Instituto de Física da USP

Dimensões generalizadas e a conjectura de Kaplan-Yorke

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Disciplina: Caos em Sistemas Dissipativos
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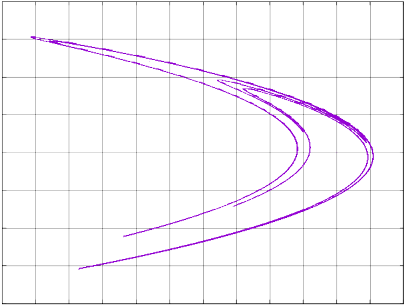
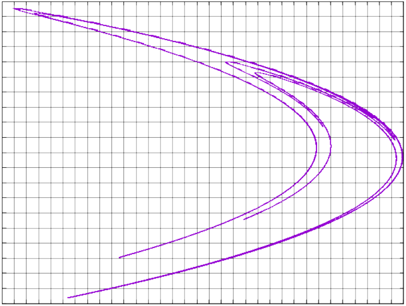
Objetivos

- Introduzir o conceito de dimensão generalizada
- Expor as dimensões mais importantes, a saber, D_0, D_1 e D_2
- Apresentar a conjectura de Kaplan-Yorke
- Discutir a relação entre geometria e dinâmica





Dimensão da contagem de caixas

$$D = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$$

Medida natural

$$\mu_i = \lim_{T\to\infty} \frac{\eta(C_i, \boldsymbol{x}_0, T)}{T}$$

$\eta(C_i, \boldsymbol{x}_0, T)$: tempo que a órbita que se origina em $\boldsymbol{x}_0$ passa na caixa C_i durante o intervalo de tempo $0 \le t \le T$.

Espectro de dimensões generalizadas

$$D_q = \frac{1}{1-q} \lim_{\epsilon\to 0} \frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)}, \qquad I(q,\epsilon) = \sum_{i=1}^{N(\epsilon)} \mu_i^q$$

$$D_{q_1} \le D_{q_2} \ \text{ se } \ q_1 > q_2$$

Dimensão D_0

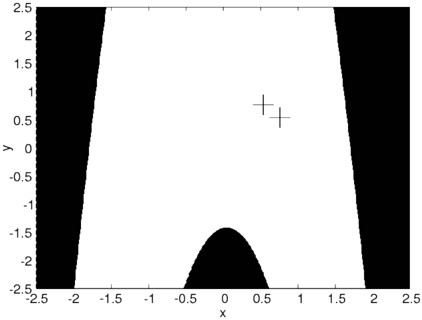
- $q = 0 \Rightarrow I(0,\epsilon) = \sum_{i=1}^{N(\epsilon)} \mu_i^0 = N(\epsilon)$

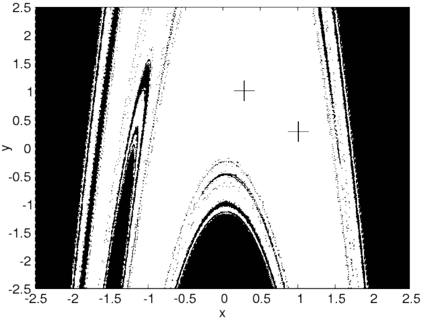
$\therefore \quad D_0 = \lim_{\epsilon \to 0} \dfrac{\ln N(\epsilon)}{\ln(1/\epsilon)} \Rightarrow D_0$ é a dimensão da contagem de caixas

- $\mu_i = \dfrac{1}{N(\epsilon)} \Rightarrow I(q,\epsilon) = \sum_{i=1}^{N(\epsilon)} \left(\dfrac{1}{N(\epsilon)}\right)^q = N(\epsilon)^{1-q}$

$\therefore \quad D_q = D_0 \quad ,\forall q$

Importância de D_0





Função incerteza

$$f(\epsilon) \sim \epsilon^{2-D_0}$$

Dimensão D_1

Definimos

$$D_1 = \lim_{q\to 1} D_q = \lim_{\epsilon\to 0}\lim_{q\to 1} \frac{1}{1-q}\frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)}, \qquad I(q,\epsilon) = \sum_{i=1}^{N(\epsilon)} \mu_i^q$$

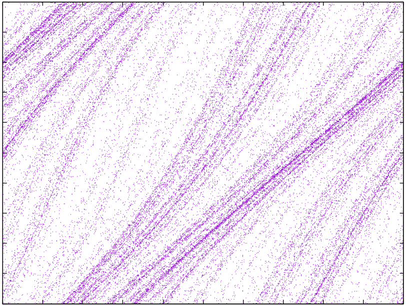
Dimensão de informação

$$D_1 = \lim_{\epsilon\to 0} \frac{\sum_{i=1}^{N(\epsilon)} \mu_i \ln \mu_i}{\ln \epsilon}$$

Importância de D_1

Mapa de Sinai

$$x_{n+1} = x_n + y_n + \delta \cos(2\pi y_n) \qquad \text{mod } 1$$
$$y_{n+1} = x_n + 2y_n \qquad \text{mod } 1$$



$$D_0 = 2$$
$$D_1 < 2$$

$$D_0(\theta) = D_1, \qquad 0 < \theta < 1$$

Dimensão D_2

"Integral" de correlação

$$C(\epsilon) = \lim_{K\to\infty} \frac{1}{K^2} \sum_{ij}^{K} \Theta(\epsilon - |\boldsymbol{x}_i - \boldsymbol{x}_j|)$$

Podemos mostrar que

$$C(\epsilon) \sim I(2,\epsilon) = \sum_{i=1}^{N(\epsilon)} \mu_i^2$$

Portanto

$$D_2 = \lim_{\epsilon\to 0} \frac{\ln C(\epsilon)}{ln\epsilon}$$

Importância de D_2

- Dados experimentais
- Alta dimensionalidade

Outro exemplo: periodicidade induzida em computadores

$$\bar{m} \sim \delta^{-D_2/2}$$

$\bar{m}$: tamanho do período
δ: erro de arredondamento

Conjectura de Kaplan-Yorke

Dimensão de Lyapunov

$$D_L = K + \frac{1}{|h_{k+1}|} \sum_{j=1}^{k} h_j, \qquad \sum_{j=1}^{k} h_j \geq 0$$

Conjectura de Kaplan-Yorke

$$D_L = D_1$$

Mapa do padeiro

Definição geral

$$\boldsymbol{B}(x_n, y_n) = \begin{cases} \begin{pmatrix} \lambda_a & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} & \text{,se } \; y < \alpha \\ \\ \begin{pmatrix} \lambda_b & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 1 - \lambda_b \\ -\frac{\alpha}{\beta} \end{pmatrix} & \text{,se } \; y > \alpha \end{cases}$$

$$h_1 = \lim_{n \to \infty} \frac{1}{n} \ln \left[\left(\frac{1}{\alpha} \right)^{n_1} \left(\frac{1}{\beta} \right)^{n_2} \right] = \alpha \ln \frac{1}{\alpha} + \beta \ln \frac{1}{\beta}$$

$$h_2 = \lim_{n \to \infty} \frac{1}{n} \ln \left[(\lambda_a)^{n_1} (\lambda_b)^{n_2} \right] = \alpha \ln \lambda_a + \beta \ln \lambda_b$$

Mapa do padeiro

Por outro lado,

$$\hat{I}(q,\epsilon) = \hat{I}_a(q,\epsilon) + \hat{I}_b(q,\epsilon)$$
$$\hat{I}_a(q,\epsilon) = \alpha^q \hat{I}_a(q,\epsilon/\lambda_a)$$
$$\hat{I}_b(q,\epsilon) = \beta^q \hat{I}_b(q,\epsilon/\lambda_b)$$
$$\hat{I}(q,\epsilon) \simeq K\epsilon^{(q-1)\hat{D}_q}$$

Equação transcendental para o mapa do padeiro

$$\alpha^q {\lambda_a}^{(q-1)\hat{D}_q} + \beta^q {\lambda_b}^{(q-1)\hat{D}_q} = 1$$

Conjectura de Kaplan-Yorke satisfeita

$$D_L = 1 + \frac{\alpha \ln(1/\alpha) + \beta \ln(1/\beta)}{\alpha \ln(1/\lambda_a) + \beta \ln(1/\lambda_b)} = D_1$$

Referências

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