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Jet quenching of the heavy quarks in the quark-gluon plasma and the nonadditive statistics

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ABSTRACT

Using the Plastino-Plastino (PP) equation, we calculate transport coefficients of the heavy-quarks traversing inside the quark-gluon plasma, and generalize their relationship with differential energy loss. The PP equation indicates anomalous diffusion of the probe particles and yields a quasi-exponential stationary distribution obtained also from the nonadditive statistics proposed by C. Tsallis. We estimate energy loss in a nonadditive quark-gluon medium, and calculate the jet quenching parameter ($\hat{q}$) for the PP dynamics. With the help of the estimate of $\hat{q}$, we calculate the nuclear suppression factor (R_{AA}) of the heavy-quarks passing through a nonadditive quark-gluon plasma using the model proposed by Dokshitzer and Kharzeev. In many a case, the parameters in the analysis are fixed from the experimental results to minimize arbitrariness. There is a good agreement between the theoretical calculation and experimental R_{AA} data, indicating that fast heavy-quarks may be subjected to anomalous diffusion inside the QGP.

1. Introduction

Transport of high-energy particles inside the hot and dense quark-gluon plasma (QGP) medium created in high-energy collisions (e.g. Pb-Pb) is an interesting and active field of research. One of the standard ways to study this phenomenon is to utilize the Boltzmann transport equation (BTE), or its approximations (e.g. the Fokker-Planck equation), that dictate the evolution of a non-equilibrium distribution (say that of the heavy quarks) inside the QGP medium of the light quarks (LQ), and gluons (g). Heavy-quarks act as 'efficient' probes because they are produced very early before the formation of the QGP, they witness the whole evolution of the QGP (thermalization to freeze-out), and they act as particles external to the QGP medium that is formed out of light-quarks and gluons. So, the heavy-quark dynamics is interesting and studied extensively in the literature. Important inputs to the transport equations are the transport coefficients like drag and diffusion coefficients that are also related to the energy loss inside the medium. Transport equations are solved using these inputs, and the solutions are used to establish a connection with the experimental observables like the nuclear suppression factor.

Transport coefficients are also important inputs to establish the stationary distributions of the transport equations [1]. Ref. [1] shows that using the perturbative QCD estimates of the drag and the diffusion coefficients of the heavy quarks (HQ), the stationary solution of the Fokker-Planck equation (that the authors initially assumed to belong to a broad class characterized by some parameters) is a quasi-exponential distribution arising from the nonadditive statistics proposed by C. Tsallis [2]. It was also shown in other works that without assuming a mathematical form of the stationary distribution, it is possible to generalize the Boltzmann transport equation [3–5] so that the same quasi-exponential function is obtained as the stationary solution in a more straightforward way.

Strongly interacting plasmas like the QGP can be characterized by the plasma parameter Γ, defined by a ratio of the average potential to kinetic energy, having a value greater than 1 [6]. In such cases, the interaction length is much greater than the Debye screening length, and the medium follows a generalized kinetic equation (like the generalized Boltzmann transport equation) that leads to equilibration. In such finite-sized systems the

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'molecular chaos hypothesis' may not be valid [7], and an interplay between the probe particle and the medium particle leads to a quasi-exponential stationary state obtained from the nonadditive statistics.

The nonadditive statistics also arises in systems with a fluctuating ambiance [8,9]. Imprints of nonadditivity are found in the experimental data obtained in high-energy collision experiments, for example, where the particle spectra can be described in terms of the following quasi-exponential function [10,11],

$$\left(1 + \frac{q-1}{T}E\right)^{-\frac{1}{q-1}},\tag{1}$$

that owes its origin to the Tsallis statistics and converges to the exponential Boltzmann-Gibbs distribution in the limit $q \to 1$. In the above equation $E = \sqrt{P^2 + m^2}$ ($P \equiv |\vec{P}|$) is the single particle energy of a particle of the mass m moving with the three-momentum $\vec{P}$, q is the entropic parameter, and T is the temperature parameter. It can be shown that the q parameter is connected to the relative variance in (inverse) temperature [8]. Also, scaling properties of the Yang-Mills theory show that q can be deduced from the field theory parameters like N_c (no. of colours) and N_f (no. of flavours) [12]. The q parameter is utilized to generalize the molecular chaos hypothesis (stosszahlansatz) used in the conventional Boltzmann transport equation. The generalized stosszahlansatz appears in the generalized Boltzmann transport equation (gBTE) for a non-equilibrium distribution f given below (see, e.g., [4]),

$$P^\mu \partial_\mu f^q = C_q[f] \quad (\mu = 0, 1, 2, 3).\tag{2}$$

It is noteworthy that the power index q on the left hand side ensures energy-momentum conservation [4], and the generalized molecular chaos is convoluted inside the collision term C_q that dictates the evolution of the distribution f. It is also possible to establish a connection between the quantity $\partial_\mu f^q$ and the fractal derivatives. In the fractal approach, the Boltzmann equation,

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_r f + \vec{F} \cdot \vec{\nabla}_p f = C[\vec{p}(t+dt), \vec{p}(t)],\tag{3}$$

is modified to incorporate the fractal momentum space by imposing [13]

$$C[\vec{p}(t+dt), \vec{p}(t)] = D_t^{\alpha_q} f,\tag{4}$$

where $D_t^{\alpha_q} f$ is the fractal derivative defined by Parvate and Gangal [14]. An appropriate continuous approximation of the fractal derivative leads to the q-deformed derivative [15,16],

$$D_t^{\alpha_q} f = f^{q-1} \frac{df}{dt},\tag{5}$$

where $\alpha_q - 1 = 1 - q$. The relaxation time approximation of the BTE yields

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_r f + \vec{F} \cdot \vec{\nabla}_p f = -\frac{f - f_0}{\tau},\tag{6}$$

where τ is the relaxation time and f_0 is the stationary state distribution. The approximation is valid for states near the stationary one. The term in the r.h.s. of the equation above is a finite difference approximation of the derivative df/dt near the stationary state. By considering the time-momentum space is fractal, this derivative must be substituted by the fractal derivative operator or, in the appropriate continuous approximation, the substitution

$$\frac{df_q}{dt} \to f_q^{q-1} \frac{df_q}{dt} \sim \frac{df_q^q}{dt},\tag{7}$$

where f_q is a distribution characterized by the q parameter. Considering that the l.h.s. of Eq. (6) is the time derivative of the distribution f, we get

$$\frac{df}{dt} = \frac{df_q^q}{dt}.\tag{8}$$

A general solution for this equation can be search with the ansatz

$$f(\vec{p}, t) = \sum_{q'} f_{q'}(\vec{p}, t).\tag{9}$$

However, for systems associated with Tsallis Statistics, it is usually found that the system is characterized by a single and well defined value q, so $f \to f_q$. Due to Eq. (6), this condition is satisfied only if $f = f_q^q$. With this result, Eq. (6) needs to be rewritten as

$$\frac{\partial f_q^q}{\partial t} + \vec{v} \cdot \vec{\nabla}_r f + \vec{F} \cdot \vec{\nabla}_p f = -\frac{f_q - f_0}{\tau},\tag{10}$$

which is the equation utilized in Ref. [17].

The Boltzmann transport equation can be subjected to different approximations like the relaxation time approximation considered above. Recently an iterative scheme has been developed to obtain approximate analytical solutions of the generalized Boltzmann transport equation [17]. Although the relaxation time approximation of the collision term is a useful first approximation for studying transport phenomena, its assumptions may be valid only for restrictive scenarios [18]. So, attempts have been made to go beyond the relaxation time approximation [19]. One may also utilize the Landau kinetic approximation [20] that observes that the most of the quark-gluon scattering is soft. In this approximation, it is possible to obtain the Fokker-Planck (FP) equation, and the Plastino-Plastino (PP) equation from the BTE, and the gBTE respectively [16,17,21].

In this work, we investigate the energy loss of the heavy quarks in the quark-gluon plasma medium dictated by the Plastino-Plastino dynamics involving anomalous diffusion of the heavy-quarks. As far as our knowledge goes, following findings and analyses are being reported for the first time in our paper:

1. A generalized estimation of drag and diffusion transport coefficients relaxing the small momentum approximation of the incoming heavy-quarks has been provided.

2. We generalize the Walton-Rafelski formula [1] connecting differential energy loss (-dE/dr), drag, and diffusion coefficients in Eq. (24), and estimate energy loss per unit length by the heavy quarks traversing through a nonadditive QGP medium.

3. We compute the jet quenching transport coefficient $\hat{q}$ (with the help of Eq. (30) obtained in this paper) that is connected to the transverse diffusion coefficient, and provide an estimate of the nuclear suppression factor using the Dokshitzer-Kharzeev (DK) formula [22] and compare the results with experimental data.

4. Using the DK formula, we also provide an analytical result connecting nuclear suppression factor (R_{AA}) and differential energy loss (-dE/dr) in Eq. (45).

A comparative study of the Fokker-Planck, and the Plastino-Plastino dynamics of the heavy quarks have been performed in a recent article [23]. However, in this work transport coefficients are estimated following Ref. [21] for small incoming momentum of the heavy quarks, and the present work generalizes those estimates. Some estimates of the nuclear suppression factor with the nonextensive initial distributions have been the subject matter in some recent works [17,24] that consider the relaxation time approximation. The present work generalizes that framework and the estimates by considering the microscopic scenario of the parton-parton interaction.

The paper is organized as follows: in the next section we provide the formalism to estimate differential energy loss of the heavy quarks from the Plastino-Plastino (PP) transport coefficients. In Section 3, we compute the jet quenching transport coefficient ($\hat{q}$), and provide an estimate of the nuclear suppression factor in the Plastino-Plastino framework. Section 4 contains analytical results establishing a relationship among nuclear suppression factor (R_{AA}), jet quenching parameter ($\hat{q}$), and differential energy loss. Section 5 contains a summary of our work, conclusions, and discussion.

2. Differential energy loss, and the Plastino-Plastino transport coefficients

2.1. PP drag and diffusion transport coefficients

In this section, we shall derive an extension of the Walton-Rafelski relationship connecting the transport coefficients, and energy loss. However, before that, we review how the transport coefficients like drag, and diffusion can be evaluated from the Plastino-Plastino equation. First, we expand the l.h.s. of the gBTE given by Eq. (2) assuming absence of an external force, and obtain the following,

$$\frac{\partial f^q(\vec{P}, \vec{r}, t)}{\partial t} + \vec{v}.\vec{\nabla}_r f^q(\vec{P}, \vec{r}, t) = E^{-1} C_q[f(\vec{P}, \vec{r}, t)], \tag{11}$$

where f is a non-equilibrium distribution (of the heavy-quarks) dependent on three-momentum $\vec{P} \equiv \{\vec{P}_{\mathrm{T}}, P_{\mathrm{L}}\}$ (T: transverse; L: longitudinal), position $\vec{r}$, and time t. For a relativistic system, energy of the heavy-quark of the mass m is given by, $E = \sqrt{P^2 + m^2}$, and its velocity is given by $\vec{v} = E^{-1}\vec{P}$.

Now, we assume that the scale of spatial variations along the longitudinal direction is sufficiently large, and the initial motion is one-dimensional along the x-direction. By using the Bjorken's assumption of longitudinal boost-invariance (at a time t, conditions at point x are the same as those at point $x = 0$) at the central rapidity region [25,26], f obeys

$$f(\vec{P}_{\mathrm{T}}, P_x, x, t) = f(\vec{P}_{\mathrm{T}}, P'_x, t), \tag{12}$$

where $P'_x = (P_x - u_x E)\gamma$, $u_x = x/t$, and $\gamma = (1 - u_x^2)^{-1/2}$. Noting that at $x = 0$, $\partial P'_x/\partial x = -E/t$, we find,

$$v_x \frac{\partial f^q}{\partial x} = -\frac{P_x}{t} \frac{\partial f^q}{\partial P_x}. \tag{13}$$

Hence, it is possible to write the gBTE for the heavy-quark distributions in the following form [17],

$$\left(\frac{\partial}{\partial t} - \frac{P_x}{t} \frac{\partial}{\partial P_x}\right) f^q(\vec{P}_{\mathrm{T}}, P_x, t) = E^{-1} C_q[f(\vec{P}_{\mathrm{T}}, P_x, t)]. \tag{14}$$

For the binary collisions of particles with four-momenta $P^\mu \equiv (E, \vec{P})$ (having the momentum distribution f), and $Q^\mu \equiv (\mathcal{E}, \vec{Q})$ (having the momentum distribution g), that change to $P'^\mu \equiv (E', \vec{P}')$, and $Q'^\mu \equiv (\mathcal{E}', \vec{Q}')$ after scattering, the collision term of the gBTE can be written as [4,5],

$$C_q = \frac{1}{2} \int \frac{d^3\vec{Q}}{(2\pi)^3 2\mathcal{E}} \frac{d^3\vec{Q}'}{(2\pi)^3 2\mathcal{E}'} \frac{d^3\vec{P}'}{(2\pi)^3 2E'} |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta^4(P + Q - P' - Q') \left[h_q\{f(\vec{P}'), g(\vec{Q}')\} - h_q\{f(\vec{P}), g(\vec{Q})\}\right]. \tag{15}$$

$|\overline{\mathcal{M}}|^2$ is the square-averaged amplitude of the heavy quark collisional processes (with the light quarks, and gluons) in a quark-gluon plasma medium given by Ref. [27]. The function h_q provides an ansatz for the generalization of the molecular chaos hypothesis given by,

$$h_q\{f, g\} = \exp_q\left[\log_q(f) + \log_q(g)\right], \tag{16}$$

where

$$\log_q f = \frac{1 - f^{1-q}}{q - 1}. \tag{17}$$

Here $\log_q$ is the q-logarithm function and becomes the conventional logarithm in the limit $q \to 1$, which also implies,

$$\lim_{q \to 1} h_q\{f, g\} = \lim_{q \to 1} \exp_q\left[\log_q(f) + \log_q(g)\right]$$
$$= \exp\left[\log(f) + \log(g)\right]$$

$$\equiv f \times g, \tag{18}$$

i.e., the distribution functions are independent, which is a consequence of the molecular chaos hypothesis. In this limit, one gets back the collision term of the conventional BTE [21],

$$C = \frac{1}{2} \int \frac{d^3\vec{Q}}{(2\pi)^3 2\mathcal{E}} \frac{d^3\vec{Q}'}{(2\pi)^3 2\mathcal{E}'} \frac{d^3\vec{P}'}{(2\pi)^3 2E'} |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta^4(P+Q-P'-Q') \left[f(\vec{P}')g(\vec{Q}') - f(\vec{P})g(\vec{Q}) \right]. \tag{19}$$

Using the Landau kinetic approximation, it is possible to obtain [17] the Plastino-Plastino equation from Eq. (14):

$$\left. \frac{\partial f}{\partial t} \right|_{P_x t} = -\frac{\partial}{\partial P_i} \left(A_i^{\mathrm{PP}} f \right) + \frac{\partial}{\partial P_i \partial P_j} \left(B_{ij}^{\mathrm{PP}} f^{2-q} \right) \qquad (i,j = 1,2,3). \tag{20}$$

B_{ij}^{PP}, and A_i^{PP}, the Plastino-Plastino diffusion and drag coefficients, are given by

$$B_{ij}^{\mathrm{PP}} = \frac{f^{q-1}}{2} \left[\frac{1}{2Eq} \int \frac{d^3\vec{Q}}{(2\pi)^3 2\mathcal{E}} \frac{d^3\vec{Q}'}{(2\pi)^3 2\mathcal{E}'} \frac{d^3\vec{P}'}{(2\pi)^3 2E'} |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta^4(P+Q-P'-Q') f^{1-q} S_{\vec{P},\vec{Q}} K_i K_j \right]$$
$$\equiv \frac{f^{q-1}}{2} \langle\langle K_i K_j \rangle\rangle^{PP}, \tag{21a}$$

$$A_i^{\mathrm{PP}} = \frac{1}{2Eq} \int \frac{d^3\vec{Q}}{(2\pi)^3 2\mathcal{E}} \frac{d^3\vec{Q}'}{(2\pi)^3 2\mathcal{E}'} \frac{d^3\vec{P}'}{(2\pi)^3 2E'} |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta^4(P+Q-P'-Q') f^{1-q} S_{\vec{P},\vec{Q}} K_i \equiv \langle\langle K_i \rangle\rangle^{PP}, \tag{21b}$$

where the 3-momentum transfer is $K_i \equiv (P - P')_i$ and the factor $S_{\vec{P},\vec{Q}}$, given by

$$S_{\vec{P},\vec{Q}} = \left[1 + \frac{\left(1 + \delta q_b \frac{|\vec{Q}|}{T_b} \right)^{\frac{\delta q}{\delta q_b}}}{\left(1 + \delta q_h \frac{E}{T} \right)^{\frac{\delta q}{\delta q_h}}} - \frac{1}{\left(1 + \delta q_h \frac{E}{T} \right)^{\frac{\delta q}{\delta q_h}}} \right]^{-\frac{1}{\delta q}}, \tag{22}$$

represents the interplay (characterized by $\delta q \equiv q - 1$) between the incoming heavy quark distribution (temperature T, entropic parameter $\delta q_h \equiv q_h - 1$), and the medium distribution (temperature T_b, entropic parameter $\delta q_b \equiv q_b - 1$). Both the distributions are parameterized by a function similar to Eq. (1) and for all the calculations in the paper we use $q_h = q_b = 1.10$, $q = 1.002$, $T_h = 0.4$ GeV, and $T_b = 0.2$ GeV. The factor $S_{\vec{P},\vec{Q}}$ arises due to generalized molecular chaos in the collision term, and converges to the exponential Maxwell-Boltzmann medium distribution (hence giving rise to molecular chaos) when q, q_h, and q_b approach 1.

2.2. Differential energy loss

Utilizing the notation used in Eq. (21a), differential energy loss of the heavy quarks in an environment dictated by the non-additive statistics can be written as below following Ref. [28]:

$$-\frac{dE}{dr} = \left\langle\!\!\left\langle \frac{E - E'}{v} \right\rangle\!\!\right\rangle^{PP} = \left\langle\!\!\left\langle \frac{E^2 - EE'}{P} \right\rangle\!\!\right\rangle^{PP}. \tag{23}$$

By defining $2B_{00}^{\mathrm{PP}} \equiv f^{q-1} \langle\langle (E - E')^2 \rangle\rangle^{PP}$, and from Eqs. (21a), and (21b) it is possible to establish a relationship between $-dE/dr$, and the heavy quark transport coefficients as shown below:

$$-\frac{dE}{dr} = \left\langle\!\!\left\langle \frac{E^2 - EE'}{P} \right\rangle\!\!\right\rangle^{PP} = \left\langle\!\!\left\langle \frac{P^2 + m^2 - EE' - \vec{P}\cdot\vec{P}' + \vec{P}\cdot\vec{P}'}{P} \right\rangle\!\!\right\rangle^{PP}$$
$$= \left\langle\!\!\left\langle \frac{1}{P} \left[P^2 - \vec{P}\cdot\vec{P}' + \frac{1}{2}\left\{ E^2 - P^2 + E'^2 - P'^2 - 2EE' + 2\vec{P}\cdot\vec{P}' \right\} \right] \right\rangle\!\!\right\rangle^{PP}$$
$$= \left\langle\!\!\left\langle \left[\frac{P_i K_i}{P} + \frac{1}{2P}\left\{ (E - E')^2 - (\vec{P} - \vec{P}')^2 \right\} \right] \right\rangle\!\!\right\rangle^{PP}$$
$$= \frac{P_i}{P} \langle\langle K_i \rangle\rangle^{PP} + \frac{1}{2P} \langle\langle (E - E')^2 \rangle\rangle^{PP} - \frac{1}{2P} \langle\!\langle (\vec{P} - \vec{P}')^2 \rangle\!\rangle^{PP}$$
$$= \frac{P_i A_i^{\mathrm{PP}} + f^{1-q} B_{00}^{\mathrm{PP}} - f^{1-q} B_{ii}^{\mathrm{PP}}}{P}. \tag{24}$$

In the above equation (also in Eqs. (28), and (42)), repeated indices are summed over. It is noteworthy that in the limit $q \to 1$, Eq. (24) converges to a similar equation obtained in Ref. [1] for the Boltzmann-Gibbs statistics. In the high-energy regime ($E, E' \gg m$) where small angle scattering is more dominant, the above equation can be approximated as,

$$-\frac{dE}{dr} \approx \gamma^{\mathrm{PP}} P, \tag{25}$$

where $A_i^{\mathrm{PP}} = \gamma^{\mathrm{PP}} P_i$, and $P_i P_i = P^2$. We evaluate the transport coefficients following the method elaborated in Ref. [21]. Fig. 1 compares the differential energy loss in the Plastino-Plastino, and the Fokker-Planck formalism. It shows that the Plastino-Plastino energy loss starts surpassing the Fokker-Planck counterpart at around 2 GeV momentum of the incoming charm quark.
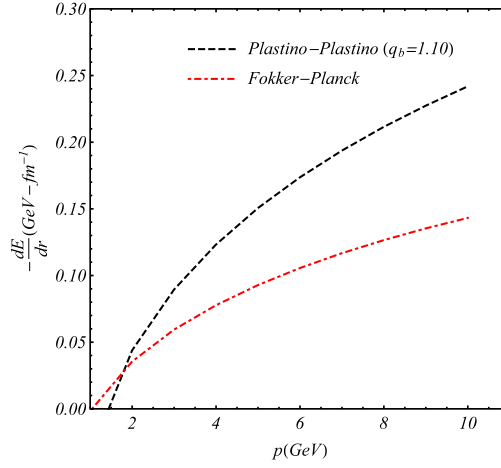
Fig. 1. Comparison of the energy loss of the charm quarks ($m = 1.5$ GeV) following the FP and PP dynamics in the QGP of $T_b = 0.2$ GeV.

Experimental evidences of energy loss in the QGP are manifest through measuring the nuclear suppression factor defined by,

$$R_{\mathrm{AA}} = \frac{\left(d^2 N / d p_{\mathrm{T}} dy\right)^{\mathrm{A+A}}}{N_{\mathrm{coll}} \times \left(d^2 N / d p_{\mathrm{T}} dy\right)^{\mathrm{p+p}}}. \tag{26}$$

N_{coll} is the number of nucleon-nucleon binary collisions (p+p) while a nucleus 'A' collides with another nucleus. $d^2 N / d p_{\mathrm{T}} dy$ denotes the differential yield within the transverse momentum and rapidity range p_{T} to $p_{\mathrm{T}} + d p_{\mathrm{T}}$ and y to $y + dy$. Theoretical estimates of R_{AA} are obtained by solving the transport equation. However, in this work, we aim to estimate R_{AA} by computing another transport coefficient $\hat{q}$, the jet quenching parameter, that yields the transverse momentum broadening per unit length. Dokshitzer and Kharzeev [22] have derived an analytical result connecting $\hat{q}$ and R_{AA}, and the rest of the article will be devoted to computing $\hat{q}$, and providing numerical estimates of R_{AA}. We also will discuss analytical results connecting the transport coefficients and nuclear suppression factor.

3. Calculating the jet quenching parameter $\hat{q}$

In the previous section, we have derived the Plastino-Plastino equation from the gBTE, and found the integral expressions for the drag and diffusion coefficients, A_i^{PP}, and B_{ij}^{PP}. The diffusion tensor B_{ij}^{PP} can be written in terms of a longitudinal and a transverse diffusion coefficient in the following way [21],

$$B_{ij}^{\mathrm{PP}} = \left(\delta_{ij} - \frac{P_i P_j}{P^2}\right) B_{\mathrm{T}}^{\mathrm{PP}}(P^2) + \frac{P_i P_j}{P^2} B_{\mathrm{L}}^{\mathrm{PP}}(P^2) \quad (P^2 \equiv |\vec{P}|^2). \tag{27}$$

Hence, the trace of the diffusion tensor (that appears in Eq. (24)) is $B_{ii}^{\mathrm{PP}} = 2 B_{\mathrm{T}}^{\mathrm{PP}} + B_{\mathrm{L}}^{\mathrm{PP}}$. From Eq. (27) we obtain,

$$
\begin{aligned}
B_{\mathrm{T}}^{\mathrm{PP}} &= \frac{1}{2} \left(\delta_{ij} - \frac{P_i P_j}{P^2}\right) B_{ij}^{\mathrm{PP}} \\
&= \frac{f^{q-1}}{4} \left(\delta_{ij} - \frac{P_i P_j}{P^2}\right) \langle\langle K_i K_j \rangle\rangle^{PP} \\
&= \frac{f^{q-1}}{4} \left\langle\!\!\left\langle K^2 - \frac{(\vec{P} \cdot \vec{K})^2}{P^2} \right\rangle\!\!\right\rangle^{PP} \\
&= \frac{f^{q-1}}{4} \langle\langle K_{\mathrm{T}}^2 \rangle\rangle^{PP},
\end{aligned}
\tag{28}
$$

where $\vec{K} \equiv (\vec{K}_{\mathrm{T}}, K_{\mathrm{L}}) \equiv (\vec{K}_{\mathrm{T}}, \vec{K} \cdot \vec{P}/P)$. Further, $\langle\langle K_{\mathrm{T}}^2 \rangle\rangle$, that gives transverse momentum broadening per unit time, can be connected to the jet quenching coefficients $\hat{q}$ that estimates average transverse momentum broadening per unit length in the following way [29]:

$$\hat{q} = \frac{\langle\langle K_{\mathrm{T}}^2 \rangle\rangle}{v_{\mathrm{L}}} = \frac{E}{P_{\mathrm{L}}} \langle\langle K_{\mathrm{T}}^2 \rangle\rangle \quad (v_{\mathrm{L}} : \text{longitudinal velocity}). \tag{29}$$

Hence, from Eq. (28),

$$\hat{q}^{\mathrm{PP}} = 4 f^{1-q} \frac{E}{P_{\mathrm{L}}} B_{\mathrm{T}}^{\mathrm{PP}}. \tag{30}$$

It is noteworthy that, in the limit $q \to 1$, a similar relationship between Fokker-Planck $\hat{q}$, and B_{T} has been derived in Ref. [30] for the kinematic region $P_{\mathrm{L}} \gg m, P_{\mathrm{T}}$ that implies $\hat{q} \approx 4 B_{\mathrm{T}}$. The jet quenching transport coefficient $\hat{q}$ is connected to the nuclear suppression factor R_{AA} that is an experimental observable. So, with our estimates of $\hat{q}$, we are also able to provide an estimate of R_{AA} in a nonadditive medium. In particular, it will be interesting to see what effect the entropic q parameter (that parameterizes the generalized molecular chaos) has on heavy-quark suppression. An

analytical formula relating $\hat{q}$ and the nuclear suppression factor $R_{\mathrm{AA}}^{\mathrm{h}}$ for the heavy quarks (denoted by 'h') has been used in Ref. [31], and derived in Ref. [22]. This formula is given by (for a path length ℓ, strong coupling α_s, and colour factor C_F),

$$R_{\mathrm{AA}}^{\mathrm{h}}(P_{\mathrm{T}}, \ell) \simeq \exp\left[-\frac{2\alpha_s C_F}{\sqrt{\pi}} \ell \sqrt{\frac{\hat{q}\mathcal{L}_h^{\mathrm{abs}}}{P_{\mathrm{T}}}} + \frac{16\alpha_s C_F}{9\sqrt{3}} \ell \left(\frac{\hat{q}m^2}{m^2 + P_{\mathrm{T}}^2} \right)^{1/3} \right]. \tag{31}$$

However, this formula for $R_{\mathrm{AA}}^{\mathrm{h}}$ is derived considering radiative processes. Hence, we need to calculate the radiative transport coefficients to be discussed in the following section, along with the evaluation of the quantity $\mathcal{L}_h^{\mathrm{abs}}$.

3.1. Estimating the jet quenching parameter $\hat{q}$: collisional and radiative

According to Eq. (30), $\hat{q}^{\mathrm{PP}} = 4(E/P_{\mathrm{L}})f^{1-q}B_{\mathrm{T}}^{\mathrm{PP}}$, and is given by the following integration in the PP case for the collisional (e.g. HQ+LQ/g $\rightarrow$ HQ+LQ/g) processes.

$$\hat{q}_{\mathrm{coll}}^{\mathrm{PP}} = \frac{1}{2qP_{\mathrm{L}}} \int \frac{d^3\vec{Q}}{(2\pi)^3 2\mathcal{E}} \frac{d^3\vec{Q}'}{(2\pi)^3 2\mathcal{E}'} \frac{d^3\vec{P}'}{(2\pi)^3 2E'} |\overline{\mathcal{M}}|_{2\to2}^2 (2\pi)^4 \delta^4(P+Q-P'-Q') S_{\vec{P},\vec{Q}} \left(P'^2 - \frac{(\vec{P}.\vec{P}')^2}{P^2} \right). \tag{32}$$

For the radiative processes (e.g. HQ+LQ/g $\rightarrow$ HQ+LQ/g+g), we follow Refs. [30,32] to calculate the radiative transport coefficients from the collisional ones. To be consistent with Eq. (31), we consider only collinear, and soft gluon (4-momentum $k_5^\mu \equiv \{E_5, \vec{k}_5\}$) radiation without the Bose enhancement factor in the present calculation. These considerations lead to the following estimate of the radiative jet transport coefficient,

$$\hat{q}_{\mathrm{rad}}^{\mathrm{PP}} = \frac{1}{2qP_{\mathrm{L}}} \int \frac{d^3\vec{Q}}{(2\pi)^3 2\mathcal{E}} \frac{d^3\vec{Q}'}{(2\pi)^3 2\mathcal{E}'} \frac{d^3\vec{P}'}{(2\pi)^3 2E'} \frac{d^3\vec{k}_5}{(2\pi)^3 2E_5} (2\pi)^4 \delta^4(P+Q-P'-Q'-k_5) S_{\vec{P},\vec{Q}} \left(P'^2 - \frac{(\vec{P}.\vec{P}')^2}{P^2} \right)$$
$$\times |\overline{\mathcal{M}}|_{2\to3}^2 \theta_1 \left(E - E_5 \right) \theta_2 \left(\tau - \tau_{\mathrm{F}} \right), \tag{33}$$

where the first θ-function prohibits gluon radiation of energy greater than that of the incoming heavy quark, and the second θ-function accounts for the additive region of the gluon spectra due to multiple scattering in which the interaction time (τ) is greater than the formation time (τ_{F}) of the gluons [33]. These two θ-functions put an upper-bound and a lower-bound respectively in the gluon 3-momentum ($\vec{k}_5 \equiv \{\vec{k}_{\mathrm{T}}, k_{\mathrm{L}}\}$) integration. The radiative matrix element for the soft, collinear gluon emission at an angle θ_g can be written in the following way [22] (for $g_s^2 = 4\pi\alpha_s$; $\alpha_s \equiv$ strong coupling),

$$|\overline{\mathcal{M}}|_{2\to3}^2 \approx |\overline{\mathcal{M}}|_{2\to2}^2 \times \frac{12g_s^2}{k_{\mathrm{T}}^2} \left(1 + \frac{m^2}{E^2\theta_g^2} \right)^{-2}. \tag{34}$$

Also, parameterizing the soft gluon 4-momentum in terms of its rapidity (η), we get,

$$\hat{q}_{\mathrm{rad}}^{\mathrm{PP}} \approx \int \text{collisional part} \times \frac{6\alpha_s}{\pi} \int d\eta \int \frac{dk_{\mathrm{T}}}{k_{\mathrm{T}}} \left(1 + \frac{m^2}{E^2\theta_g^2} \right)^{-2} \theta_1 \left(E - E_5 \right) \theta_2 \left(\tau - \tau_{\mathrm{F}} \right). \tag{35}$$

Now, using the parameterization $E_5 = k_{\mathrm{T}} \cosh\eta$, putting $\tau_{\mathrm{F}} = \cosh\eta/k_{\mathrm{T}}$, and using $\theta_g = 2\tan^{-1}\left[\exp(-\eta)\right]$,

$$\hat{q}_{\mathrm{rad}}^{\mathrm{PP}} \approx \int \text{collisional part} \times \frac{6\alpha_s}{\pi} \int d\eta \int_{\Lambda \cosh\eta}^{\frac{E}{\cosh\eta}} \frac{dk_{\mathrm{T}}}{k_{\mathrm{T}}} \left[1 + \frac{m^2}{E^2 \left(2\tan^{-1}\left[\exp(-\eta)\right] \right)^2} \right]^{-2}. \tag{36}$$

In the above integration, the interaction rate Λ is evaluated from the following integral,

$$\Lambda = \frac{1}{2qE} \int \frac{d^3\vec{Q}}{(2\pi)^3 \mathcal{E}} \frac{d^3\vec{Q}'}{(2\pi)^3 \mathcal{E}'} \frac{d^3\vec{P}'}{(2\pi)^3 E'} |\overline{\mathcal{M}}|_{2\to2}^2 (2\pi)^4 \delta^4(P+Q-P'-Q') f^{1-q} S_{\vec{P},\vec{Q}}. \tag{37}$$

In Fig. 2 we show the variation of the (Plastino-Plastino and Fokker-Planck) jet transport coefficient $\hat{q}^{\mathrm{PP/FP}} = \hat{q}_{\mathrm{coll}}^{\mathrm{PP/FP}} + \hat{q}_{\mathrm{rad}}^{\mathrm{PP/FP}}$ with the incoming HQ transverse momentum p_{T}.

3.2. Estimating $\mathcal{L}_h^{\mathrm{abs}}$ and R_{AA}, and comparison with experimental data

For evaluating $\mathcal{L}_h^{\mathrm{abs}}$, we shall follow Ref. [22] that defines

$$\mathcal{L}_h = \frac{d}{d\ln p_{\mathrm{T}}} \ln\left[\frac{d\sigma^{\mathrm{vac}}}{dp_{\mathrm{T}}^2} (p_{\mathrm{T}}) \right], \tag{38}$$

and uses the following vacuum differential cross-section for the D-mesons [34]

$$\frac{d\sigma^{\mathrm{vac}}}{dp_{\mathrm{T}}^2} = C \left(\frac{1}{bm^2 + p_{\mathrm{T}}^2} \right)^{\frac{n}{2}}, \tag{39}$$

for $b = 1.4 \pm 0.3$, $n = 10.0 \pm 1.2$, and $m = 1.5$ GeV. Hence, the quantity $\mathcal{L}_h$ can be evaluated to be,
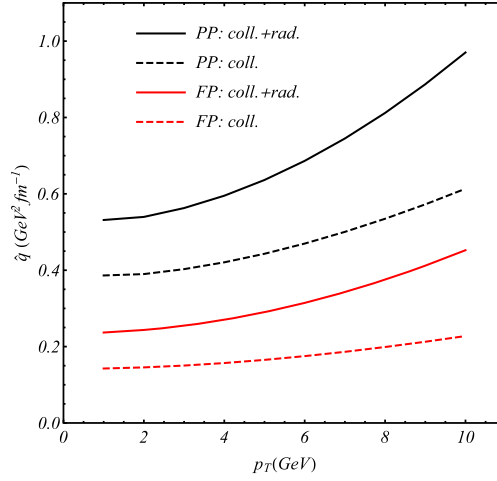
Fig. 2. Comparison of total (collisional+radiative), and collisional jet quenching parameters ($\hat{q}$) of the charm quarks ($m = 1.5$ GeV; $P_L = 10$ GeV) following the FP and PP ($q_b = 1.10$) dynamics in the QGP of $T_b = 0.2$ GeV.
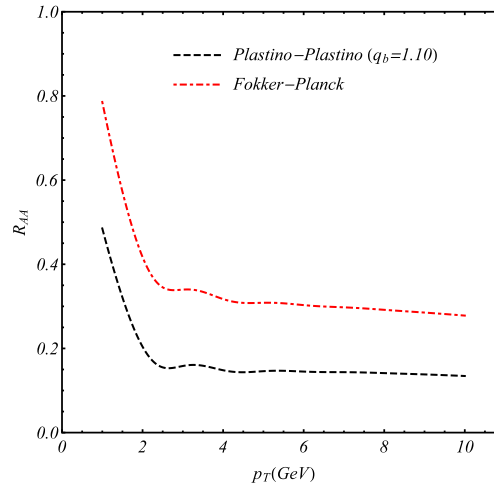

Fig. 3. Comparison of the nuclear suppression factor (R_{AA}) of the charm quarks ($m = 1.5$ GeV; $P_L = 10$ GeV) following the FP and PP dynamics in the QGP of $T_b = 0.2$ GeV calculated using Eq. (31). We put $b = 1.7$, $n = 11$, $\ell = 3$ fm.

$$\mathcal{L}_h = -\frac{np_T^2}{bm^2 + p_T^2} \Rightarrow \mathcal{L}_h^{abs} = |\mathcal{L}_h|. \tag{40}$$

We plot the variation of the charm quark nuclear suppression factor with its transverse momentum in Fig. 3.

We also compare the theoretical model calculations with the experimental data [35] of the D^0 mesons in Fig. 4. To this end, we integrate Eq. (31) with the following function [36] fragmenting heavy-quarks ('h') to the hadrons ('H') with the momentum fraction z of the partons:

$$D_h^H(z) \propto \frac{1}{z\left(1 - \frac{1}{z} - \frac{\epsilon_h}{(1-z)}\right)^2}. \tag{41}$$

4. Relationship among jet quenching parameter, differential energy loss, and nuclear suppression factor

From Eqs. (24), and (30), one can establish a relationship between $-dE/dr$, and $\hat{q}$ in the following way:

$$
\begin{aligned}
-\frac{dE}{dr} &= \frac{P_i A_i^{PP} + f^{1-q} B_{00}^{PP} - f^{1-q} B_{ii}^{PP}}{P} \\
&= \frac{P_i A_i^{PP} + f^{1-q}(B_{00}^{PP} - 2B_T^{PP} - B_L^{PP})}{P} \\
&= P\gamma^{PP} + f^{1-q}\left(\frac{B_{00}^{PP} - B_L^{PP}}{P}\right) - \frac{\hat{q}^{PP} P_L}{2PE}.
\end{aligned}
\tag{42}
$$

In the Boltzmann-Gibbs (BG) limit (in which case we simply remove the superscript 'PP'), the above equation becomes,

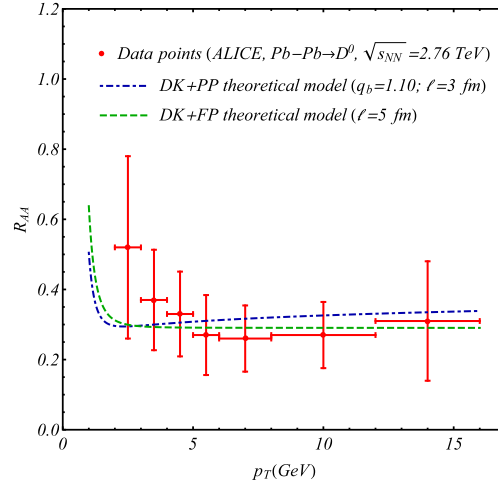$$-\frac{dE}{dr} = P\gamma + \frac{B_{00} - B_L}{P} - \frac{\hat{q} P_L}{2PE}. \tag{43}$$

Fig. 4. Comparison of nuclear suppression factor of the D^0 mesons calculated using Eqs. (31), and (41) for the PP and FP dynamics with experimental data obtained by the ALICE collaboration [35]. We put $T_b = 0.2$ GeV, $b = 1.7$, $n = 11$, and $\epsilon_h = 0.056$.

The transport coefficients γ, B_{00}, $\hat{q}$ and B_L are the conventional Fokker-Planck transport coefficients. Eq. (42) also implies,

$$\hat{q}^{\mathrm{PP}} = \frac{2PE}{P_L}\left[P\gamma^{\mathrm{PP}} + f^{1-q}\left(\frac{B_{00}^{\mathrm{PP}} - B_L^{\mathrm{PP}}}{P}\right) + \frac{dE}{dr}\right] \quad \text{(Plastino-Plastino), and}$$

$$\hat{q} = \frac{2PE}{P_L}\left(P\gamma + \frac{B_{00} - B_L}{P} + \frac{dE}{dr}\right) \quad \text{(Fokker-Planck).} \tag{44}$$

From Eq. (44), it is possible to establish a connection between nuclear suppression factor, and differential energy loss in the following way:

$$R_{\mathrm{AA}}^h(P_T, \ell) \simeq \exp\left[-\frac{2\alpha_s C_F}{\sqrt{\pi}}\ell\sqrt{\frac{\frac{2PE}{P_L}\left(P\gamma^{\mathrm{PP}} + \frac{f^{1-q}(B_{00}^{\mathrm{PP}} - B_L^{\mathrm{PP}})}{P} + \frac{dE}{dr}\right)\mathcal{L}_h^{\mathrm{abs}}}{P_T}}\right.$$

$$\left. +\frac{16\alpha_s C_F}{9\sqrt{3}}\ell\left(\frac{\frac{2PE}{P_L}\left(P\gamma^{\mathrm{PP}} + \frac{f^{1-q}(B_{00}^{\mathrm{PP}} - B_L^{\mathrm{PP}})}{P} + \frac{dE}{dr}\right)m^2}{m^2 + P_T^2}\right)^{1/3}\right], \tag{45}$$

that, in the BG limit ($q \to 1$), becomes,

$$R_{\mathrm{AA}}^h(P_T, \ell) \simeq$$

$$\exp\left[-\frac{2\alpha_s C_F}{\sqrt{\pi}}\ell\sqrt{\frac{\frac{2PE}{P_L}\left(P\gamma + \frac{B_{00} - B_L}{P} + \frac{dE}{dr}\right)\mathcal{L}_h^{\mathrm{abs}}}{P_T}} + \frac{16\alpha_s C_F}{9\sqrt{3}}\ell\left(\frac{\frac{2PE}{P_L}\left(P\gamma + \frac{B_{00} - B_L}{P} + \frac{dE}{dr}\right)m^2}{m^2 + P_T^2}\right)^{1/3}\right]. \tag{46}$$

5. Discussion and conclusion

In this work, we revisit the derivation of the Plastino-Plastino equation from a generalized Boltzmann transport equation. The index q generalizing the molecular chaos hypothesis also appears in Ref. [13] that connects the Plastino-Plastino equation to the fractal space approach. We calculate the transport coefficients like the diffusion and the jet quenching parameter of the charm quarks, and differential energy loss in a nonadditive QGP medium. We have generalized some previously used relationships (Eqs. (24), and (30)), and presented analytical formulae establishing a connection between R_{AA}, an experimental observable, and the transport coefficients using Ref. [22] (Eqs. (45), and (46)). Although the present paper considers R_{AA} data, the application of the present formalism is not limited to evaluating R_{AA} only. Our calculation also enables to estimate the elliptic flow parameter by defining path-length-dependent R_{AA} for the in-plane and out-of-plane directions [37]. Another method of estimating elliptic parameter may involve solving the Plastino-Plastino equation and utilizing the evolved distribution. We reserve the discussion on elliptic flow for a future study.

We observe that when there is an interplay between the incoming heavy-quarks and the medium particle distribution, manifested by a generalized molecular chaos, numerical values of transport coefficients, energy loss, and suppression of the heavy-quarks increase compared to their Boltzmann-Gibbs counterparts. Given that for finite systems generalized molecular chaos may be a more realistic scenario [7], our estimates in this paper will be relevant for the QGP system. The $\hat{q}$ values obtained using the PP dynamics are around 1 GeV^2-fm^{-1} which is comparable to the value obtained in Ref. [31] and less compared to the 'experimentally favoured range' for the RHIC data [38]. However, the DK formula considers soft, eikonal

and collinear gluon radiation approximation, and it will be interesting to investigate how generalizing this approximation considered in Ref. [22], following Ref. [39], will impact the estimate of $\hat{q}$, and other experimental observables. With these modifications, it can also be expected that the contribution of radiation in $\hat{q}$ will increase, more so for the PP case where $\hat{q}_{rad}$ is around 40-60% of $\hat{q}_{coll}$. A few salient points that come out of the discussion are listed below:

- Calculation of R_{AA} is based on calculating the jet transport coefficient evaluated using the Plastino-Plastino equation that takes into account anomalous diffusion of the heavy-quarks in the medium.
- In many a case, the parameters in the analysis are fixed from the experimental results to minimize arbitrariness.
- For the given parameter values, there is a good agreement between the theoretical calculation and R_{AA} data, indicating that fast heavy-quarks may be subjected to anomalous diffusion inside the QGP [40,41].

Our calculation of R_{AA} using the PP dynamics captures the trend of decreasing suppression (considering the central values) at higher transverse momenta (Fig. 4), but the FP dynamics does not show this trend. It can be intuitively understood that R_{AA} depends on ℓ that may be equalled to the average path length traversed by the heavy-quarks. This quantity ℓ can introduce a centrality dependence in our calculations. We observe that for a 0-20% central collision between nuclei with the same radius (e.g. Pb on Pb), the average path length of the heavy-quarks varies between 55% to 75% of the radius of the produced system. So, our analysis with the PP dynamics indicates that the diameter of the QGP system produced may be around 8-10 fm, but the FP dynamics indicates a larger diameter of around 13-18 fm. Nevertheless, we think that a more detailed analysis involving centrality dependent experimental data and interferometry may be required. For the lower p_T values, more involved investigation (perhaps involving collectivity) may benefit the study. This also begs a more rigorous approach to establish a connection between theory and experiment by utilizing a numerical solution of the Plastino-Plastino equation with the transport coefficients as inputs. In this extension, the collective behaviour of medium (that is not considered in the present work) should be included for a more reliable comparison with experimental data. We reserve these works for the future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data used in the analysis can be found in the cited papers.

Acknowledgements

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