Ergodic Hierarchy

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Content

Boltzmann vs. Gibbs

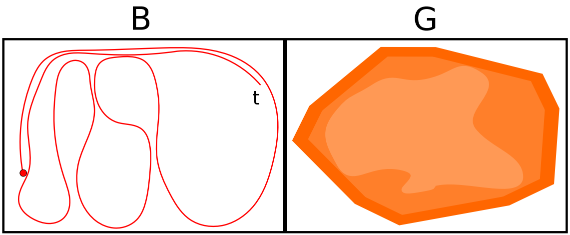
- Boltzmann: Trajectory Approach

  $\vec{X}(\{x_0\}, t = \epsilon) = T\vec{X}(\{x_0\}, t = 0)$

  $\lim_{t\to\infty}\langle F(\vec{X})\rangle_t = F_B \implies$ Equilibrium (H-Theorem)

- Gibbs: Ensemble Approach

  $\vec{X} \neq \vec{X}(t)$, $\langle F(\vec{X})\rangle_{\cup s^i} = F_G, \vec{X} = \{x\} \in s^i, \lim_{i\to\infty} \cup s^i = S$



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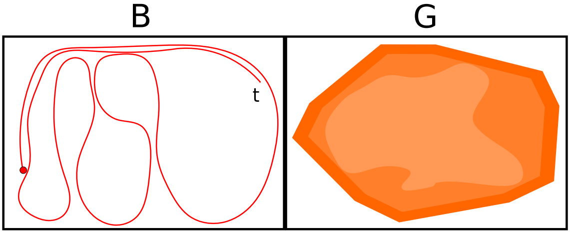

Boltzmann vs. Gibbs

Boltzmann:

- Almost any trajectory contains all information
- When $t \to \infty$, *F might still depend on* x_0

Gibbs:

- $F(\vec{X})$ dependend on $\vec{X} = \{x\}$, *not t*

  $F = \int_X F(\vec{X}) d\mu$, where μ is a measure over X
- "Analog" $t \to \infty$ limit: $\{s^i\}$ is a proper partition of the σ-algebra S of X ($\cup s^i = S \approx X$)

"Naïve" Ergodic Theorem:
The equilibrium properties of **some** systems **are independent of the description chosen**

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are independent of the description chosen

Formal Definition

- Automorphic Transformations

  $(T \mid \forall s_i \in S, Ts_i \in S)$

  Dynamical systems: $\vec{X}(x_1, \dot{x}_1, \ddot{x}_1, \ldots, x_2, \dot{x}_2, \ddot{x}_2, \ldots)$

  Measure-preserving: $T^{-1}s_i \in S$, $\mu(T^{-1}s_i) = \mu(s_i)$

- Estabilish a measure through a characteristic function f_{s_i}:

  $f_{s_i}(\vec{X}) = 1 \iff \vec{X} \in s_i \subset S$

  Then,

  $\int_S f_{s_i}(\vec{X})d\mu = \mu(s_i) \qquad (\mu(S) \equiv 1)$

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Equivalent Definition

- Birkhoff Pointwise ET

  $(X, S, \mu) \rightarrow$ space, σ-algebra and ergodic measure* μ.

  *Given a transformation T and its invariant subset $S' = T^{-1}S'$, either $\mu(S') = 0$ or $\mu(S - S') = 0$

  $f : X \rightarrow X$ measure-preserving

  $\phi : X \rightarrow \mathbb{R}$ (absolutely) integrable

  Then, for μ-almost $\forall \vec{X} \in X$,

  $\lim_{n\rightarrow\infty} \frac{1}{n} \sum_{k=1,n-1} \phi(f^k \vec{X})$ exists and is $\int \phi \, d\mu$

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Physical Measure

Basin of a physical measure:

$$\mathcal{B}(\mu) := \Big\{ \vec{X} \in X; \;\; \forall \phi : X \to \mathbb{R} \in C^1$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1,n-1} \phi(f^k x) = \int \phi d\mu \neq 0 \Big\}$$

finite statistical description means, for *finite, invariant* $\mu_1, \mu_2, \ldots, \mu_n$,

$$\cup^i \mu_i = S - S', \;\; B(\mu(S')) = 0$$

(S' has zero-volume basin)

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Physical Measure

Summing it up:

- Define a measure through a characteristic function for a given partition $s_i \in S$
- Sets that collapse to a finite union of points have measure zero
- Apart from these sets, there can be a finite number of invariant measure sets that cover the σ-algebra S of X, which are the physical measures

"Ergodic" means there is no finite partition of S that separates the measures, i.e. there is always a point from a different set "nearby" $\rightarrow$ "Metrically Indecomposable"

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Extreme Cases

- Predictable Flows:
  Initial conditions give information about *arbitrary* time in the future
  (e.g. Newtonian two-body problem)
- Bernoulli Flow
  Observing a state gives *no information* about previous or later states
  e.g. the baker transformation, which is isomorphic to **coin tossings**! (i.e., to a Bernoulli shift)

Intermediate Cases

There are cases in-between

(e.g., logistic map with $a \neq 4$ has non-homogeneous probabilities)

In general, there is a hierarchy

Ergodic $\supset$ Weak Mixing $\supset$ Strong Mixing $\supset$ Kolmogorov $\supset$ Bernoulli

(Kolmogorov are usually called “K-Systems”)

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Mixing

Assymptotic behaviour of transformations!

Strong Mixing: for subsets $A, B \in S$ and a transformation T:

$$\lim_{n \to \infty} \mu(T^n B \cap A) = \mu(A)\mu(B)$$

Weak Mixing: fluctuations exist but average out ,

$$\lim_{n \to \infty} \frac{1}{n} \sum |\mu(T^n B \cap A) - \mu(A)\mu(B)| = 0$$

K-Systems

K-Systems are very complicated, but for completeness:

$$\lim_{n\to\infty} \sup_{s\in\sigma_{n,r}} |\mu(s\cap A_0) - \mu(s)\mu(A_0)| = 0$$

(Cornfeld, Fomin & Sinai,1982)

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Bernoulli

Partition: $A = \{\alpha\}$, $\alpha_i \cap \alpha_j = 0$, $i \neq j$, $\cup_{i=1,N} \; \alpha_i = S$

$$\mu(\sigma_i \cap \alpha_j) = \mu(\sigma_i)\mu(\alpha_j)$$

$$\forall \{\sigma_i\} \in T^k S \;\;, \;\; \{\alpha_j\} \in T^\ell S \;\;, \;\; k \neq \ell$$

Bernoulli

Bernoulli shift in the baker's transformation B

Write the binary expasion for x, y:

$x = 0.a_1 a_2 a_3 \ldots, y = 0.b_1 b_2 \ldots$

$B(0.a_1 a_2 a_3 \ldots, 0.b_1 b_2 b_3 \ldots) = (0.a_2 a_3 \ldots 0.a_1 b_1 b_2 b_3 \ldots),$

which is a Bernoulli shift on
$\ldots b_3 b_2 b_1 .a_1 a_2 a_3 \cdots \to \ldots b_3 b_2 .b_1 a_1 a_2 a_3 \ldots$

(Taking the axiom of choice)

Ergodicity Breaking

Why All That?

Breaking ergodicity may *break the thermodynamic limit*!

Simple case: Ferromagnets! Spontaneus symmetry breaking

Time average $(+m) \neq X$ average $(\pm m)$

One can introduce a *symmetry breaking field*, avoid the critical point, regain ergodicity, all fine.

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Ergodicity Breaking

Spin Glasses/Frustrated Disordered Systems

Hierarchy of time scales $t_1 \ll t_2 (\ll t_3 \ldots)$

Broken ergodicity *without* breaking any symmetry of the Hamiltonian! No obvious "field" can be applied

Option 1: Trajectories become confined in local minima for infinite time: Statistical averages and order parameters might be wrong, dynamics are mostly unnafected

Option 2: Trajectories *never stop evolving*, but navigate a complicated landscape: **aging**
(e.g. Mean-Field Sherrington-Kirkpatrick Spin-Glass)

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Conclusion

- Breaking of ergodicity leads to many unsolved problems
- “Mere” ergodicity is just the begining of a hierarchy
- Understanding the hierarchy can help understand the transitions in disordered phases through correlation function behaviour
- But the mathematical formalism can be dense

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