XI - Bifurcações

Referência Principal: *Chaos*

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I - Introdução

•Bifurcação: mudança do atrator com variação do parâmetro de controle.

•Bifurcações ocorrem em sequência com a variação do parâmetro de ocntrole.

•Mesmas bifurcações são observadas em diferentes sistemas dinâmicos.

•Identificar bifurcações é importante no estudo de sistemas dinâmicos.

II – Bifurcações
Sela – Nó
Duplicação de Períodos

Parâmetro de bifurcação : valor em que ocorre a perda da estabilidade da solução.

Órbita de bifurcação : órbita correspondente ao parâmetro de bifurcação.

Definição

$f_a(\vec{v})$: mapa em R^n (espaço de fase) dependente de um parâmetro a definido em um espaço de parâmetros I, $a \in I$.

$f_a(\vec{v})$ ou $f(a, \vec{v})$

Mapa unidimensional

$$f_a' \equiv \frac{df_a}{dx}$$

$$D\, f_a$$

$$\text{Bifurcações básicas:} \begin{cases} \text{Bifurcações sela-nó: surgimento de pontos fixos} \\ \\ \text{Duplicação de período: surge órbita com período} \\ \text{duplicado} \end{cases}$$

Bifurcação Sela - Nó

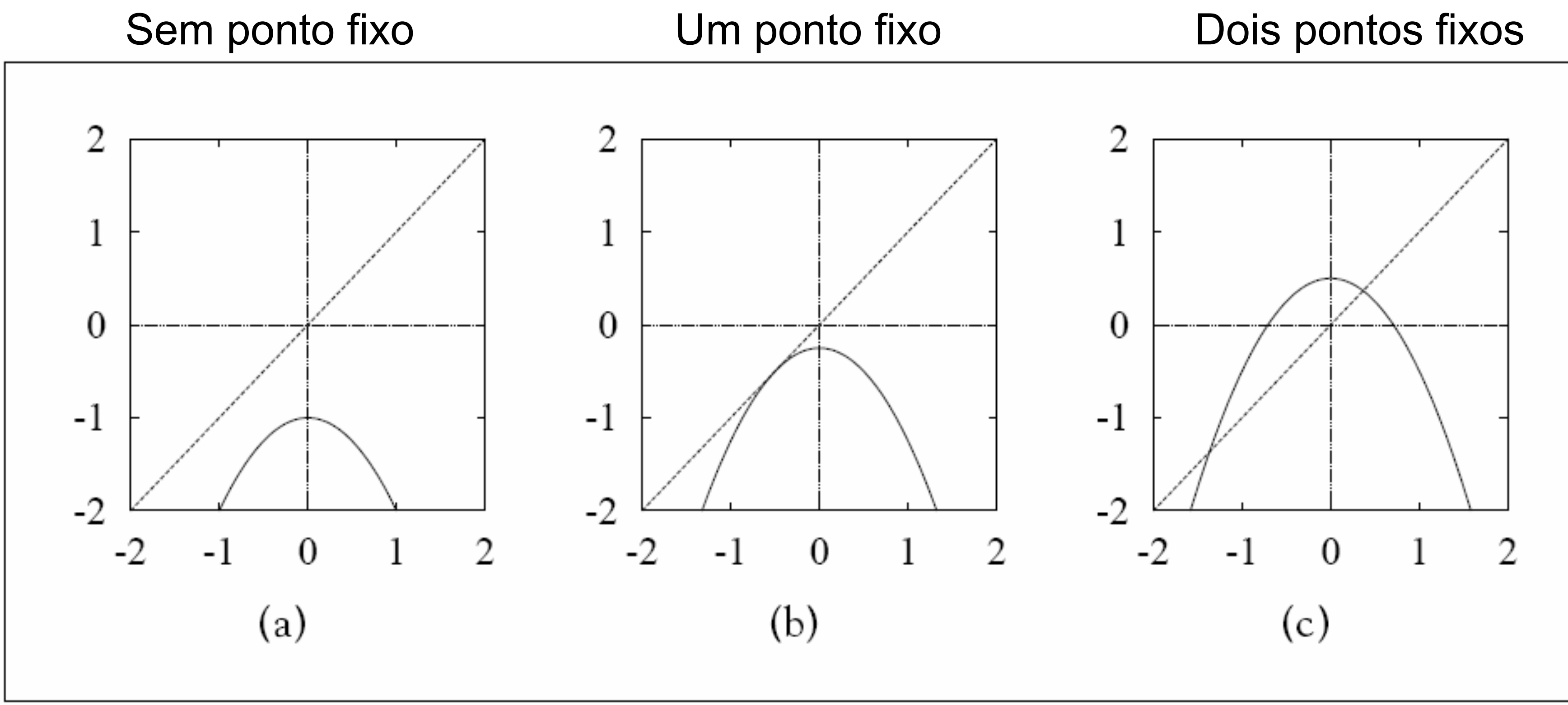

Figure 11.1 A saddle-node bifurcation.
The graph of the quadratic family $f(a, x) = a - x^2$ before, at, and following a saddle-node bifurcation is shown. (a) At $a = -1$, the graph does not intersect the dotted line $y = x$. (b) At $a = -0.25$, the graph and the line $y = x$ intersect in one point, the point of tangency; for $a > -0.25$, they intersect in two points. (c) At $a = 1/2$, f has a repelling fixed point and an attracting fixed point.

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Diagrama de Bifurcação
Bifurcações Sela – Nó e Duplicação de Período

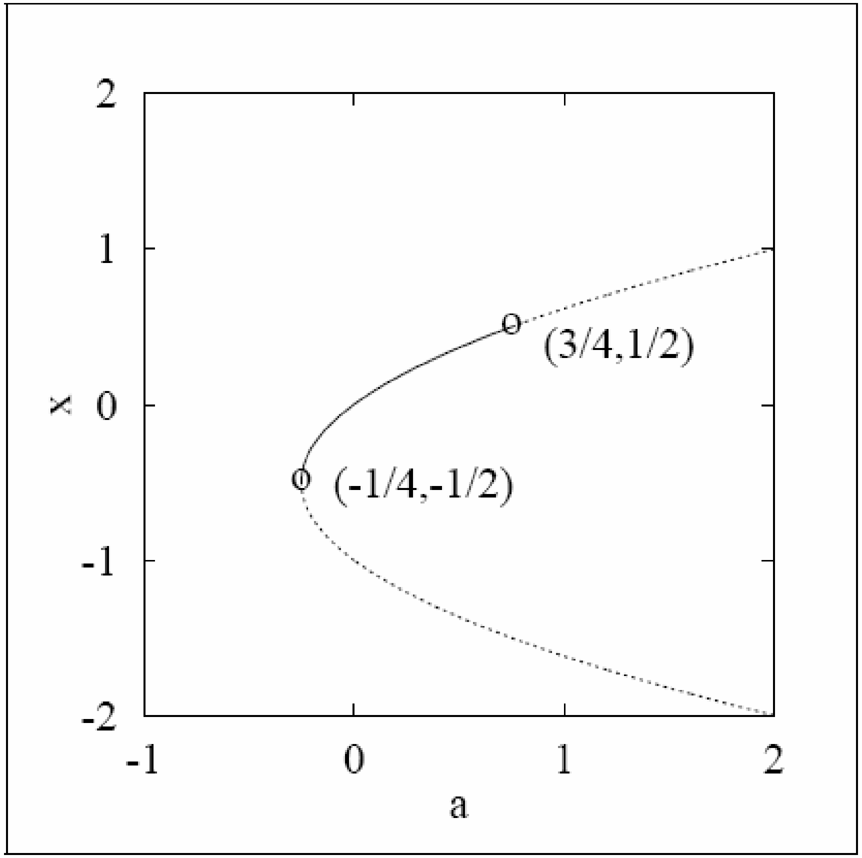

Figure 11.2 Bifurcation diagram of a saddle-node bifurcation.
Fixed points of the quadratic family $f(a, x) = a - x^2$ are shown. Attracting fixed points (sinks) are denoted by the solid curve, and repelling fixed points (sources) are on the dashed curves. The circle at $a = -1/4$ denotes the location of a saddle-node bifurcation. The other circle denotes where a period-doubling bifurcation occurs.

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Bifurcação: Duplicação de Período

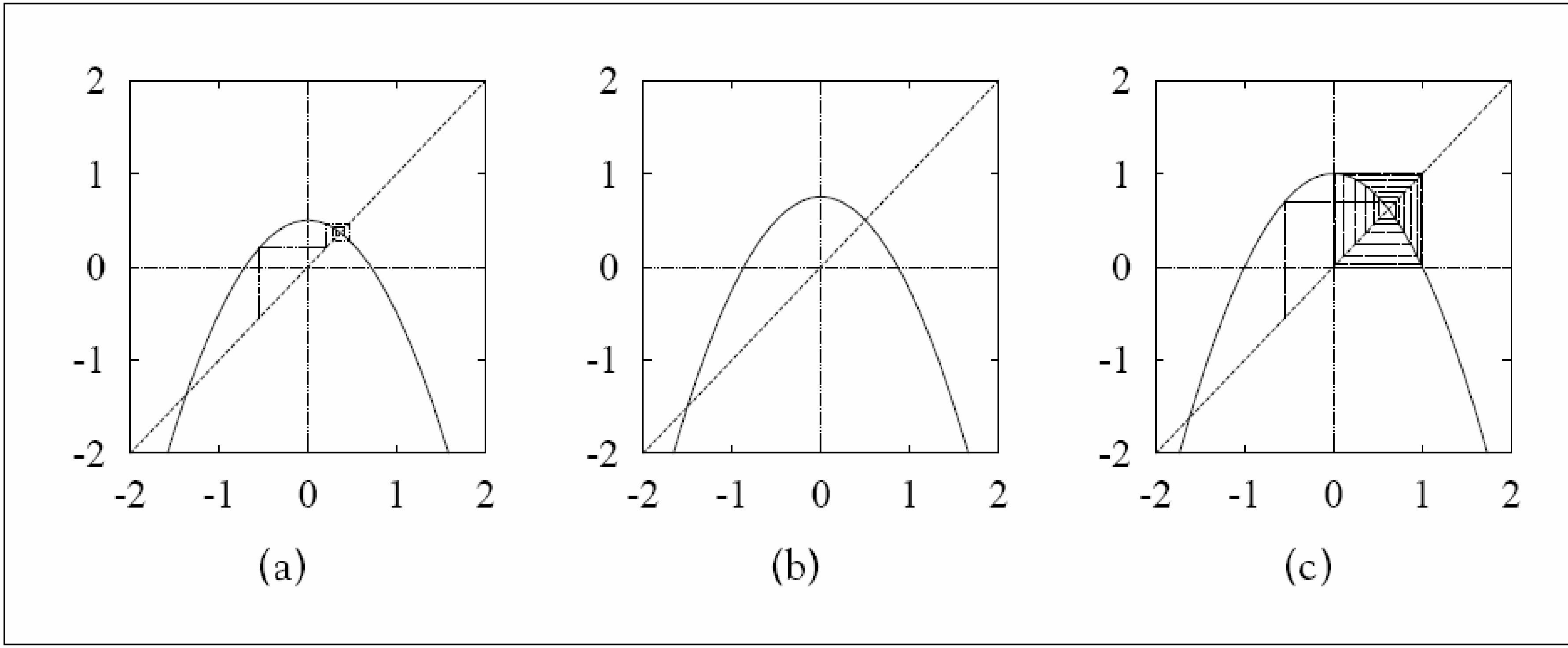

Figure 11.3 The quadratic map at a period-doubling bifurcation.
Graphs of the quadratic family $f_a(x) = a - x^2$ before, during and after a period-doubling bifurcation. (a) $a = 0.5$. Orbits flip back and forth around the sink as they approach. (b) $a = 0.75$. At the bifurcation point, $x = 1/2$ still attracts orbits, although the derivative of f_a is -1. (c) $a = 1$. Beyond the bifurcation, orbits are attracted by the period-two sink.

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Duplicação de Período

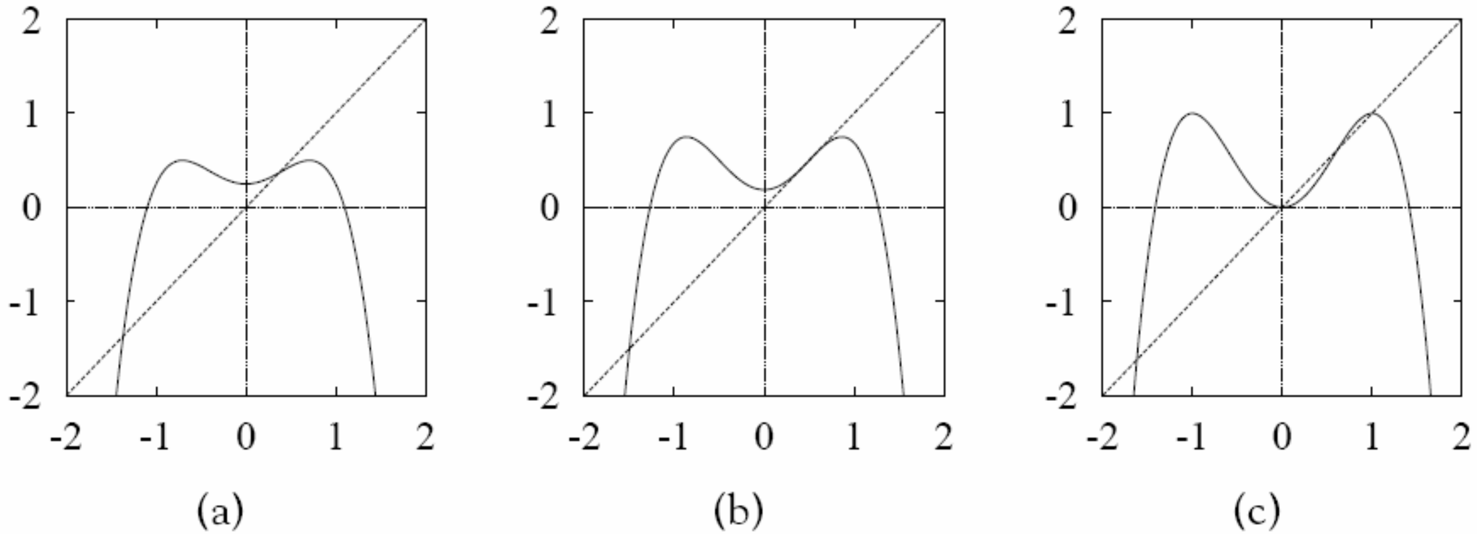

Figure 11.4 The second iterate of the quadratic map at a period-doubling bifurcation.
(a) The second iterate f_a^2 of the quadratic map $f_a(x) = a - x^2$ is graphed at $a = 0.5$, before the bifurcation. The fixed point above zero is an attractor. (b) The function f_a^2 is graphed at $a = .75$, the bifurcation parameter. The line $y = x$ is tangent to the graph at the positive fixed point, which is an attractor (although not a hyperbolic one). (c) At $a = 1$, following the bifurcation, there are three nonnegative fixed points of f_a^2: a flip repeller fixed point of f_a surrounded by a period-two attractor of f_a.

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Diagrama de Bifurcação na Bifurcação Duplicação de Período

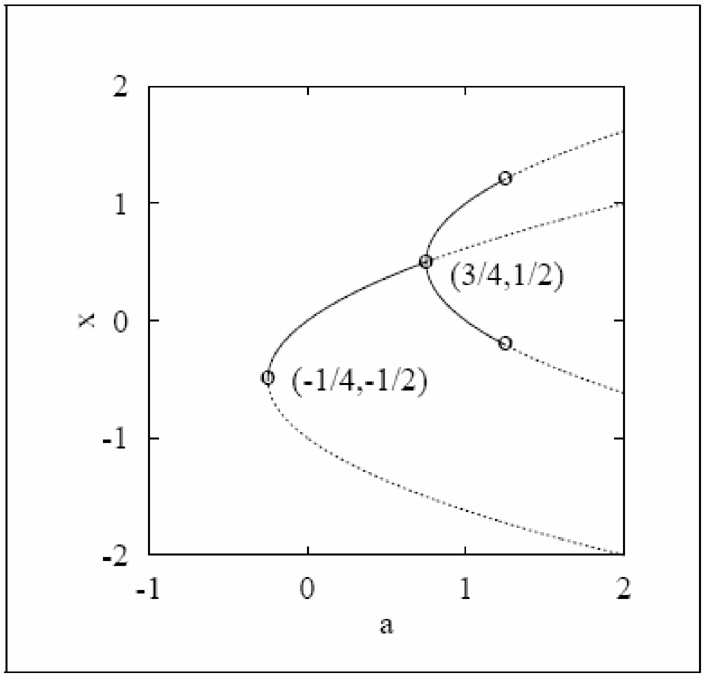

Figure 11.5 Diagram of a period-doubling bifurcation.
Fixed points and period-two orbits of the quadratic family $f(a, x) = a - x^2$ are shown. Attractors are on solid curves, while repellers are on dashed curves. The circles denote the location of the bifurcation orbits.

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Bifurcação Sela – Nó
Órbita de Período 3

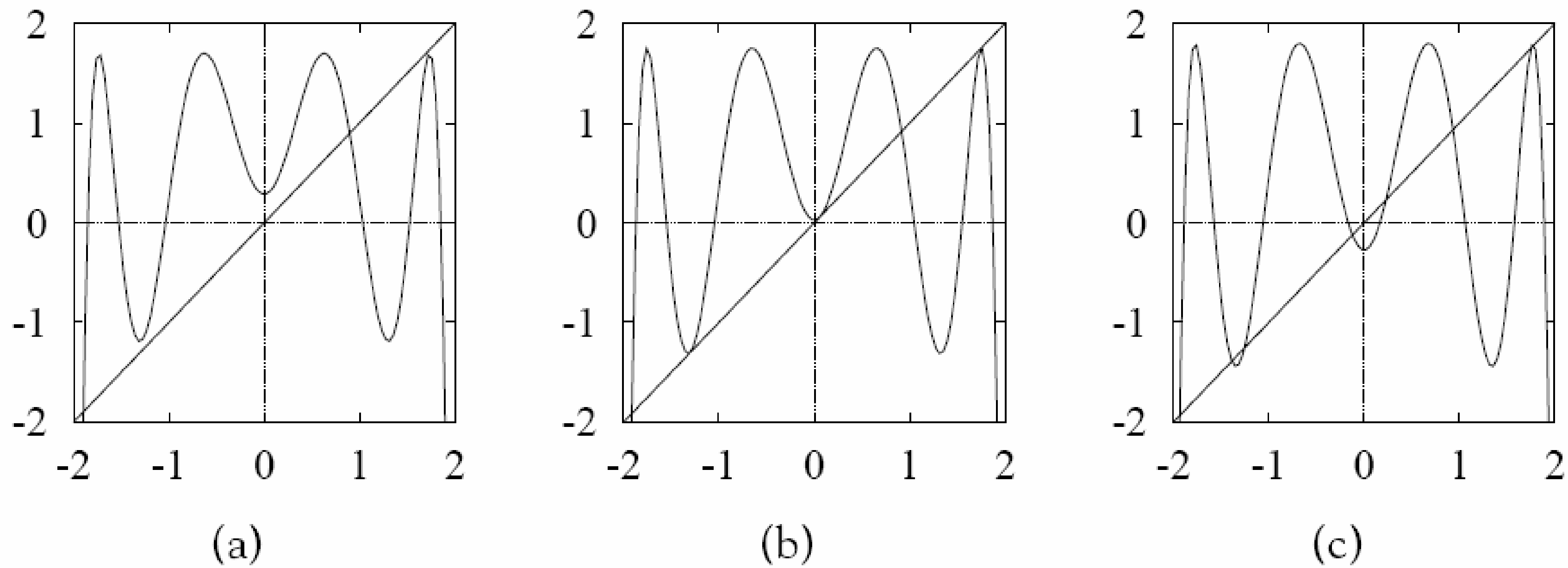

Figure 11.6 A period-three saddle-node bifurcation.
Graphs of the third iterate of the quadratic map $f_a(x) = a - x^2$ before, at, and following a period-three saddle-node bifurcation. (a) At $a = 1.7$, there are two solutions of $f_a^3(x) = x$. (b) The bifurcation point $a = 1.75$. (c) At $a = 1.8$, there are eight solutions of $f_a^3(x) = x$.

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Diagrama de Bifurcação
Órbita de Período 3

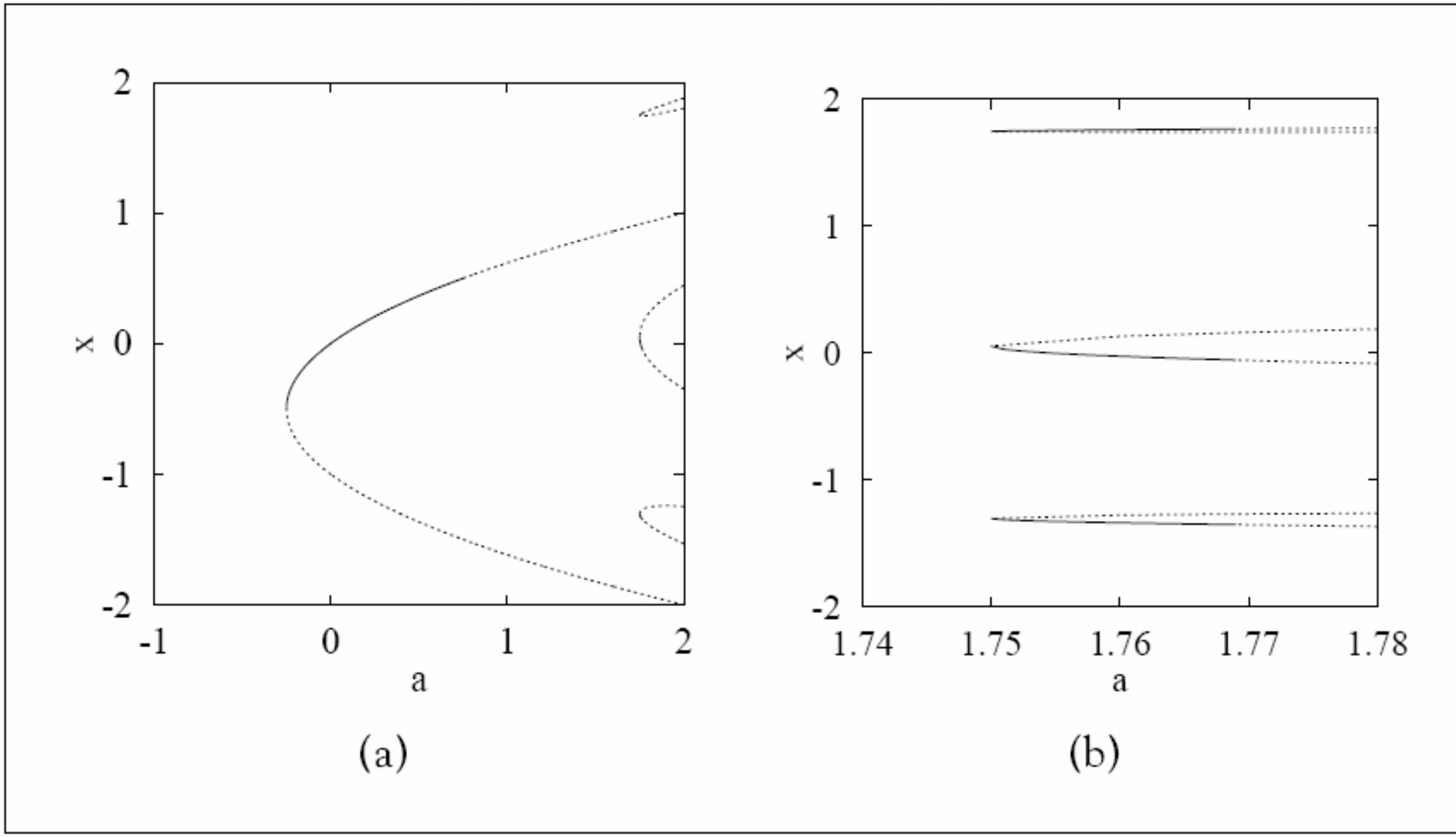

Figure 11.7 The bifurcation diagram of a period-three saddle-node bifurcation.
(a) Fixed points of f^3 are shown for the quadratic family $f(a, x) = a - x^2$; (b) a magnification of (a) near the saddle node shows only the period-three orbits.

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Esquemas das Bifurcações

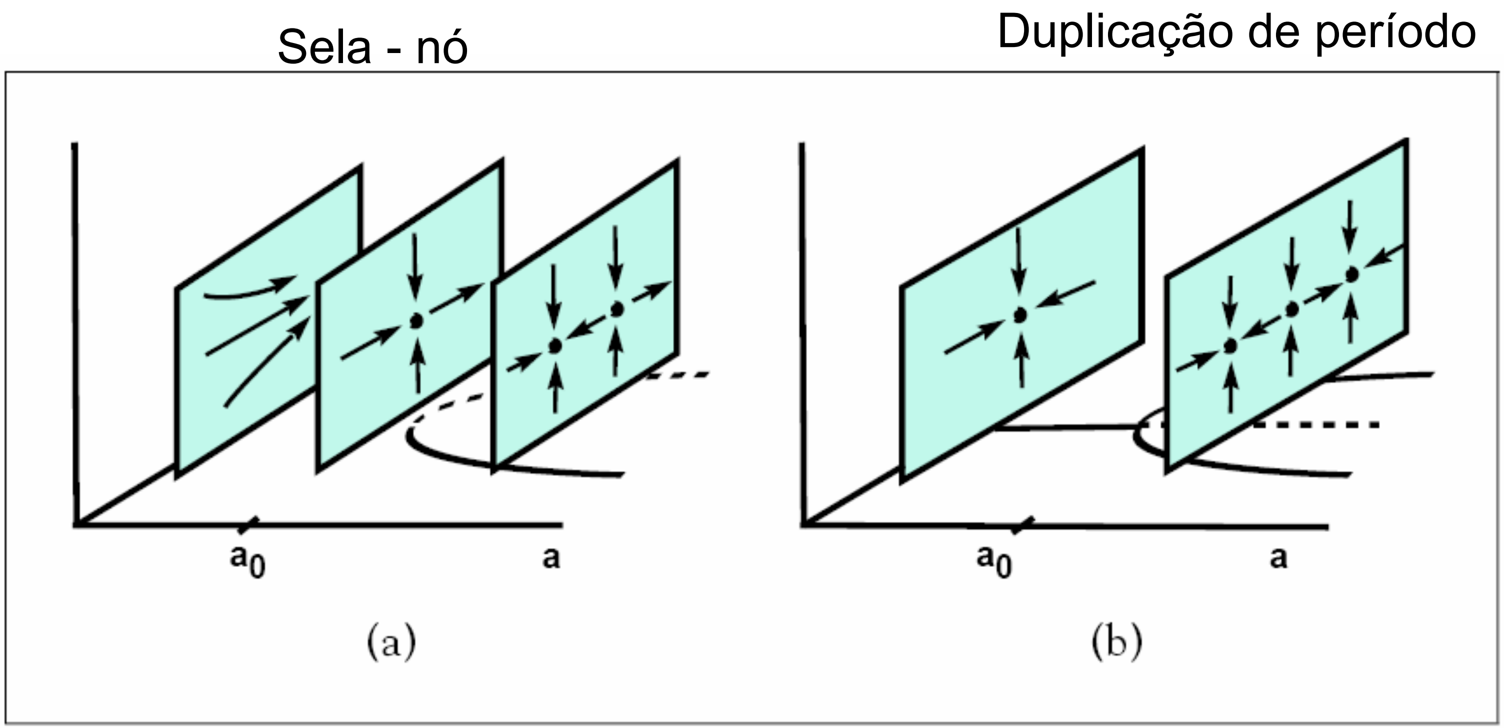

Figure 11.8 Bifurcations in a parametrized map of the plane.
(a) A fixed point appears at parameter $a = a_0$ in a saddle-node bifurcation. For $a > a_0$ there is an attracting fixed point and a saddle fixed point. The cross-sectional figures depict the action of the planar map at that parameter value. (b) An attracting fixed point loses stability at $a = a_0$ in a period-doubling bifurcation. For $a > a_0$ there is a saddle fixed point and a period-two attractor.

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Diagramas de Bifurcação
Mapa de Hénon

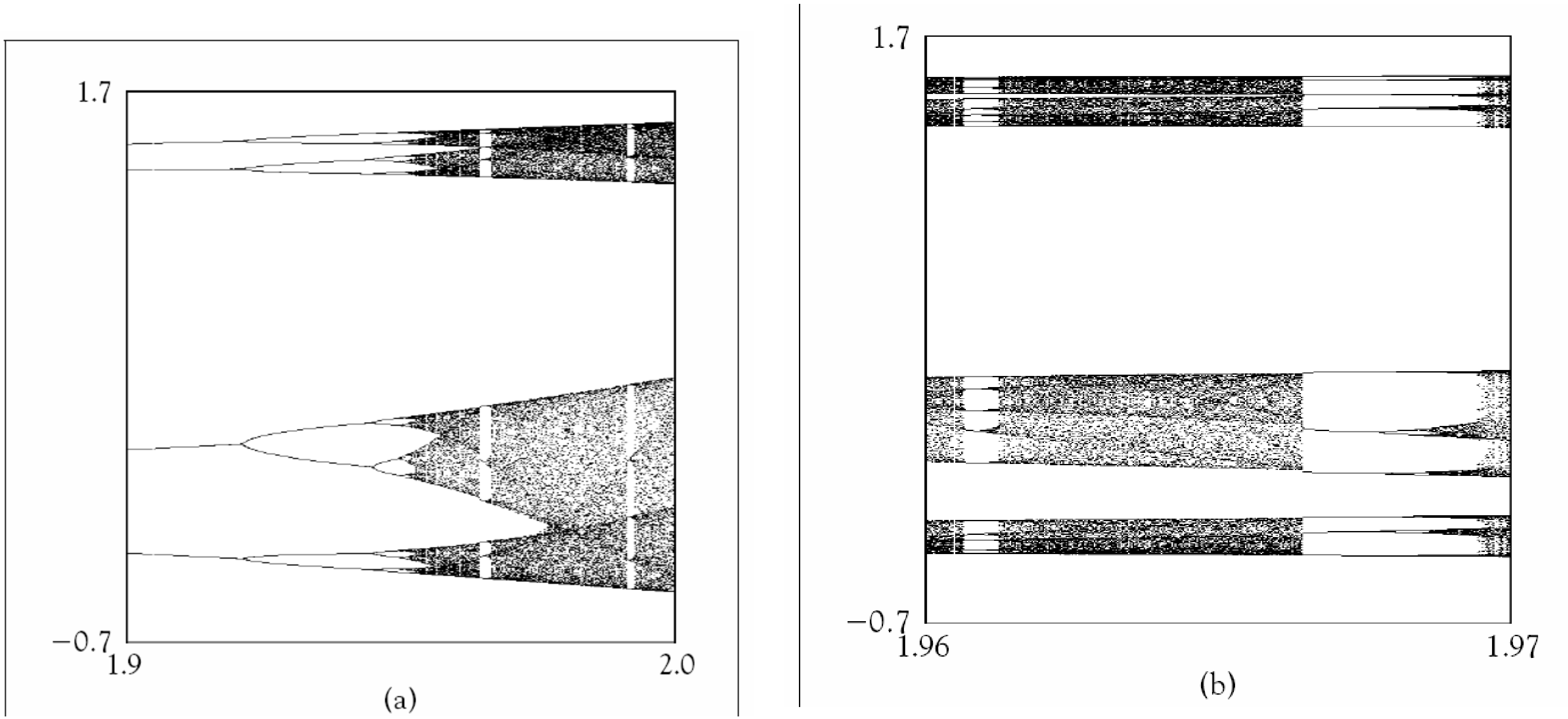

Figure 11.9 Bifurcation diagrams for the Hénon map.
The x-coordinate is plotted vertically, and the horizontal axis is the bifurcation parameter a for the family $\mathbf{h}_a(x, y) = (a - x^2 - 0.3y, x)$. (a) A period-four orbit at $a = 1.9$ period-doubles to chaos. At $a = 2$ the attractor is a two-piece chaotic attractor. (b) A magnification of the period-twelve window in (a).

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Mapa de Hénon

$$h_a = (a - x^2 + b\,x,\ x) \qquad |b| < 1, \quad b \text{ fixo}$$

$$\text{Pontos fixos } \vec{p} \text{ e } \vec{q}: \begin{cases} a - x^2 + b\,x = x \\ x = x \end{cases}$$

$$\vec{p} = \left(\frac{-(1-b) + \sqrt{(1-b)^2 + 4a}}{2}, \frac{-(1-b) + \sqrt{(1-b)^2 + 4a}}{2} \right)$$

$$\vec{q} = \left(\frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2}, \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2} \right)$$

Nota: *Para* $a < \dfrac{-(1-b)^2}{4} \Rightarrow$ não há ponto fixo

$$a* = \frac{-(1-b)^2}{4} \Rightarrow \text{bifurcação de ponto de sela, } \vec{p} = \vec{q}$$

Mapa de Hénon

$$x_{n+1} = a - x_n^2 + b\,x_n$$

$$y_{n+1} = x_n$$

$$D\,h = \begin{pmatrix} -2\,x_n & b \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -2\,x_n - \lambda & b \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda = -\,x_n \pm \sqrt{x_n^2 + b}$$

$$Para\ a* = \frac{-(1-b)^2}{4} \Rightarrow x* = \frac{-\,(1-b)}{2} \Rightarrow \lambda = \begin{cases} 1 \\ -b \end{cases}$$

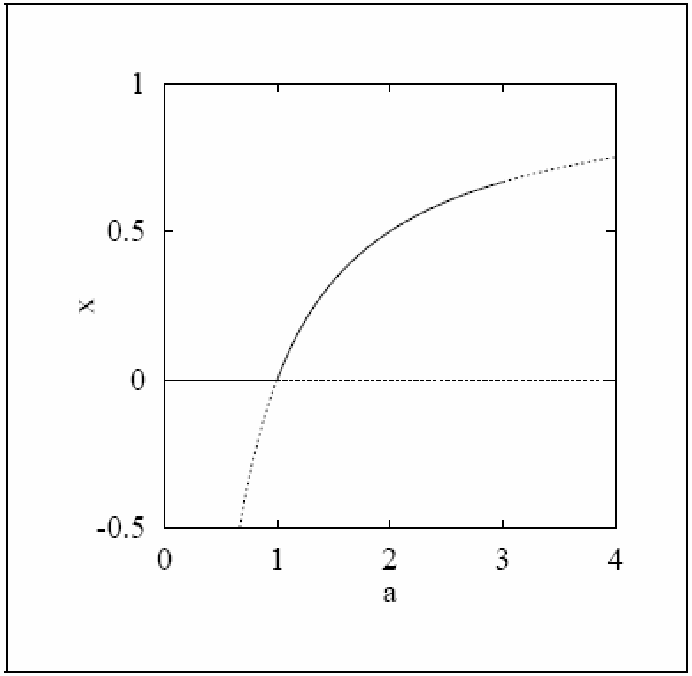
$$a > \frac{-\,(1-b)^2}{4} \Rightarrow \vec{p}\ \acute{e}\ atrator, \vec{q}\ \acute{e}\ repulsor.\ Sela - n\acute{o}.$$

$$a* = \frac{3\,(1-b)^2}{4} \Rightarrow bifurcação\ com\ duplicação\ de\ período\ no\ ramo\ \vec{p}$$

Isso ocorre em $(x*, y*) = (\frac{1-b}{2}, \frac{1-b}{2})$, *com* $\lambda = -1$

Para $b = -0.3 \Rightarrow a* \cong 1.3$

Bifurcação Transcrítica



Mapa logístico

$$y = a\,x\,(1 - x)$$

$$\text{Pontos fixos} \begin{cases} a = 1 \Rightarrow x = 0 \text{ raiz dupla} \\ a \neq 1 \Rightarrow \text{duas raizes para } 0 < a < 4 \end{cases}$$

Figure 11.10 Transcritical bifurcation of fixed points in the logistic family. The bifurcation shown here is atypical in that it is neither a saddle-node nor a period-doubling bifurcation. This type of bifurcation occurs since the graph is constrained to pivot about the fixed point (0, 0) as the parameter changes.

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Bifurcação da Forquilha

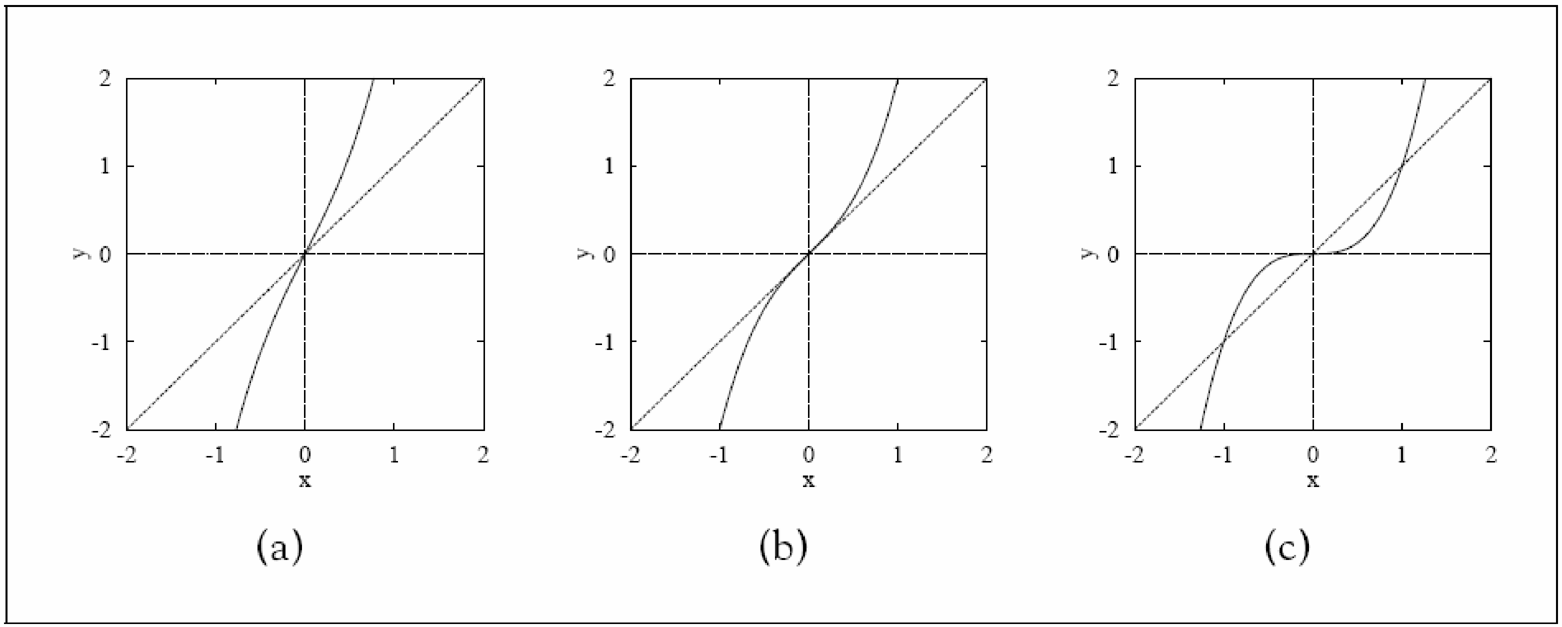

Figure 11.11 A pitchfork bifurcation.
The graphs of the cubic maps $f_a(x) = x^3 - ax$ before, during and after a pitchfork bifurcation. (a) $a = -2$ (b) $a = -1$ (c) $a = 0$.

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Bifurcação da Forquilha

Diagrama de Bifurcação

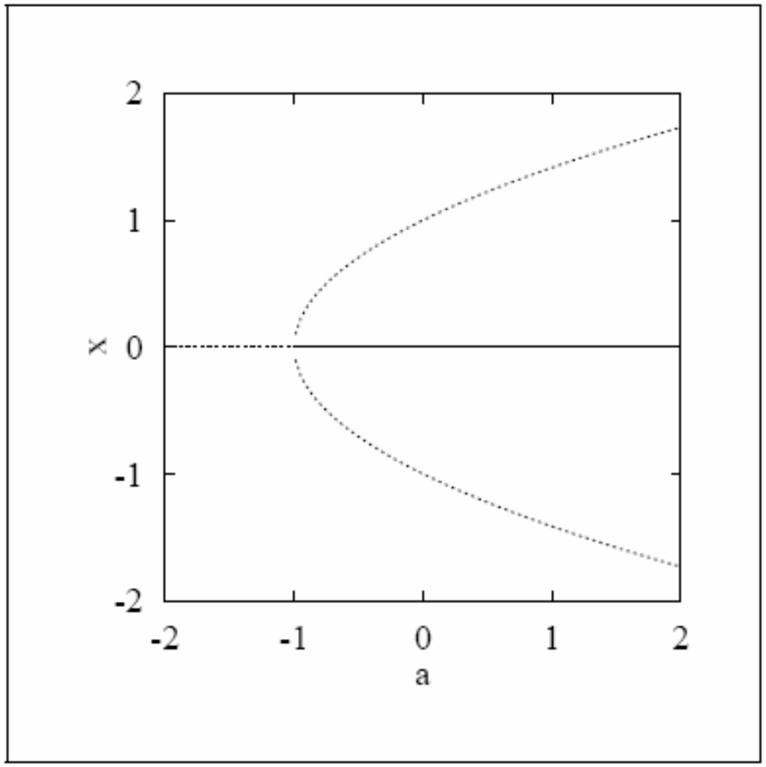

Figure 11.12 The bifurcation diagram for a pitchfork bifurcation.
Fixed points of the family of cubic maps $f_a(x) = x^3 - ax$ are shown. This type of bifurcation is not unusual for the special class of maps which are odd functions, for which $f_a(x) = -f_a(-x)$.

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III – Continuação de Pontos Fixos

Variação contínua do ponto fixo de um mapa unidimensional até uma bifurcaçao
(deixa de existir ou duplica de período).

Isso ocorre desde que $f_a' \neq 1$.

Definição

f_a em R^n, ponto fixo $f_{a*}(\vec{v}*) = \vec{v}*$

$\vec{v}*$ é continuável se $\vec{v}*$ de f_a, para $a \cong \bar{a}$, variar em trajetória contínua

Um conjunto de pontos fixos de f na vizinhança de

$(a*-d, a*+d) \text{ x } N_\varepsilon(\vec{v}*)$

é o gráfico de uma função contínua g de $(a*-d, a*+d)$ para $N_\varepsilon(\vec{v}*)$

$\Rightarrow$

$(a_c, \vec{v}_c)$ é a continuação de $(a*, \vec{v}*)$ em $a = \bar{a}$

Continuação do Ponto Fixo

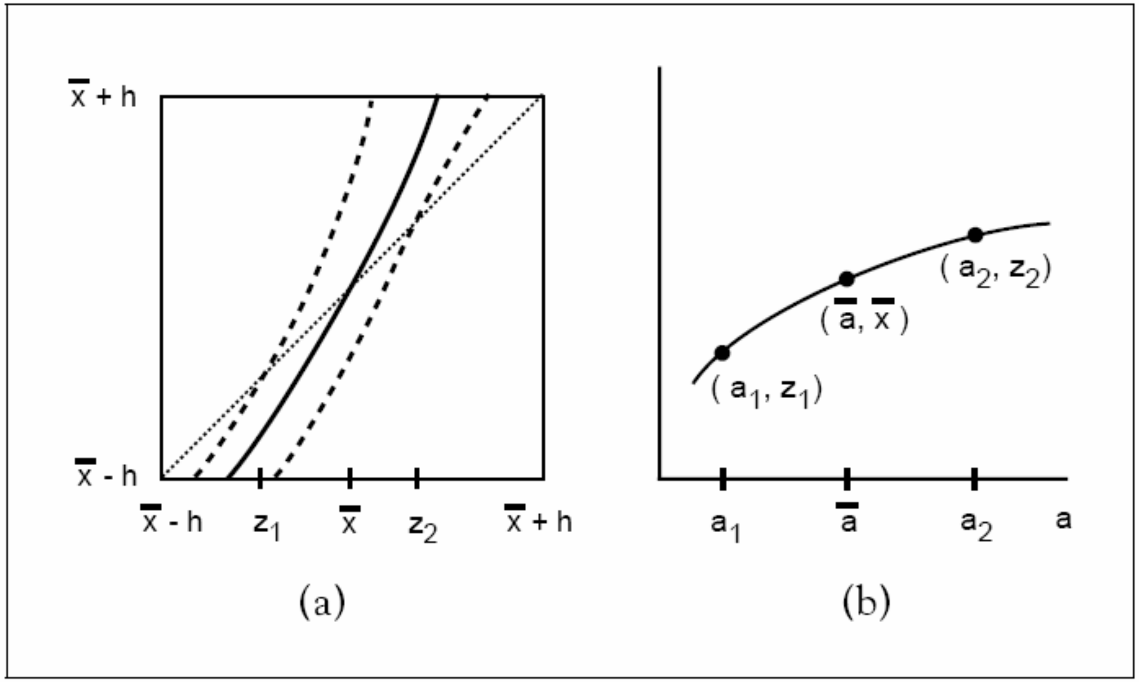

Figure 11.13 Graph of a scalar map f with a hyperbolic fixed point.
(a) An interval on which $f_a' > 1$ is shown. For small perturbations of $f_{\bar{a}}$ (small changes in a) the graph and derivatives of points near the fixed point remain close to those of the original map. The graph of $f_{\bar{a}}$ is indicated with a solid curve. The dashed curves are the graphs of maps f_{a_1} and f_{a_2}, where a_1 and a_2 are near $\bar{a}$, and $a_1 < \bar{a} < a_2$. The map f_{a_1} has one fixed point at z_1, and the map f_{a_2} has one fixed point at z_2. (b) The bifurcation diagram shows a continuous path of fixed points through $(\bar{a}, \bar{x})$.

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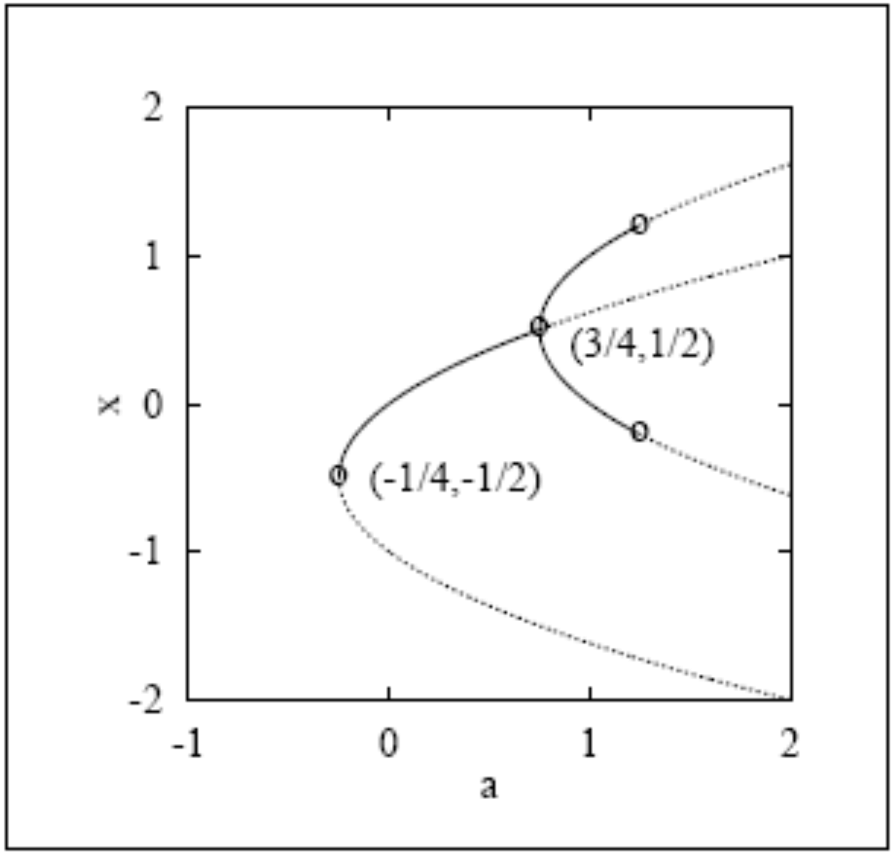
Exemplo



Variação contínua do ponto fixo no intervalo -1/4 < a< 3/4

Figure 11.5 Diagram of a period-doubling bifurcation.
Fixed points and period-two orbits of the quadratic family $f(a, x) = a - x^2$ are shown. Attractors are on solid curves, while repellers are on dashed curves. The circles denote the location of the bifurcation orbits.

No exemplo do slide anterior,

Mapa $f = a - x^2$

Bifucação sela-nó em $(a, x) = (-1/4, -1/2)$

$\Rightarrow$ sem continuação para $x < -1/4$

Mapa f com bifurcação dobra de período em $(a, x) = (3/4, 1/2)$

$f'_{3/4}(1/2) = -1 \Rightarrow$ ponto fixo com continuação

Mapa f^2 com bifurcação em $(a, x) = (3/4, 1/2)$

${f^2}'_{3/4}(1/2) = f'_{3/4}(1/2)\, f'_{3/4}(1/2) = (-1)(-1) = 1 \Rightarrow$

ponto fixo sem continuação

Teorema

a) f_a mapa suave definido em R, $f_a(\bar{x}) = \bar{x}$

$f_a'(\bar{x}) \neq 1 \Rightarrow (\bar{a}, \bar{x})$ tem continuação

b) f_a suave em R^n, $n > 1$

$f_a(\bar{\vec{v}}) = \bar{\vec{v}}$

λ: autovalor da matriz jacobiana $Df_a(\bar{\vec{v}})$ no ponto $\bar{\vec{v}}$

$\lambda \neq 1 \Rightarrow (\bar{a}, \bar{x})$ tem continuação

IV – Bifurcações em Mapas Unidiminesionais

$f_a' = 1 \quad \Rightarrow$ bifurcação

Variação descontínua do ponto fixo de f_a na bifurcaçao (deixa de existir ou duplica de período).

Se $\quad f_a' = -1 \quad \Rightarrow \quad f_a'^2 = 1$

Duplicação de período

Continuidade do ponto fixo de f_a

Descontinuidade do ponto fixo de f_a^2

Teorema (bifurcação sela - nó)

f_a : mapa suave, unidimensional, um parâmetro

$f_a(\bar{x}) = \bar{x}, \quad f_a'(\bar{x}) = 1$

$$A = \frac{\partial f(\bar{a}, \bar{x})}{\partial a} \neq 0 \quad \text{e} \quad D = \frac{\partial^2 f(\bar{a}, \bar{x})}{\partial x^2} \neq 0$$

$\Rightarrow$

Duas curvas de pontos fixos emanam de $(\bar{a}, \bar{x})$

$\exists$ pontos fixos para $a > \bar{a}$ se $DA < 0$

Não existem pontos fixos para $a > \bar{a}$ se $DA > 0$

Theorem 11.9 (Saddle-node bifurcation.) *Let f_a be a smooth one-parameter family of one-dimensional maps. Assume that $\overline{x}$ is a fixed point of $f_{\overline{a}}$ such that $f_{\overline{a}}'(\overline{x}) = 1$. Assume further that*

$$A = \frac{\partial f}{\partial a}(\overline{a}, \overline{x}) \neq 0 \quad \textit{and} \quad D = \frac{\partial^2 f}{\partial x^2}(\overline{a}, \overline{x}) \neq 0.$$

Then two curves of fixed points emanate from $(\overline{a}, \overline{x})$. The fixed points exist for $a > \overline{a}$ if $DA < 0$ and for $a < \overline{a}$ if $DA > 0$.

Ilustração do Teorema

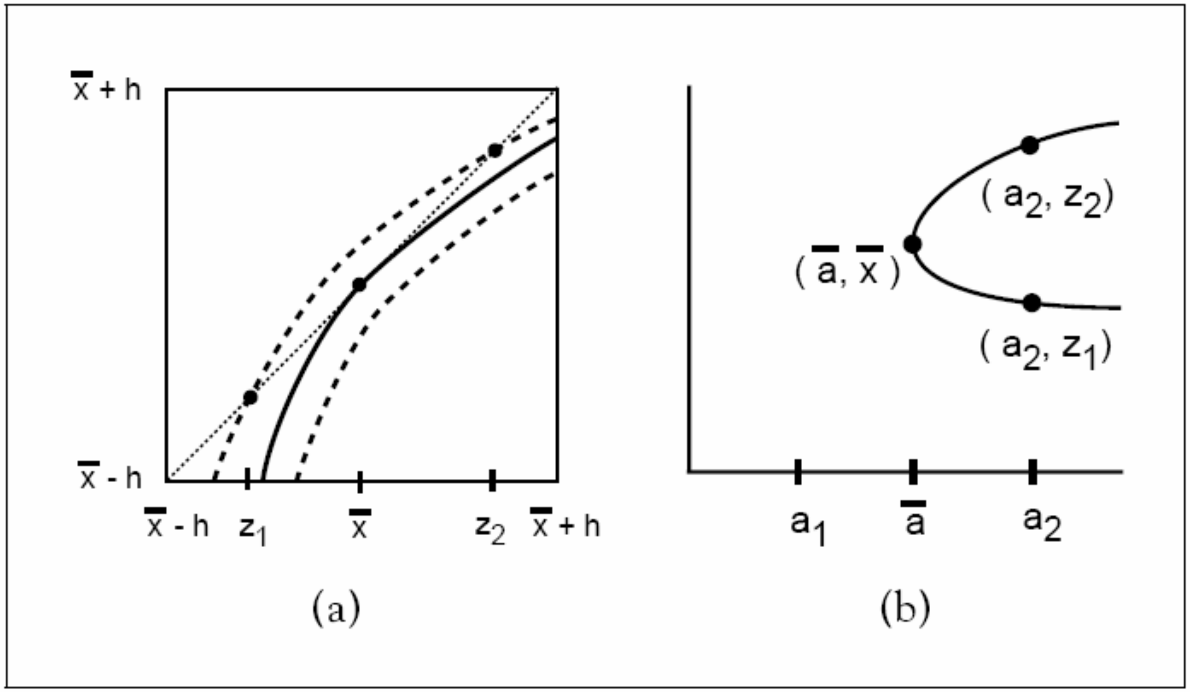


Figure 11.14 Graphs of a family of maps near a saddle-node bifurcation. (a) The solid curve is the graph of $f_{\bar{a}}$, and the dashed curves show the graphs of maps f_{a_1} and f_{a_2} for a_1 and a_2 near a, and $a_1 < \bar{a} < a_2$. The map f_{a_1} has no fixed points, the map $f_{\bar{a}}$ has one fixed point (the bifurcation point), and the map f_{a_2} has two fixed points, z_1 and z_2. (b) In the bifurcation diagram, two paths of fixed points are created at $a = \bar{a}$.

Bifurcação sela- nó

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Ocorrência de Bifurcação

Theorem 11.10 *Let f_a be a smooth one-parameter family of one-dimensional maps, and assume that there is a path of fixed points parametrized by a through $(\overline{a}, \overline{x})$. If $f_a{}'(x)$ evaluated along the path crosses $+1$ at $(\overline{a}, \overline{x})$, then every neighborhood of $(\overline{a}, \overline{x})$ in $\mathbb{R} \times \mathbb{R}$ contains a fixed point not on the path.*

Ilustração do Teorema

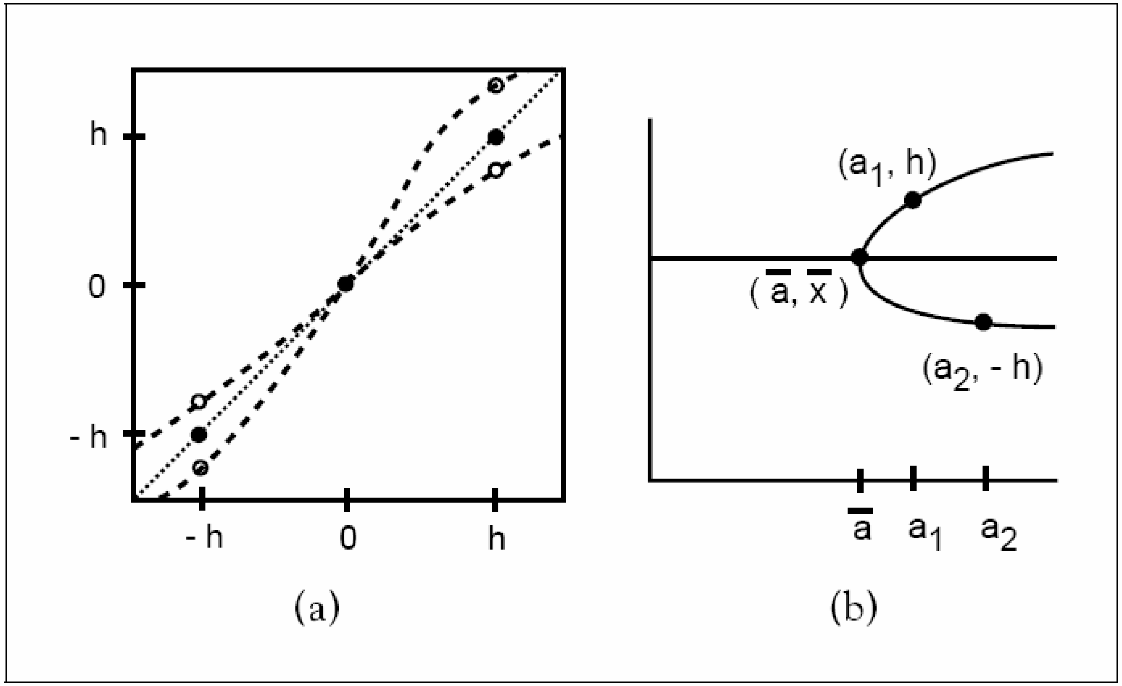

Figure 11.15 A family of maps where the derivative crosses 1.
(a) The dashed curves denote $f_{\bar{a}-h}$ and $f_{\bar{a}+h}$ in Theorem 11.10. The open circles straddle the diagonal, implying that f_a has fixed points other than 0 for parameter a between $\bar{a} - h$ and $\bar{a} + h$. (b) In the bifurcation diagram, a continuous path of fixed points bifurcates into paths of new fixed points. This is one of several possibilities when the derivative along a path of fixed points crosses 1.

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Theorem 11.11 *Let f be a smooth one-parameter family of maps. Assume that $\overline{x}$ is a fixed point of $f_{\overline{a}}$ such that $f_{\overline{a}}'(\overline{x}) = -1$. If $f_a'(x)$ evaluated along the path of fixed points through $(\overline{a}, \overline{x})$ crosses -1 at $(\overline{a}, \overline{x})$, then every neighborhood of $(\overline{a}, \overline{x})$ (in $\mathbb{R} \times \mathbb{R}$) contains a period-two orbit.*

Bifurcações em Mapas Unidimensionais

Bifurcation	$\frac{\partial f}{\partial x}(\overline{a}, \overline{x})$	$\frac{\partial f}{\partial a}(\overline{a}, \overline{x})$
saddle node	1	$\neq 0$
pitchfork, transcritical	1	0
period-doubling	-1	$\neq 0$

Table 11.1 Identification of common bifurcations in one-dimensional maps. When the derivative at the fixed point is 1 or -1, the type of bifurcation depends on other partial derivatives.

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V – Bifurcações em Mapas Bidimensionais Dissipativos

Mapa $\quad f_a = (\, g_a(x),\ 0.2\, y)\ ; \qquad g_a = a\, x\, (1 - x)$

$$Df_a = \begin{pmatrix} a - 2\, a\, x & 0 \\ 0 & 0.2 \end{pmatrix}$$

$$\begin{vmatrix} a - 2\, a\, x - \lambda & 0 \\ 0 & 0.2 - \lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda = \begin{cases} g_a'(x) \\ 0.2 \end{cases}$$

Bifurcação ocorre para $g_a'(x) = 1$

g_a : atratores, selas

f_a :atratores e fontes

Definições:

f mapa suave em R^2, $Df(\vec{v})$: matriz jacobiana

f é um mapa dissipativo se $\left|\det Df(\vec{v})\right| < 1, \ \forall \ \vec{v} \in R^2$

f é um mapa conservativo se $\left|\det Df(\vec{v})\right| = 1, \ \forall \ \vec{v} \in R^2$

Exemplo:

$f = (a - x^2 + by, \ x)$

$$\left|\det Df(\vec{v})\right| = \begin{vmatrix} 2x & b \\ 1 & 0 \end{vmatrix} = \left|-b\right|$$

Esse mapa é conservativo para $b = 1$

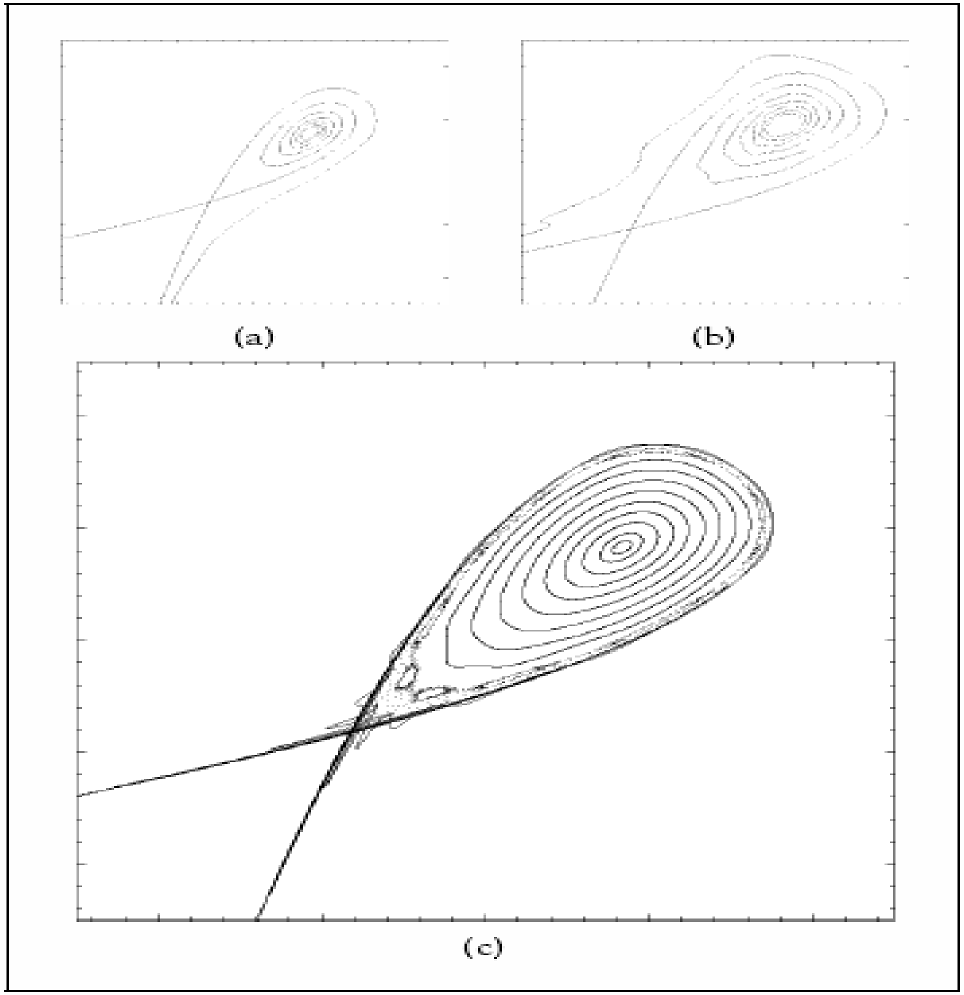

Figure 11.16 The continuation of two fixed points in the Hénon family. Various orbits of the map $h_b(x, y) = (-.75 - x^2 + by, x)$ are shown for (x, y) in $[-2.5, 5.0] \times [-2.5, 5.0]$. (a) At $b = -0.9$, the map is area-contracting and there is an attracting fixed point. The stable and unstable manifolds of a nearby saddle fixed point are depicted. One branch of the unstable manifold spirals into the attractor. (b) At $b = -1.1$, the map is area-expanding and there is a repelling fixed point (the continuation of the elliptic point at this parameter). Now the stable manifold of the saddle is shown to spiral out of the repeller. (c) At the in-between value $b = -1.0$, the map is area-preserving and there is an elliptic fixed point (the continuation of the attractor at this parameter). The stable and unstable manifolds of the saddle now cross in infinitely many homoclinic points.

Mapa de Hénon

a)Dissipativo
b)Expansivo
c)Conservativo

$-0.9 < b < -1.1$

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VI – Bifurcações no Plano
Mapas Conservativos

Propriedade dos mapas conservativos

$\exists$ auto-valor $\lambda_1 = a + bi \Rightarrow \exists$ auto-valor $\lambda_2 = a - bi$

Exemplo:

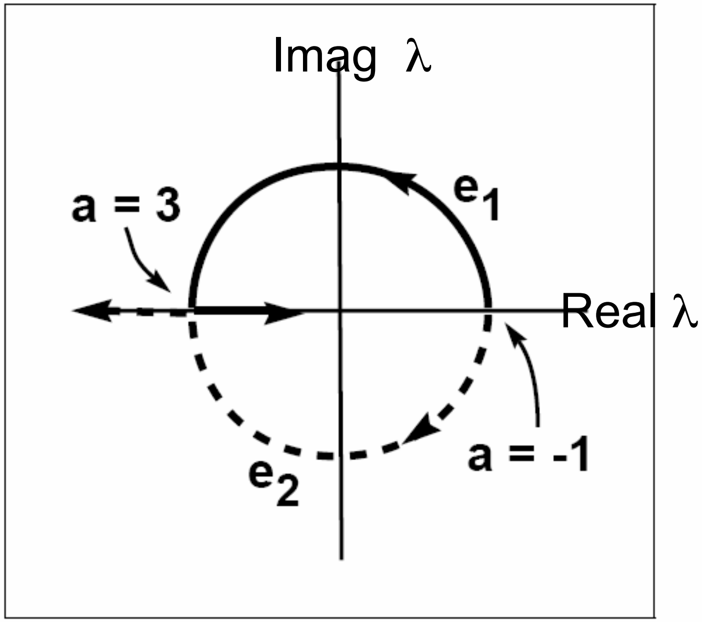
Mapa de Hénon

$h_b = (-1 - x^2 + by, x)$

$\mathrm{Det}\, Dh_b = -b = 1$ mapa conservativo

$(-0.5, -0.5)$ ponto fixo $\Rightarrow \lambda = 0.5 \pm \frac{\sqrt{3}}{2} i$

Mapa de Hénon Conservativo



Não há ponto fixo
para $a < -1$

Pontos elíptico e hiperbólico surgem em $a = -1$
(dois auto-valores $\lambda = 1$)

Em $a = 3$ ocorre dobramento de período.
Ponto elíptico se transforma em
ponto de sela (um auto-valor $\lambda > 1$)

Figure 11.17 Complex eigenvalues for a family of elliptic fixed points. Paths of eigenvalues in the complex plane for the continuation of an elliptic fixed point of the area-preserving Hénon family $\mathbf{h}(a, x, y) = (a - x^2 - y, x)$ are depicted schematically. The elliptic point is born at $a = -1$, at which point both eigenvalues are $+1$. Then the paths split and move around the unit circle as complex conjugates. The paths join at a period-doubling bifurcation when $a = 3$. There the fixed point loses stability, becoming a saddle, as one path of eigenvalues moves outside the unit circle.

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Exempo

Mapa de Hénon conservativo $(b = -1)$

$h_a = (a - x^2 - y, x)$

$a < -1 \Rightarrow$ não há pontos ffixos

$a = -1 \Rightarrow$ ponto fixo em $(-1, -1)$, $\lambda_1 + \lambda_2 = 1$

$a > -1 \Rightarrow$ dois pontos fixos: um ponto de sela e um ponto elíptico

$\lambda_{1,2} = a \pm ib$

$a = 3 \Rightarrow$ bifurcação com dobra de período

(duplicação de ilha)

Mapa Padrão

$$y, x \in [\pi, -\pi]$$

$$S_a \equiv \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n + y_n \\ y_n + a \operatorname{sen}(x_n + y_n) \end{pmatrix}$$

Mapa periódico com período $T = 2\pi$ *em x e y*

$$a = 0 \Rightarrow \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n + y_n \\ y_n \end{pmatrix}$$

Número de rotação $\iota \equiv \Delta x \equiv x_{n+1} - x_n = y_n \quad (Fig.\, 11.18a)$

$a = 0.97 \Rightarrow$ sup*erfícies KAM destruidas* $(Fig.\, 11.18 f)$

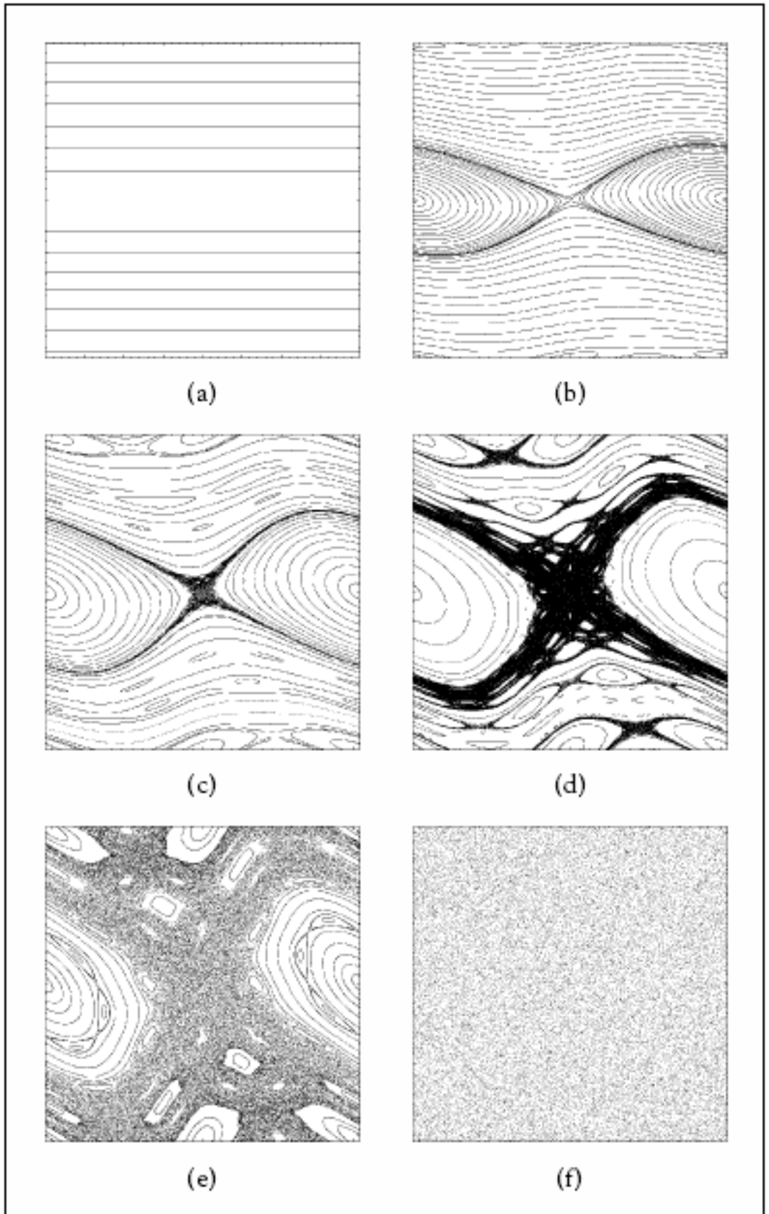

Figure 11.18 Standard maps at selected parameters.
Typical orbits of the standard family of maps $S_a(x, y) = (x + y \bmod 2\pi, y + a\sin(x + y) \bmod 2\pi)$ are shown for (x, y) in $[-\pi, \pi] \times [-\pi, \pi]$ and parameter values (a) $a = 0$ (b) $a = 0.3$ (c) $a = 0.6$ (d) $a = 0.9$ (e) $a = 1.2$ and (f) $a = 7.0$. In all but (f), several orbits are plotted. In (f), one orbit is shown.

Mapa Padrão

Variação com o parâmetro de controle

a) - g) várias órbitas

f) uma órbita

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VII – Bifurcações em Equações Diferenciais

Para equações diferenciais $\dot{\vec{v}} = f(\vec{v})$ em R^n

analisaremos o mapa de Poincaré em R^{n-1}

Definição: os auto-valores da matriz jacobiana, $(n-1)x(n-1)$, $D_{\vec{v}}T(\vec{v}_0)$ *são denominados de multiplicadores (de Floquet) da órbita periódica* γ.

λ_i : *auto-valores do mapa t*

$|\lambda_i| < 1 \Rightarrow$ *atrator*

$|\lambda_i| < 1$ *e* $|\lambda_i| > 1 \Rightarrow$ *ponto de sela*

Órbita periódica em $R^3 \Rightarrow$ *órbita periódica em* R^2

Mapa de Poincaré

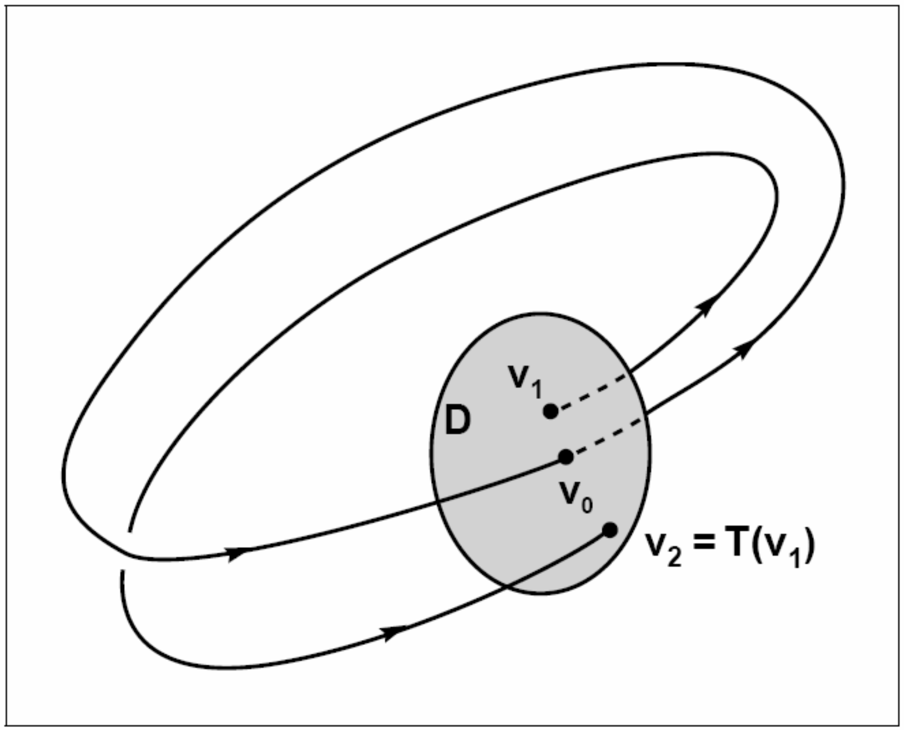

Figure 11.19 A Poincaré map.
The Poincaré map, called $\mathbf{T}$ here, is defined on a surface D which is transverse to the flow direction on a periodic orbit. The point $\mathbf{v}_0$ on the periodic orbit maps to itself under the Poincaré map, while $\mathbf{v}_1$ maps to $\mathbf{v}_2$.

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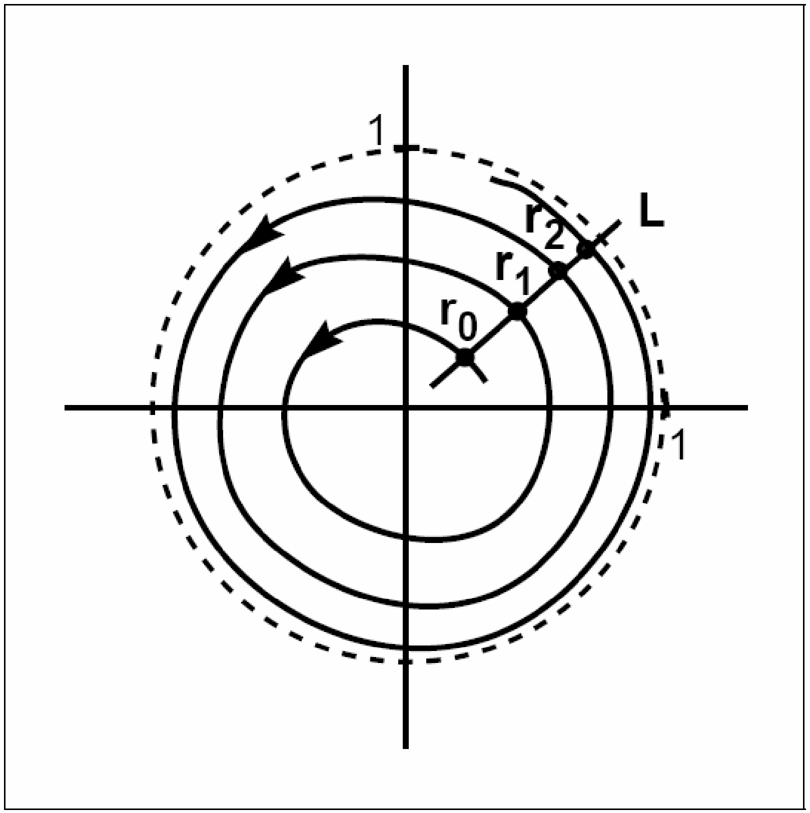
Exemplo de Mapa de Poincaré

$$\dot{r} = br(1-r)$$

$$\dot{\theta} = 1$$



Mapa de Poincaré:

$\{r_0, r_1, r_2, \ldots\}$

Atrator:
ciclo limite com r = 1

Figure 11.20 Poincaré map for a limit cycle of a planar system.
The system $\frac{dr}{dt} = 0.2r(1-r)$, $\frac{d\theta}{dt} = 1$, has a limit cycle $r = 1$. The line segment L is approximately perpendicular to this orbit. Successive images $r_1, r_2, \ldots,$ of initial point r_0 under the Poincaré map converge to $r = 1$.

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Bifurcação Sela-Nó

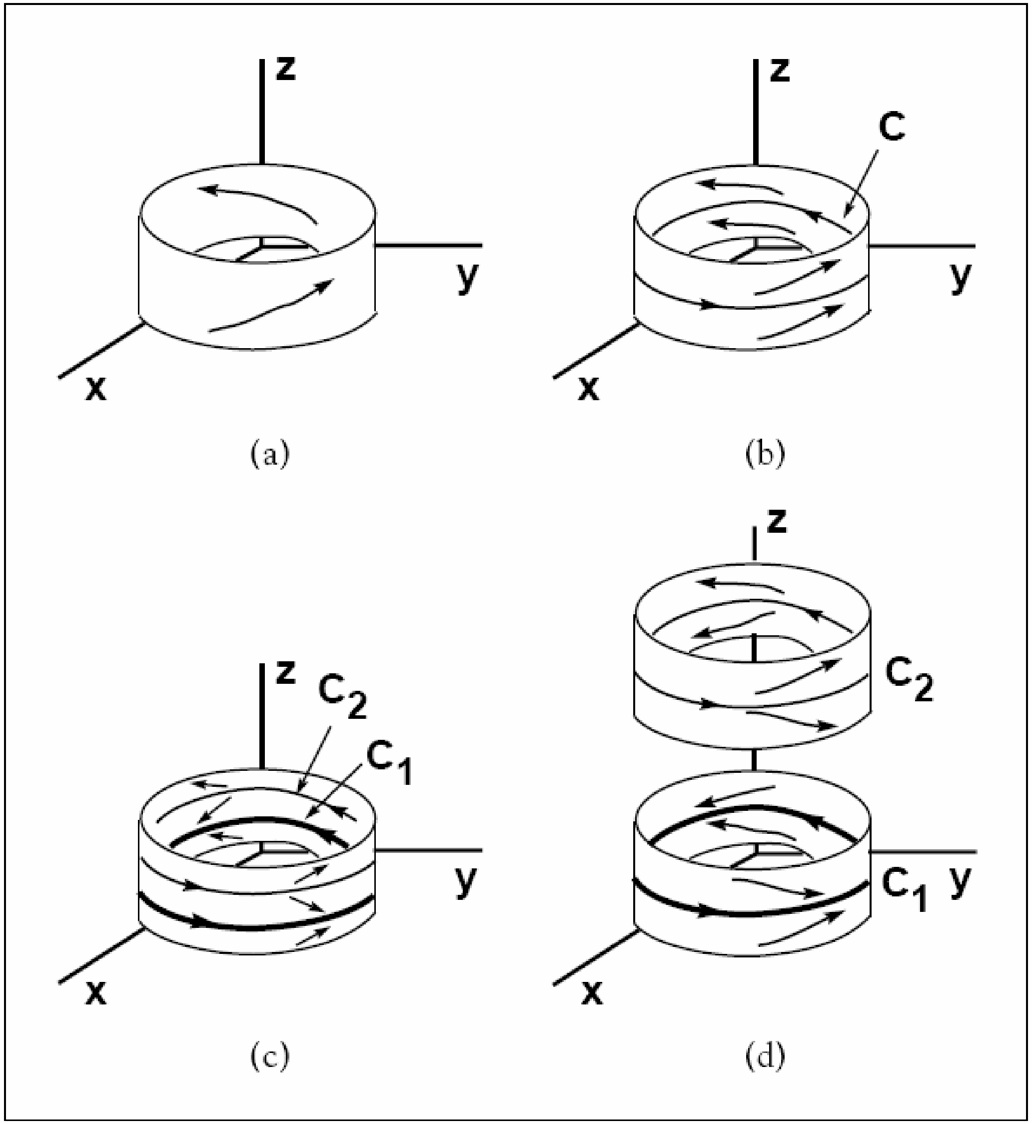

Figure 11.21 A saddle-node bifurcation for a three-dimensional flow. (a) The system begins with no periodic orbits. (b) The parameter is increased, and a saddle-node periodic orbit C appears. (c) The saddle-node orbit splits into two periodic orbits C_1 and C_2, which then move apart, as shown in (d).

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Dobramento de Período

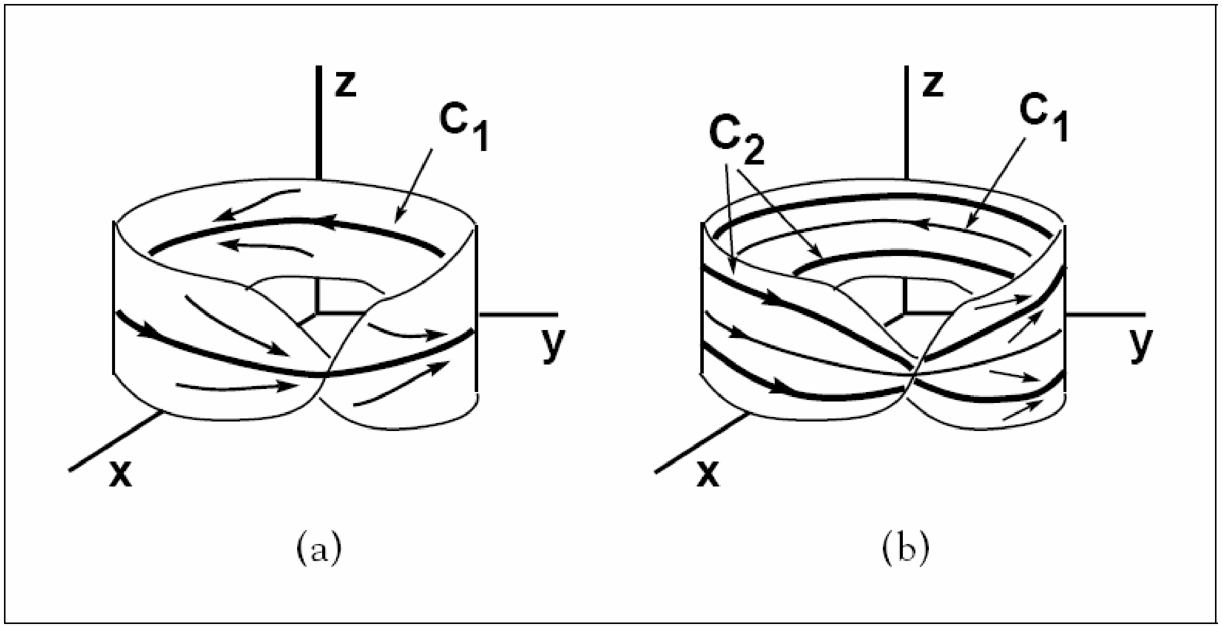

Figure 11.22 A period-doubling bifurcation for a three-dimensional flow. The system begins with a periodic orbit C_1, which has one multiplier between 0 and -1, as shown in (a). As the parameter is increased, this multiplier crosses -1, and a second orbit C_2 of roughly twice the period of C_1 bifurcates. This orbit wraps twice around the Möbius strip shown in (b).

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Bifurcação de Hopf

Exemplo

$$\dot{x} = -y + x(a - x^2 - y^2)$$

$$\dot{y} = x + y(a - x^2 - y^2)$$

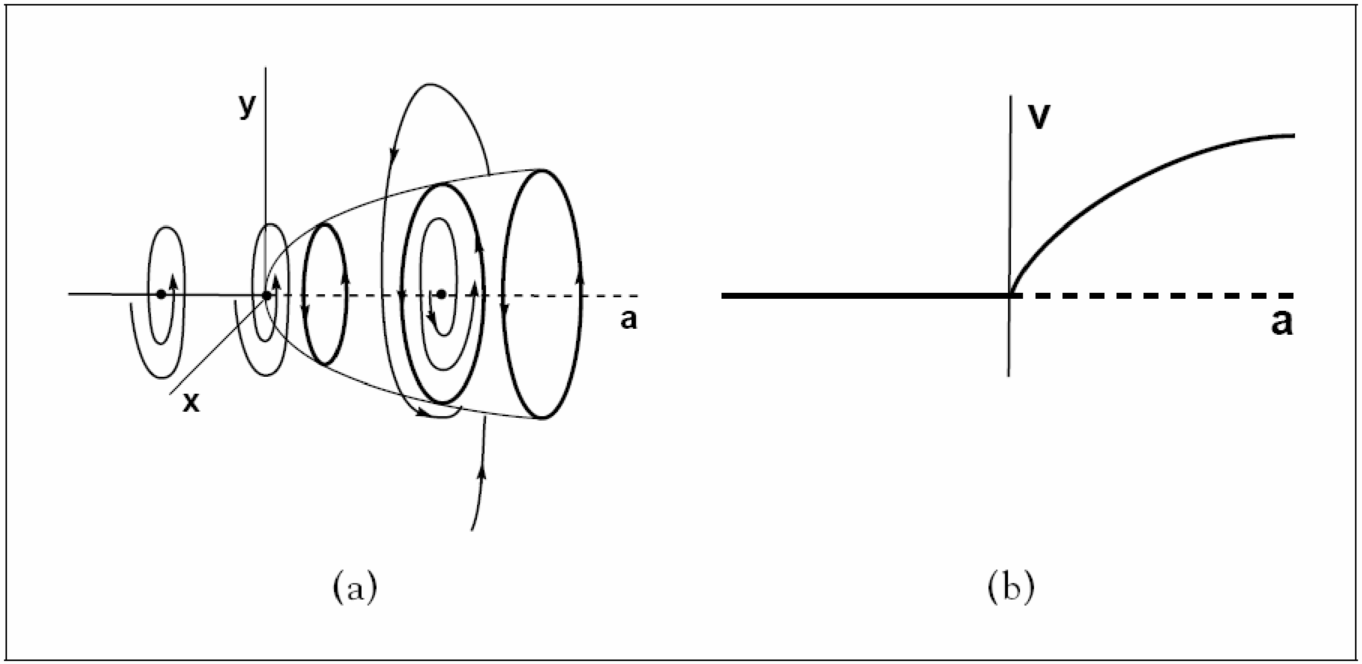

Figure 11.23 Supercritical Hopf bifurcation.
(a) The path $\{(a, 0, 0)\}$ of equilibria changes stability at $a = 0$. A stable equilibrium for $a < 0$ is replaced by a stable periodic orbit for $a > 0$. (b) Schematic path diagram of the bifurcation. Solid curves are stable orbits, dashed curves are unstable.

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Ilustração da Bifurcação de Hopf

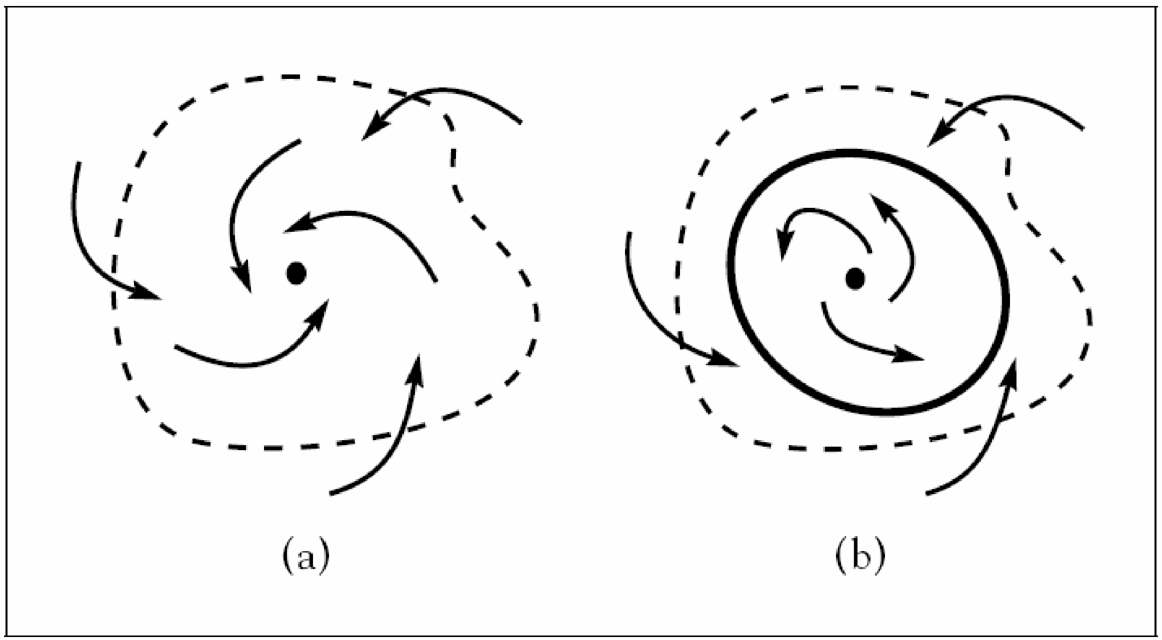

Figure 11.24 A periodic orbit bifurcates from an equilibrium in the plane. As the equilibrium goes from attracting (a) to repelling (b), a periodic orbit appears.

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Bifurcação de Hopf Sub-Crítica

$$\dot{r} = ar + 2r^3 - r^5$$

$$\dot{\theta} = 1$$

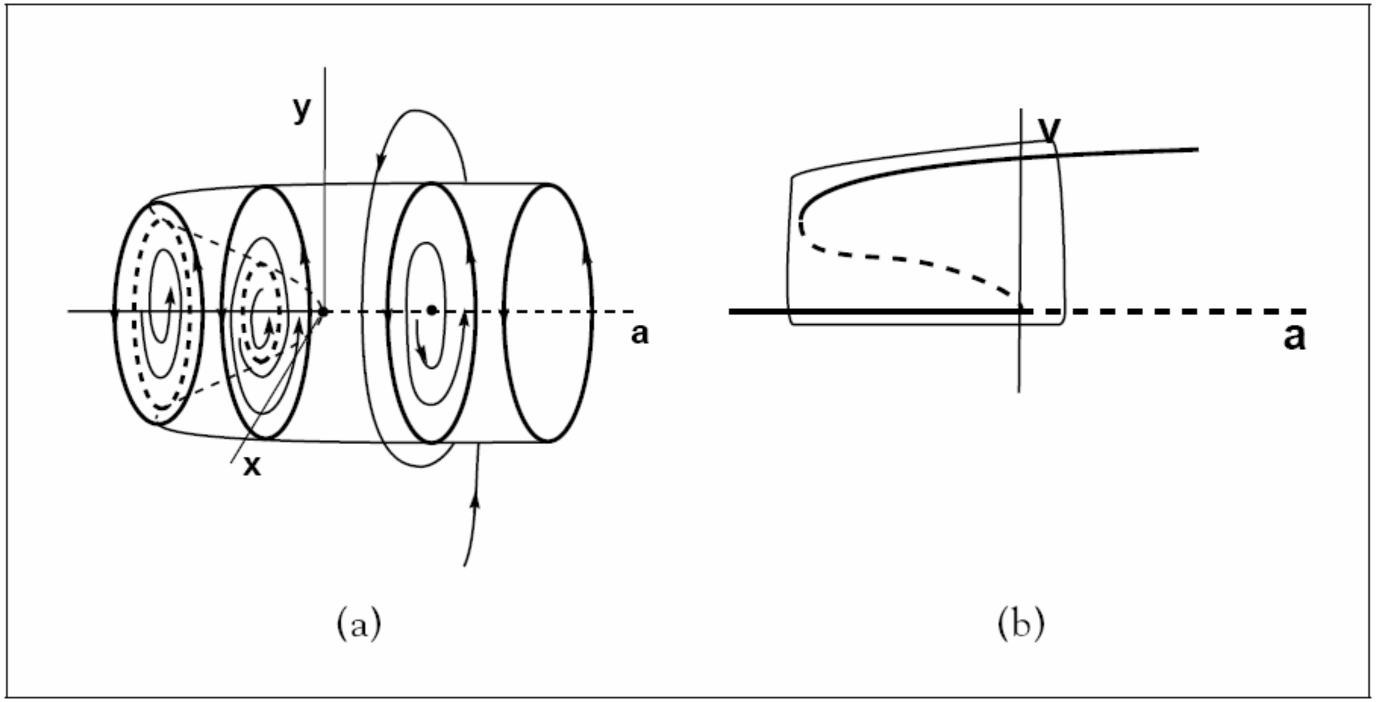

Figure 11.25 A subcritical Hopf bifurcation with hysteresis.
(a) There is a bifurcation at $a = 0$ from the path $r = 0$ of equilibria. At this point the equilibria go from stable to unstable, and a path of unstable periodic orbits bifurcates. The periodic orbits are unstable and extend back through negative parameter values, ending in saddle node at $a = -1$. An additional path of attracting periodic orbits emanates from the saddle node. (b) Schematic diagram of bifurcation

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Bifurcação de Hopf Sub-Crítica nas Equações de Lorenz

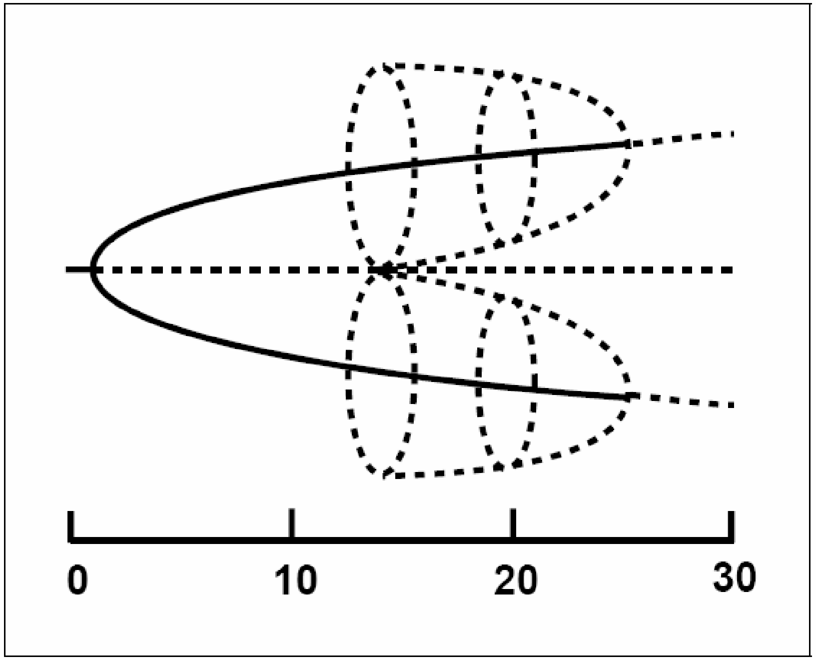

Figure 11.26 A Hopf bifurcation in the Lorenz equations.
A bifurcation diagram of the Lorenz equations for $\sigma = 10, b = 8/3$, and $0 \le r \le 30$, is shown. At $r = 1$, the origin goes from stable to unstable, as two attracting equilibria bifurcate. Paths of stable orbits are indicated by solid curves, while paths of unstable orbits are represented by dashed curves and circles. A subcritical Hopf bifurcation occurs at $r = 24.74$, at which point two families of unstable orbits bifurcate simultaneously as the two attracting equilibria lose their stability. Typical trajectories in computer simulations then move toward the chaotic attractor, first observed to occur at the crisis value $r = 24.06$.

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