Nonlocal coupling among oscillators mediated by a diffusing substance

Ricardo L. Viana

Departamento de Física, Universidade Federal do Paraná
Curitiba, Paraná, Brasil

Workshop on Complex Dynamics of Plasmas (Funded by CAPES/COFECUB) July 26, 2018, So Paulo, Brazil: Universidade de São Paulo

Thanks to:

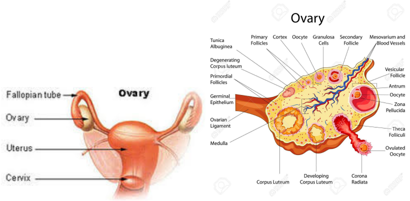
- Carlos Adalberto Schneider Batista (UFPR - Pontal do Paraná)
- José Danilo Szezech Jr. (UEPG - Ponta Grossa)
- Antônio Marcos Batista (UEPG - Ponta Grossa)
- Elbert E. Nehrer Macau (UNIFESP - São Paulo)
- Fabio Alliguieri dos Santos Silva (IFPR - Paranaguá)
- Raul Palma Aristides (UFPR - PhD student)

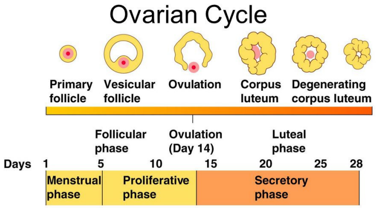
Introduction

- there are many problems in physical chemistry and biology involving the interaction among individuals mediated by the diffusion of some chemical in the medium
- this chemical is both released and absorbed by the individuals, often depending on dynamical processes occuring within them
- the release, diffusion and absorption processes create a long-range coupling among individuals that affects their dynamics
- in this work we explore the consequences of this basic idea into models of non-locally coupled nonlinear oscillators
- coupling problems typically involve more than one timescale

Ovarian cycles

- surge of luteinizing hormone from the anterior pituitary
- LH luteinizes several follicles in the ovary causing each to rupture and release its effective
- at the site of rupture a *corpus luteum* forms and secretes progesterone
- just after ovulation a new set of follicles begins to mature and releases estrogen (ovarian steroid)
- estrogen primes the system for another surge of LH
- cycle period = 28 days for humans





(c) Ovarian cycle

Figure 16.12c

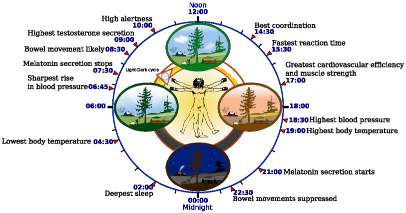
Synchronization of ovarian cycles

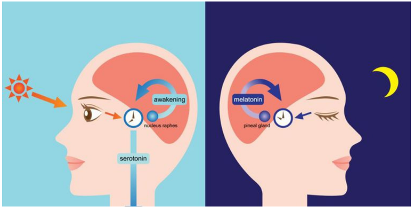
J. Schank and M. McClintock, J. Theor. Biol. **157**, 317 (1992)

- McClintock effect: women that live together synchronize their ovarian cycles
- the synchronization of ovarian cycles is mediated by airborne chemosignals called pheromones
- two distinct chemicals: one advances and another delays the phase of the ovarian cycle
- pheromones diffuse in the atmosphere: there must be a common air supply

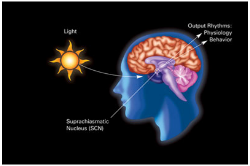
Circadian rhythms

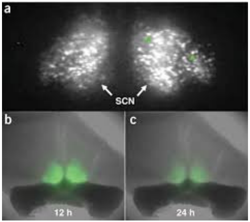
- biological processes displaying an endogenous oscillation of about 24 hours (“circa diem”)
- present in animals, plants, fungi, and cyanobacteriae
- these 24-hour rhythms are driven by circadian clocks cells
- circadian rhythms can be adjusted (entrained) to the local environment by external cues (zeitgebers)





Suprachiasmatic nucleus

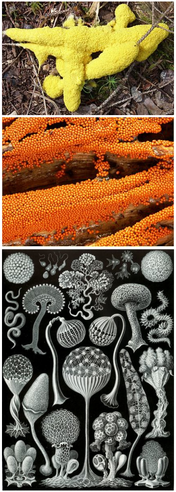




- region of the brain (hypothalamus) responsible for controlling circadian rhythms
- receives input from specialized photosensitive ganglion cells in the retina
- maintains control by synchronizing their own near-24-hour rhythms and control circadian phenomena in local tissues
- it contains around 20,000 neurons (clock cells)
- collective rhythm possible because of frequency synchronization induced by coupling among clock cells
- coupling is mediated by a neurotransmitter (GABA) diffusing through intercell medium

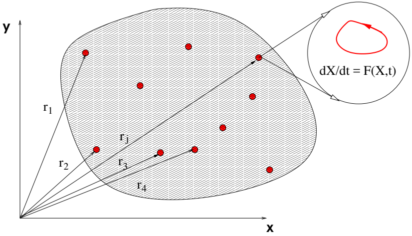
Chemotaxis of *Dictyostelium*

- chemotaxis: motion of an organism in response to a chemical stimulus
- *Dictyostelium*: amoeabae ("slime mold")
- in absence of food about 10^5 cells release signal molecules of chemoattractant cyclic adenosine monophosphate (cAMP) in the environment
- they can find other cells and move to create clusters (releasing every 6 min during periods reaching 5 to 6 hours after starvation)



Coupling model

Y. Kuramoto, Prog. Theor. Phys. **94**, 321 (1995)



- N pointlike oscillator cells located at $\vec{r}_j$, $(j = 1, 2, \cdots N)$ in a d-dimensional Euclidean space
- each oscillator has an internal dynamics governed by the flux $\mathbf{F}(\mathbf{X}, t)$, where $\mathbf{X} = (x_1, x_2, \ldots x_M)^T$ in a M-dim. phase space
- the time evolution of each oscillator is affected by the local concentration of a chemical $A(\mathbf{r}, t)$

$$\dot{\mathbf{X}}_j = \mathbf{F}(\mathbf{X}_j, t) + \mathbf{g}(A(\vec{r}_j, t))$$

Coupling model in the adiabatic limit

- the concentration satisfies a inhomogeneous diffusion equation

$$\varepsilon\dot{A} = -\eta A + D\nabla^2 A + \sum_k h(\mathbf{X}_k)\delta(\vec{r} - \vec{r}_k)$$

- η: coefficient of chemical degradation, D: diffusion coefficient
- $\varepsilon \approx 0$: diffusion timescale much faster than oscillator period
- concentration fast-relaxes to a stationary value

$$A(\vec{r}_j) = \sum_k \sigma(\vec{r}_j - \vec{r}_k) h(\mathbf{X}_k)$$

- $\sigma(\mathbf{r})$ is the Green function of the diffusion equation

$$(\eta - D\nabla^2)\sigma(\vec{r}_j - \vec{r}) = \delta(\vec{r}_j)$$

- chemical coupling in the adiabatic limit

$$\dot{\mathbf{X}}_j = \mathbf{F}(\mathbf{X}_j, t) + \sum_k \sigma(\vec{r}_j - \vec{r}_k)\mathbf{g}\left(h(\mathbf{X}_k)\right).$$

Types of chemical coupling

- linear coupling: $\mathbf{g}(h(\mathbf{X}_k)) = \mathbf{A}\mathbf{X}_k$
- future coupling: $\mathbf{g}(h(\mathbf{X}_k)) = \mathbf{A}\mathbf{F}(\mathbf{X}_k)$
- nonlinear coupling: $\mathbf{g}(h(\mathbf{X}_k)) = \mathbf{A}\mathbf{H}(\mathbf{X}_k)$
- Green function for isotropic systems ($r \equiv |\vec{r}_j - \vec{r}|$)

$$\sigma(r) = C \begin{cases} \exp(-\gamma r), & \text{if } d = 1, \\ K_0(\gamma r), & \text{if } d = 2, \\ \frac{\exp(-\gamma r)}{\gamma r}, & \text{if } d = 3 \end{cases}$$

- inverse coupling length: $\gamma = \sqrt{\eta / D}$,
- C is determined from the normalization condition $\int d^d r \sigma(\vec{r}) = 1$

Limiting cases

- $\gamma \to 0$: global type of coupling (all-to-all)

$$\dot{\mathbf{X}}_j = \mathbf{F}(\mathbf{X}_j, t) + \mathbf{A}\overline{\mathbf{X}}$$

- mean field of all oscillators (except itself)

$$\overline{\mathbf{X}} = \frac{1}{N-1} \sum_{k=1, k\neq j}^{N} \mathbf{X}_k$$

- large γ: local (diffusive) type of coupling (nearest neighbors)

$$\dot{\mathbf{X}}_j = \mathbf{F}(\mathbf{X}_j, t) + \mathbf{A}\left(\mathbf{X}_{j-1} + \mathbf{X}_{j+1}\right)$$

- varying γ we can pass continuously from global to local coupling (in numerical studies of long-range coupling)

Phase oscillators with chemical coupling

- for phase oscillators $\mathbf{X} \to \theta$ and $\mathbf{F} \to \omega$
- nonlinear coupling (extended Kuramoto model)

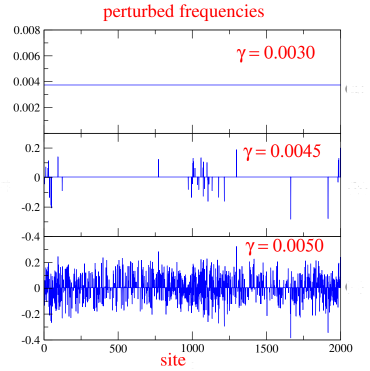
$$\dot{\theta}_j = \omega_j + K \sum_k \sigma(\vec{r}_j - \vec{r}_k) \sin(\theta_k - \theta_j)$$

- for $d = 1$ the Green function is $\sigma(x_i - x_j) = e^{-\gamma|x_i - x_j|}$ with $\gamma = \sqrt{\eta/D}$ (inverse coupling length), and normalization constant determined by $\int dx\, \sigma(x) = 1$
- one-dimensional lattice with periodic boundary conditions
- normalization factor $\kappa(\gamma) = 2\sum_{\ell=1}^{N'} e^{-\gamma\ell}$

$$\dot{\theta}_j = \omega_j + \frac{K}{\kappa(\gamma)} \sum_{\ell=1}^{N'} e^{-\gamma\ell} \left[\sin(\theta_{j-\ell} - \theta_j) + \sin(\theta_{j+\ell} - \theta_j)\right].$$

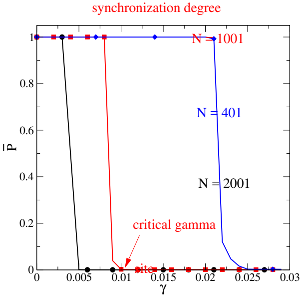
Frequency synchronization

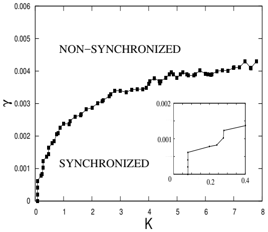
C. Batista *et al.*, Physica A **470**, 236 (2017)



- frequency without coupling: ω_j, randomly chosen according to a zero-mean gaussian PDF $g(\omega)$
- frequency after coupling: $\Omega_j = \dot{\theta}_j$
- small γ: nearly global coupling = complete synchronization
- intermediate γ: partial synchronization
- large γ: nearly local coupling = no synchronization
- competition between disorder and diffusion

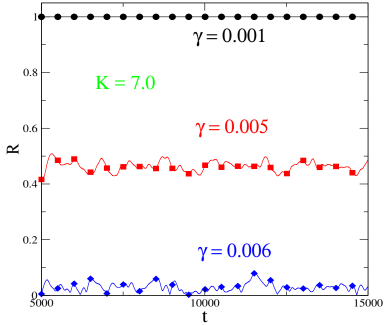
Partial frequency synchronization





- N_i: length of ith sync plateau
- N_p: number of plateaus
- average plateau size: $\langle N \rangle = (1/N_p) \sum_{i=1}^{N_p} N_i$
- sync degree: $P = \langle N \rangle / N$
- completely sync state: $P = 1$
- completely non-sync state: $P \approx 0$
- critical γ for increasing K
- small K: no sync for any γ
- global coupling ($\gamma = 0$): sync after $K_c = 2/\pi g(0) = 0.08$
- local coupling (large γ): no sync, even with large K

Phase synchronization

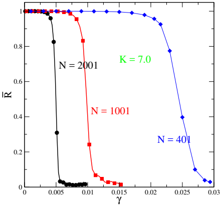


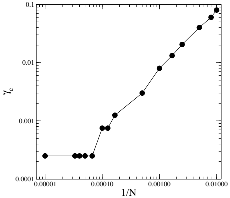
- equality of phases (stronger than frequency sync)
- complex order parameter

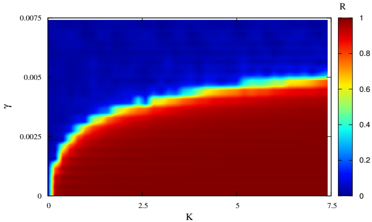
$$z(t) = R(t)e^{i\varphi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}$$

- completely synchronized state: $R = 1$, → small γ (global)
- non-synchronized state: $R \approx 0$, → intermediate γ
- partially synchronized state: $0 < R < 1$, → large γ (local)

Transition to phase synchronization



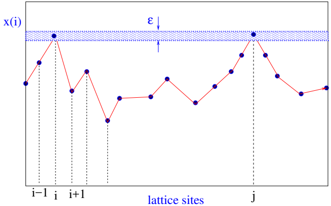




- $\bar{R}$: time-averaged order parameter magnitude
- synchronization transition for a critical γ_c (fixed K)
- γ_c decreases with increasing lattice size N
- thermodynamical limit ($N \to \infty$): $\gamma_c = 0.00025$
- coupling parameter plane: deep blue: no phase sync, red: complete phase sync

Spatial recurrence matrix

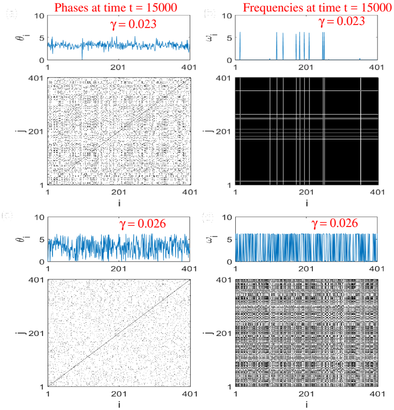
D. Vasconcelos *et al.*, Phys. Rev. E **73**, 56207 (2006)



- a tool for characterization of complex spatial patterns
- one-dimensional spatial pattern with N sites: $\{x_k\}$, $k = 0, \dots N$
- spatial recurrence: two sites i and j have the same height, up to some precision ε
- spatial recurrence matrix: $R_{ij} = 1$ if $|x_i - x_j| \leq \varepsilon$, and 0 otherwise

$$R_{ij} = \Theta(|x_i - x_j| - \varepsilon)$$

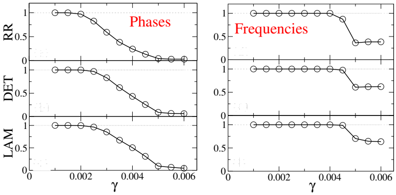
Spatial recurrence plots



- spatial recurrence plot: graphical representations of the spatial recurrence matrix with elements R_{ij}
- characterizes the existence of spatially homogeneous (synchronized) states in phase and frequency
- characterizes the existence of spatially inhomogeneous states (chimeras)

Recurrence quantification analysis

M. Santos *et al.*, Phys. Lett. A **379**, 2128 (2015)



- recurrence rate: fraction of recurrent points: $RR = \frac{1}{N^2}\sum_{i,j} R_{ij}$
- determinism: fraction of points belonging to diagonal structures
- laminarity: fraction of points belonging to horizontal structures
- transitions to phase and frequency synchronization as γ is varied from large (short range) to small (long range)

Model for biological clock cells

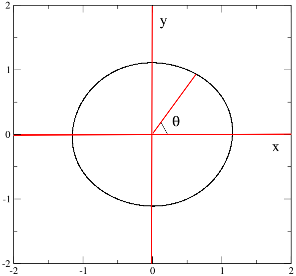
R. Kronauer *et al.*, J. Biol. Rhythms **14**, 501 (1999)

- model for coupled circadian clock cells in the SCN
- each clock cell with its own period $\tau_j \approx 24h$ (Gaussian PDF)
- Van der Pol-type oscillator describing the clock cell dynamics

$$\mathbf{X}_j = \frac{12}{\pi}\begin{pmatrix} x_j \\ y_j \end{pmatrix}, \mathbf{F}(\mathbf{X}_j, t) = \begin{pmatrix} y_j + \epsilon\left(x_j - \frac{4}{3}x_j^3\right) + B(t) - C_x x_j \\ -\left(\frac{24}{\tau_j}\right)^2 x_j + B(t) y_j - C_y y_j \end{pmatrix}$$

- $C_{x,y}$ are coupling parameters
- photic stimulation (zeitgeber): $B(t) = C\left(1 - m\langle x\rangle\right)\left[I(t)\right]^{1/3}$
- spatial average: $\langle x\rangle = (1/N)\sum_{i=1}^{N} x_i$

Phase and frequency of clock cells



- oscillator dynamics displays a limit cycle encircling the origin $(x_{k0} = 0, y_{k0} = 0)$
- geometrical phase

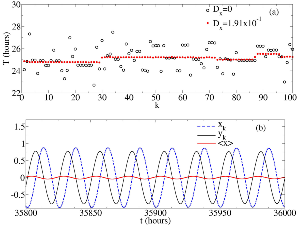
$$\theta_k(t) = \arctan\left(\frac{y_k(t) - y_{k0}}{x_k(t) - x_{k0}}\right)$$

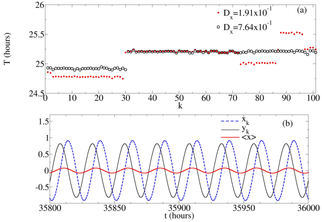
- oscillator frequency and period

$$\Omega_k = \frac{2\pi}{T_k} = \lim_{T\to\infty} \frac{\theta_k(T) - \theta_k(0)}{T}$$

Synchronization of clock cells under constant darkness

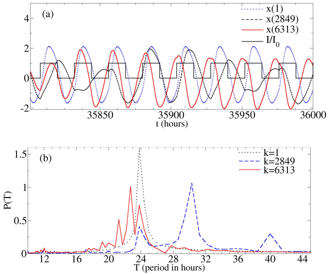
F. Silva *et al.*, CNSNS **35**, 37 (2016)





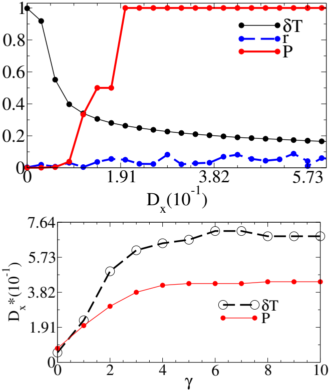
- system is evolved in constant darkness ($I = 0$)
- top: (a) distribution of periods of SCN cells with $\gamma = 1$ for $D_x = 0$ (open circles) and 0.191 (open circles)
- (b) time series for the x and y variables of one typical SCN cell and mean field $\langle x \rangle$
- bottom: D_x increased to 7.64×10^{-1} (open circles)

Synchronization of clock cells under dark-bright cycles



- system experiences dark-bright cycles of duration $\Delta t = 12h$ and constant light intensity $I_0 = 1000$
- top: (a) time series for the x variables of three SCN cells selected out of a lattice with 10201 cells with $\gamma = 1$ for $D_x = 7.64 \times 10^{-3}$
- (b) power spectra of the response of the three cells depicted in (a)

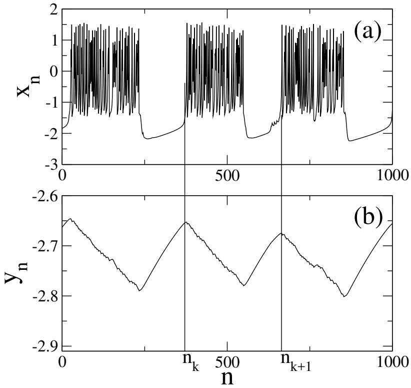
Transition to period synchronization of clock cells



- top: dependence of the period dispersion (black), order parameter magnitude (blue) and synchronization degree (red) with the coupling strength
- D_x^*: critical coupling strength for onset of period synchronization
- bottom: critical coupling strength vs. exponent γ considering the behavior of the period variance (black) and synchronization degree (red).

Autonomous bursting: the Rulkov map

N. Rulkov, Phys. Rev. E **65**, 41922 (2002)



- x_n: "action potential" (fast)
- y_n: "modulating variable" (slow)

$$x_{n+1} = \frac{\alpha}{1 + x_n^2} + y_n$$

$$y_{n+1} = y_n - \sigma x_n - \beta$$

- bursting phase

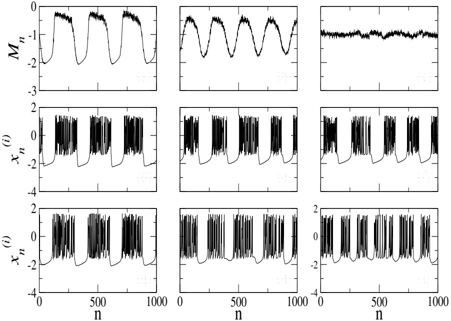
$$\varphi(n) = 2\pi k + 2\pi \frac{n - n_k}{n_{k+1} - n_k}$$

- bursting frequency $\Omega = \dot{\varphi}$

Bursting synchronization

R. L. Viana *et al.*, CNSNS **17**, 2924 (2012)

N = 251, ε = 0.1, σ = 0.001, β=0.001



γ = 0.005 γ = 0.0125 γ = 0.05

- coupled Rulkov maps

$$x_{n+1}^{(j)} = \frac{\alpha^{(j)}}{1+\left(x_n^{(j)}\right)^2} + y_n^{(j)} +$$

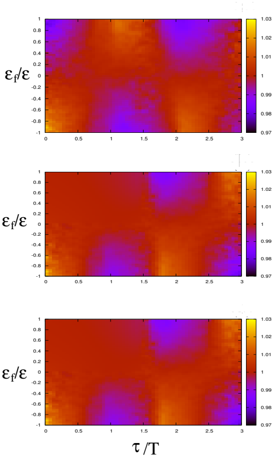
$$+\varepsilon C \sum_{\ell=1}^{N'} e^{-\gamma\ell}\left[x_n^{(j-\ell)} + x_n^{(j+\ell)}\right]$$

$$y_{n+1}^{(j)} = y_n^{(j)} - \sigma x_n^{(j)} - \beta$$

- parameter α randomly distributed in $[4.1, 4.3]$
- mean field of the lattice $M_n = \frac{1}{N}\sum_j x_n^{(j)}$

Suppression of bursting synchronization

M. Rosenblum and A. Pikowsky, Phys. Rev. E **70**, 41904 (2004)



$\gamma = 0.005$

N = 111

$\gamma = 0.0125$

$\varepsilon = 0.1$

$\gamma = 0.025$

- delayed feedback control

$$\varepsilon X_n + \varepsilon_f X_{n-\tau}$$

- weighted mean field

$$X_n = C \sum_{\ell=1}^{N'} e^{-\gamma \ell} \left(x_n^{(j-\ell)} + x_n^{(j+\ell)} \right)$$

- suppression coefficient
$S = \sqrt{var(X)/var(X_f)}$

Chemical coupling of chaotic maps

- if the local dynamics is governed by a one-dimensional map $x \mapsto f(x)$ we have (future coupling)

$$x_{n+1}^{(i)} = (1-\varepsilon) f(x_n^{(i)}) + \varepsilon \sum_{j=1}^{N} \sigma(\vec{r}_i - \vec{r}_j) f(x_n^{(j)}).$$

- non-locally coupled map lattice in one spatial dimension (where $N' = (N-1)/2$ for N odd)

$$x_{n+1}^{(i)} = (1-\varepsilon) f\left(x_n^{(i)}\right) + \frac{\varepsilon}{\kappa(\gamma)} \sum_{s=1}^{N'} e^{-\gamma s} \left[f\left(x_n^{(i-s)}\right) + f\left(x_n^{(i+s)}\right) \right],$$

- normalization condition

$$\sum_{k=1}^{N} \sigma(|\vec{r}_j - \vec{r}|) = 1 \Rightarrow \kappa(\gamma) = 2 \sum_{s=1}^{N'} e^{-\gamma s},$$

- periodic boundary conditions: $x_n^{(i \pm N)} = x_n^{(i)}$

Lyapunov spectrum

C. Anteneodo *et al.*, Phys. Lett. A **326**, 277 (2004)

- Lyapunov exponents of a coupled map lattice: $\lambda_k = \ln \Lambda_k$, where $\{\Lambda_k\}_{k=1}^{N}$ are the eigenvalues of the matrix

$$\hat{\boldsymbol{\Lambda}} = \lim_{n\to\infty} \left(\boldsymbol{\tau}_n^T \boldsymbol{\tau}_n \right)^{1/2n},$$

- ordered product of the jacobian matrices

$$\boldsymbol{\tau}_n = \mathbf{T}_{n-1}\mathbf{T}_{n-2}\ldots\mathbf{T}_1\mathbf{T}_0, \qquad T_n^{(ij)} = \frac{\partial x_{n+1}^{(i)}}{\partial x_n^{(j)}}$$

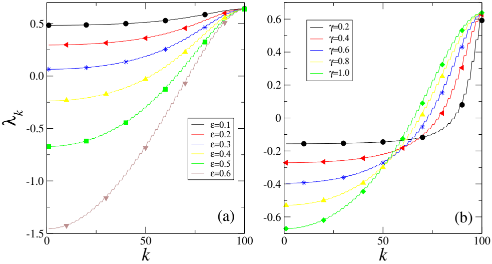
- for the chemical coupling in one spatial dimension

$$T_n^{(ik)} = (1-\varepsilon) f'\left(x_n^{(i)}\right)\delta_{ik} + \frac{\varepsilon}{\kappa(\gamma)} \exp\left(-\gamma r_{ik}\right) f'\left(x_n^{(k)}\right)(1-\delta_{ik}),$$

- where $r_{ij} = \min_\ell |i - j + \ell N|$

Lyapunov spectrum of coupled Bernoulli maps

R. Viana *et al.*, Nonlinear Dynamics **87**, 1589 (2017)



- $f(x) = \beta x$, mod 1, strongly chaotic for $\beta > 1$
- Lyapunov spectrum

$$\lambda_k = \ln \beta + \ln \left| (1-\varepsilon) + \frac{\varepsilon}{\kappa(\gamma)} b_k \right|$$

- periodic boundary conditions → jacobian matrices are circulant

$$b_k = 2 \sum_{m=1}^{N'} e^{-\gamma m} \cos \left(\frac{2\pi k m}{N} \right)$$

Stability of a completely synchronized state

C. Anteneodo *et al.*, Phys. Rev. E **68**, 045202 (2003)

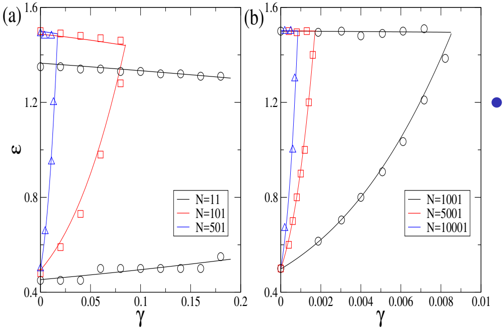
- $x_n^{(1)} = x_n^{(2)} = \ldots = x_n^{(N)} = x_n^*$: defines a synchronization manifold $\mathcal{S}$ in the phase space
- Lyapunov spectrum of the completely synchronized state

$$\lambda_k^* = \lambda_U + \ln \left| (1-\varepsilon) + \frac{2\varepsilon}{\kappa(\gamma)} \sum_{m=1}^{N'} e^{-\gamma m} \cos\left(\frac{2\pi km}{N}\right) \right|$$

- λ_U: Lyapunov exponent of the uncoupled map
- the completely synchronized state is transversely stable if $\lambda_2^* \leq 0$, such that $\varepsilon_c \leq \varepsilon \leq \varepsilon_c'$, where

$$\varepsilon_c = \left(1 - e^{-\lambda_U}\right)\left(1 - \frac{b_1}{\kappa(\gamma)}\right)^{-1}, \varepsilon_c' = \left(1 + e^{-\lambda_U}\right)\left(1 - \frac{b_{N'}}{\kappa(\gamma)}\right)^{-1}$$

Synchronization of coupled Ulam maps



- Ulam map
  $f(x) = 4x(1-x)$, $x \in [0,1)$,
  strongly chaotic ($\lambda_U = \ln 2$)
- completely synchronized state is transversely stable if $\varepsilon_c \leq \varepsilon \leq \varepsilon'_c$,

$$\varepsilon_c = \frac{1}{2\Delta}, \varepsilon'_c = \frac{3}{2\Delta'}$$

$$\Delta = 1 - \frac{b_1}{\kappa(\gamma)}, \Delta' = 1 - \frac{b_{N'}}{\kappa(\gamma)}$$

Conclusions

- oscillator coupling mediated by a diffusing substance reduces, in the one-dimensional case, to an exponentially decaying (non-local) coupling (in the adiabatic limit)
- it allows to pass from a global (all-to-all) to a local (laplacian) coupling by varying a single parameter
- frequency and phase synchronization were analyzed in terms of the coupling parameter plane (strength vs. range)
- transition to frequency and phase synchronization in terms of both K and γ
- recurrence quantification analysis used to characterize phase and frequency spatial patterns
- Lyapunov spectrum can be obtained and it is possible to study the transversal stability of the completely synchronized state

Future work

- in many situations the timescale of diffusion can be compared with the oscillator main period → adiabatic approximation no longer holds!
- Kuramoto model for chemical coupling can be generalized for time-dependent interaction kernels
- R. L. Viana and R. P. Aristides: "Nonlocal coupling among oscillators mediated by a slowly diffusing substance", arXiv:1803.10795 [nlin.AO]
- we can include motion of the oscillators → three timescales (oscillator, diffusion and motion): "active molecular dynamics" → challenges for computer simulations
- chemotaxis: oscillator motion depends on chemotactic forces, which are proportional to the local gradient of the concentration $\nabla A(\mathbf{r}, t)$

Thank you very much