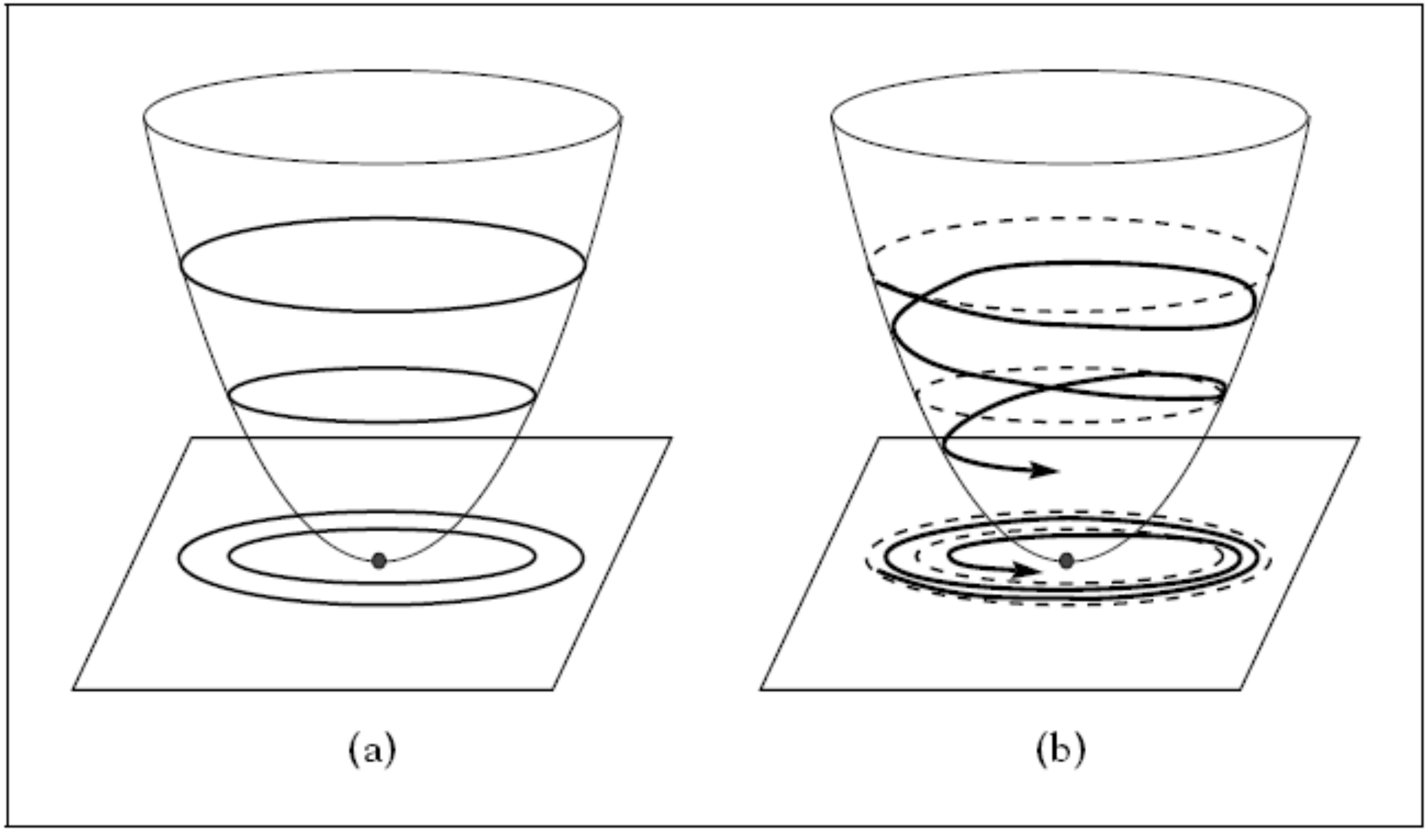


Figure 7.17 Behavior of solution trajectories with a Lyapunov function. The bowl in the figure is the graph of E. In both parts, we plot $E(\mathbf{v}(t))$ versus the solution trajectory $\mathbf{v}(t)$, which lies in the horizontal plane. (a) An equilibrium is at the critical point of the graph of the Lyapunov function E. The equilibrium is (Lyapunov) stable, since any nearby solution cannot go uphill, and can move away only a bounded distance dictated by its original energy level. (b) For a strict Lyapunov function, energy of solutions must continually decrease toward zero, cutting through energy level sets. The equilibrium is asymptotically stable.

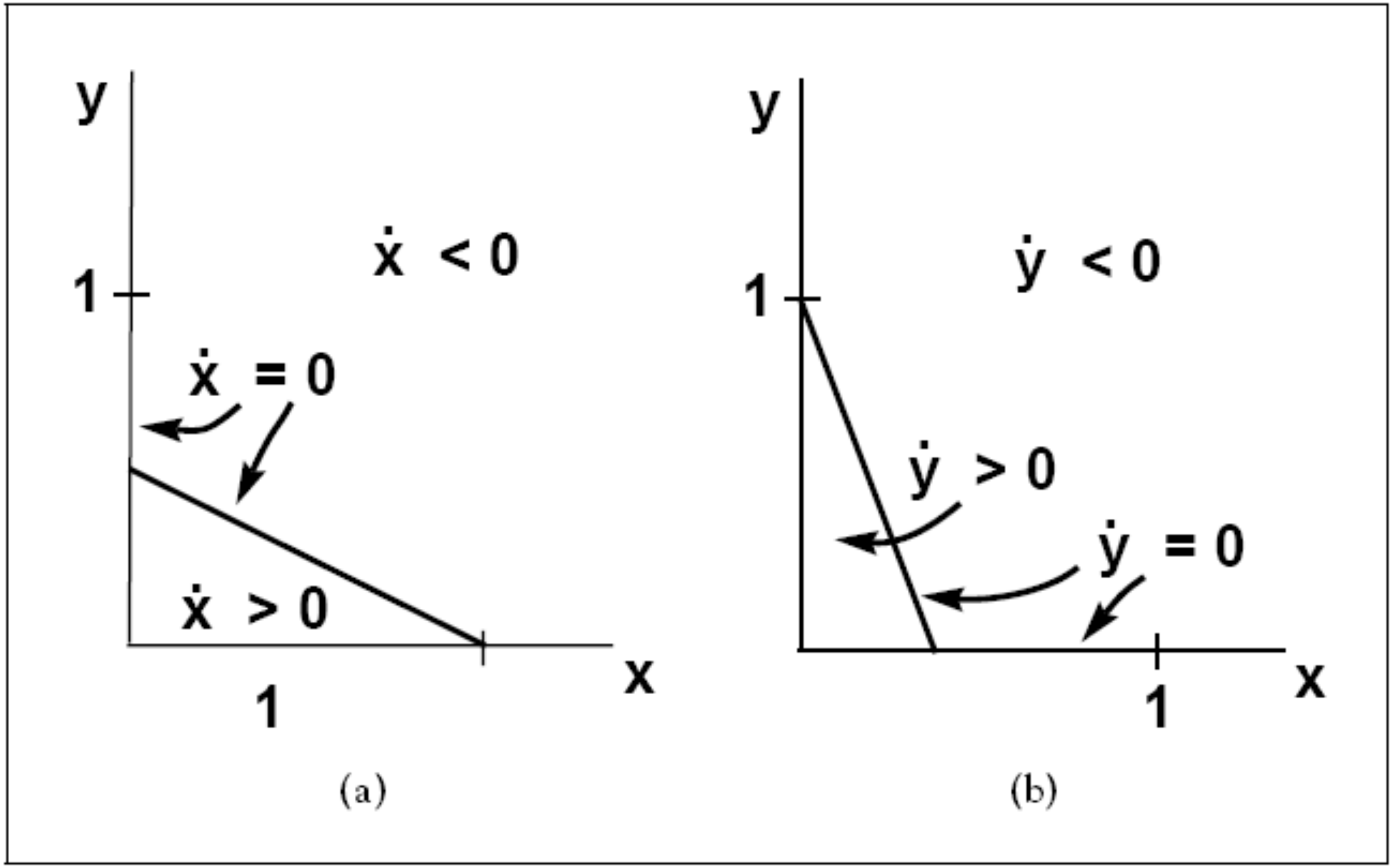


Figure 7.18 Method of nullclines for competing species.
The straight line in (a) shows where $\dot{x} = 0$, and in (b) it shows where $\dot{y} = 0$ for $a = 1$, $b = 2$, $c = 1$, and $d = 3$ in (7.47). The y-axis in (a) is also an x-nullcline, and the x-axis in (b) is a y-nullcline.

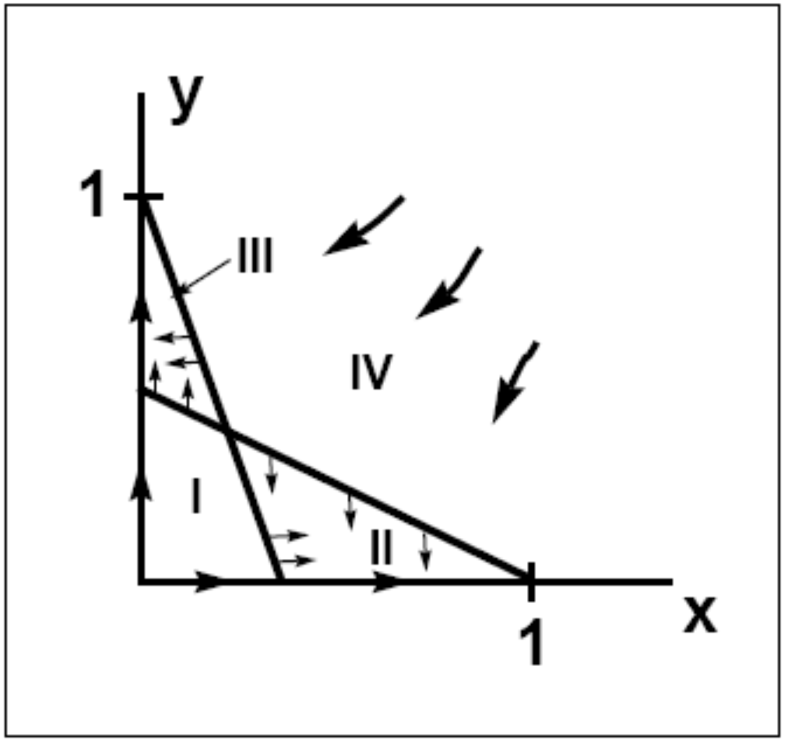


Figure 7.19 Competing species: Nullclines.
The vectors show the direction that trajectories move. The nullclines are the lines along which either $\dot{x} = 0$ or $\dot{y} = 0$. In this figure, the x-axis, the y-axis, and the two crossed lines are nullclines. Triangular regions II and III are trapping regions.

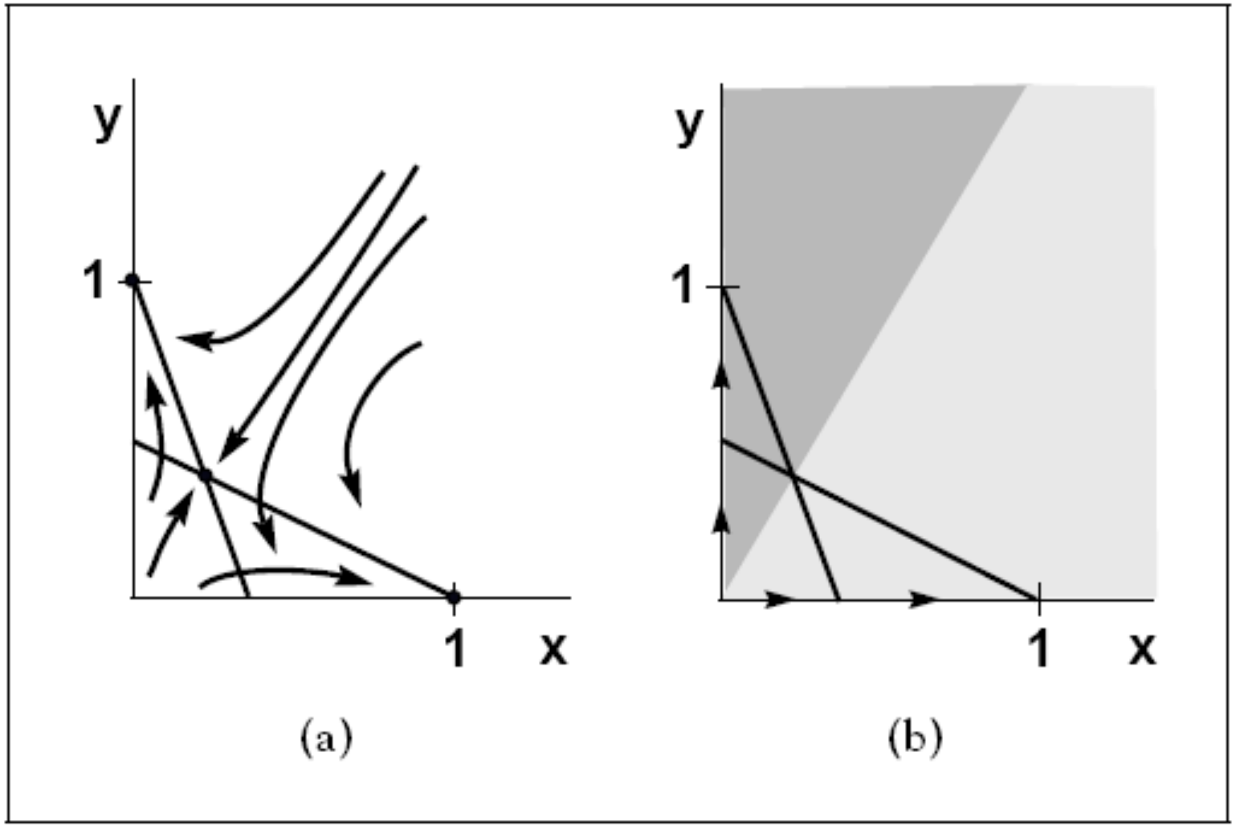


Figure 7.20 Competing species, extinction.
(a) The phase plane shows attracting equilibria at $(1,0)$ and $(0,1)$, and a third, unstable equilibrium at which the species coexist. (b) The basin of $(0, 1)$ is shaded, while the basin of $(1,0)$ is the unshaded region. One or the other species will die out.

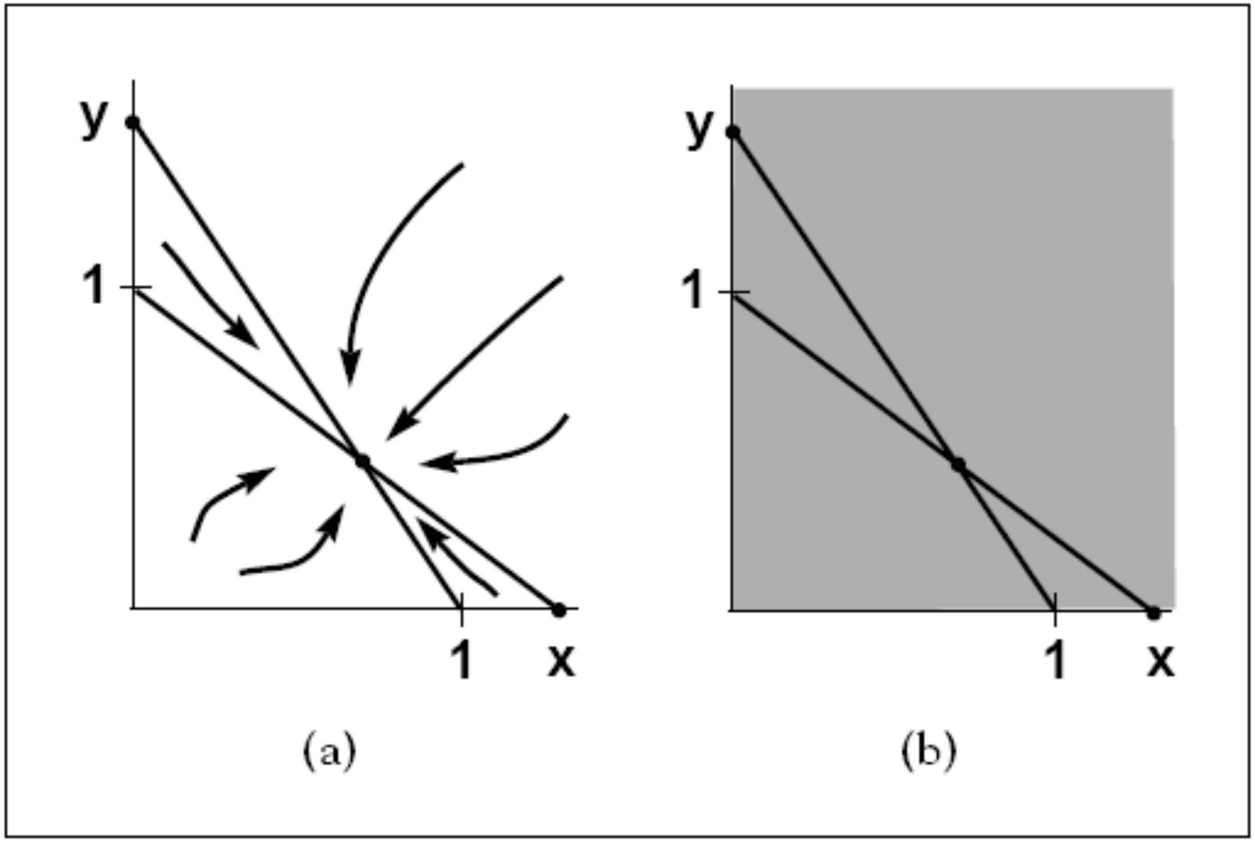


Figure 7.21 Competing species, coexistence.
(a) The phase plane shows an attracting equilibrium in which both species survive. The x-nullcline $y = \frac{3}{2} - \frac{3}{2}x$ has smaller x-intercept than the y-nullcline $y = 1 - \frac{3}{4}x$. According to Exercise T7.18, the equilibrium $(\frac{2}{3}, \frac{1}{2})$ is asymptotically stable. (b) All initial conditions with $x > 0$ and $y > 0$ are in the basin of this equilibrium.

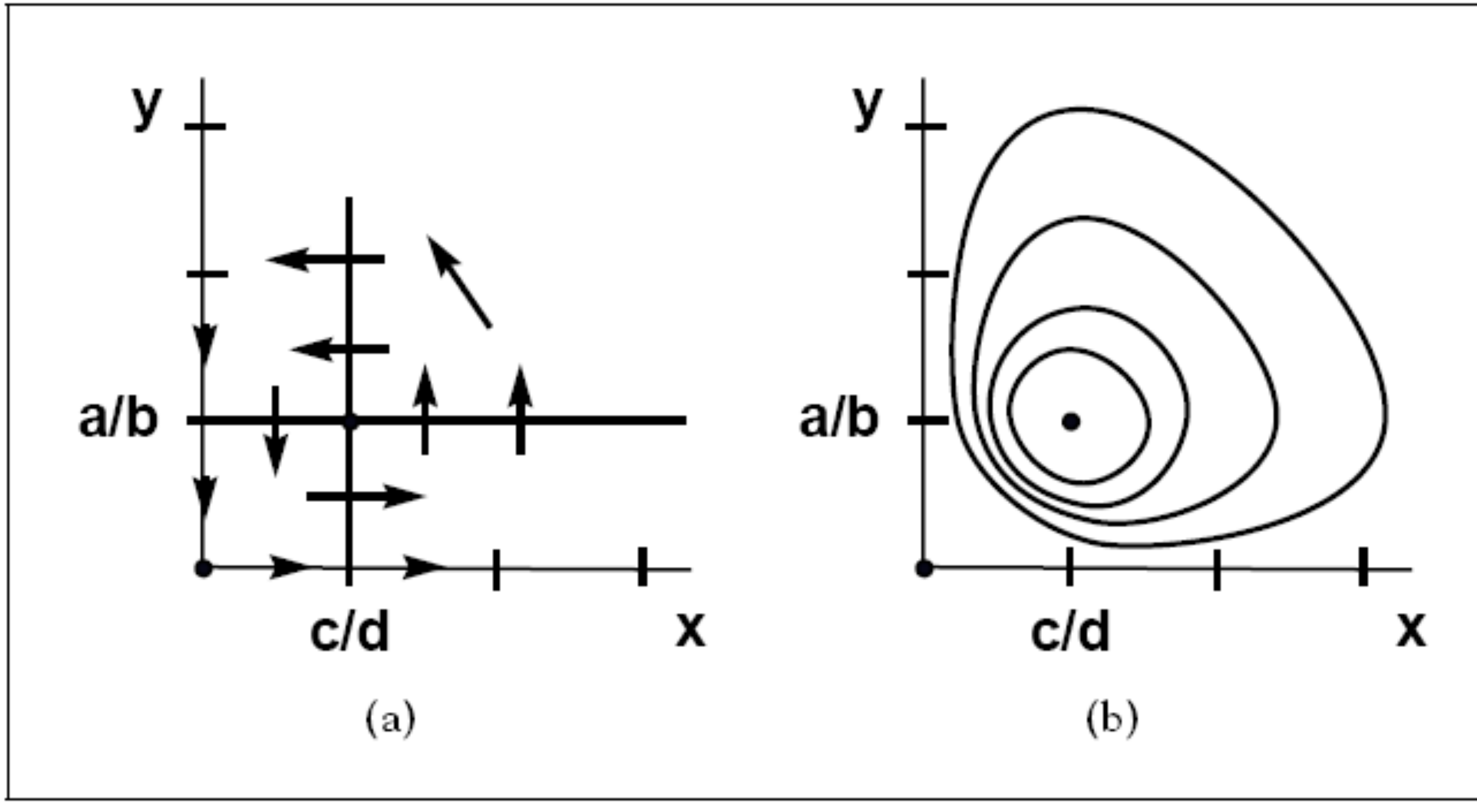


Figure 7.22 Predator-prey vector field and phase plane.
(a) The vector field shows equilibria at $(0, 0)$ and $(\frac{c}{d}, \frac{a}{b})$. Nullclines are the x-axis, the y-axis, and the lines $x = \frac{c}{d}$ and $y = \frac{a}{b}$. There are no trapping regions. (b) The curves shown are level sets of the Lyapunov function E. Since $\dot{E} = 0$, solutions starting on a level set must stay on that set. The solutions travel periodically around the level sets in the counterclockwise direction.