

# Fractal Structure in QCD and Tsallis Statistics

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# Summary

What is a fractal

Thermofractals

Yang-Mills fields and thermofractals

Dynamic of heavy-quarks in the QGP

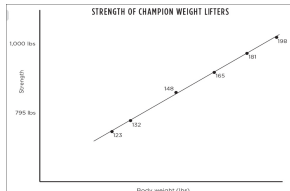
Thermofractal in hadron structure? z-Scaling.

Fractal and fractional derivatives.

# Scale and Self-Similarity



## SCALING



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## SELF-SIMILARITY

# Thermofractals

A thermofractal is a system defined by the following properties:

I- It is a system in thermodynamical equilibrium with total energy given by  $U = K + E$ , where  $K$  corresponds to the kinetic energy of  $N$  constituent subsystems and  $E$  describes the internal energy of those subsystems, which are endowed with an internal substructure.

II- The constituent particles are thermofractals which can be divided in two subclasses: Type I and Type II. For each subclass, the corresponding associated ratio,  $E/K$  or  $E/U$ , can vary according to a self-similar distribution,  $P(U)$ . This means the distribution of the internal energy is independent of the hierarchic level of the fractal structure.

III- At some level  $n$  in the hierarchy of subsystems the phase space is so narrow that one can neglect their internal structure and assume the following expression to the probability:  $P(U_n) dU_n = \rho dU_n$

$$P(\varepsilon) = A \left[ 1 + (q - 1) \frac{\varepsilon}{Nk_B\tau} \right]^{-\frac{1}{q-1}}$$

# Fractals in Yang-Mill fields

A.D., E. Megías, D.P. Menezes PRD (2020)

Complex structure of the effective parton:

Self-energy interaction:

Self-energy diagrams:

Many-loops diagrams:

Yang-Mills theory is renormalizable:  $\Gamma(p, m, g) = \lambda^{-D} \Gamma(p, \mu, \bar{g})$

F. Dyson, PR 75 (1949) 1736 Stuekelberg and Petermann, Helv. Phys. Acta 26 (1953) 499

M. Gell-Mann and F.E. Low, PR 95 (1954) 1300

Callan-Symanzik Equation

Renormalization group equation:  $\left[ M \frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial \bar{g}} + d \right] \Gamma = 0$

Effective coupling constant  $\bar{g}$

Effective mass  $\mu$

Scaling properties are present in YMF

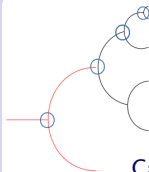
K. Symanzik, Comm. Math. Phys. 18 (1970) 227

C.G. Callan Jr., PRD 2 (1970) 1541

# Non extensivity in gauge field theory

$$P(\varepsilon) = G^n [1 - (q - 1) \frac{\varepsilon}{\lambda}]^{\frac{1}{q-1}}$$

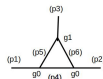
$q$  - the number of internal degrees of freedom in the fractal structure that are relevant in the process of energy transfer to the effective parton



Describes how momentum and energy are distributed at each vertex:

$$\bar{g} = \prod_i G [1 - (q - 1) \frac{\varepsilon_i}{k_T}]^{\frac{1}{q-1}} \quad \text{effective coupling}$$

Calculation of  $q$  from gauge field parameters: 1-loop approx. (QCD)



Fractal method:  $\beta_g = -\frac{1}{q-1} g^{N'+1}$

CS equation:  $\frac{1}{q-1} = d - \gamma$

$$\Rightarrow \frac{1}{q-1} = \frac{11}{3} N_c - \frac{4}{3} N_f \Rightarrow q = 1.14$$

QCD:  $d - \gamma = [\frac{11}{3} c_1 - \frac{4}{3} c_2]$

# Comparison with experiments

Extended Hagedorn theory to non extensive statistics: [AD, Physica A 391 \(2012\) 6380](#)

use of Tsallis factor: 
$$P(\varepsilon) = A[1 + (q - 1)\frac{\varepsilon}{kT}]^{-\frac{1}{q-1}}$$

[L. Marques, E. Andrade-II, AD, PRD 87 \(2013\) 114022](#)

[L. Marques, J. Cleymans, AD, PRD 91 \(2015\) 054025](#)

**Experimental value  $q = 1.14 \pm 0.01$**

Outline

Fractal

Thermofractal

Scales in  
YMtheory

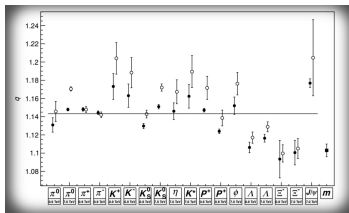
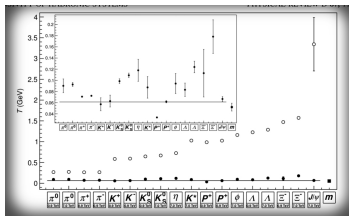
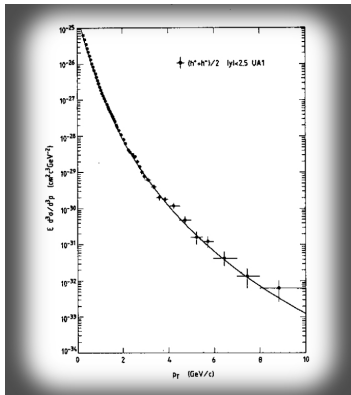
Comparison with  
experiments

Fractal structure  
of hadrons

Non-linear  
Fokker-Planck  
Equation

Fractal &  
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derivatives

Conclusions

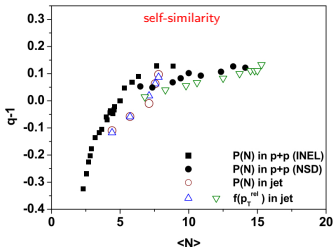


# Experimental verification

Scale invariance of gauge theory

leads to fractal structure

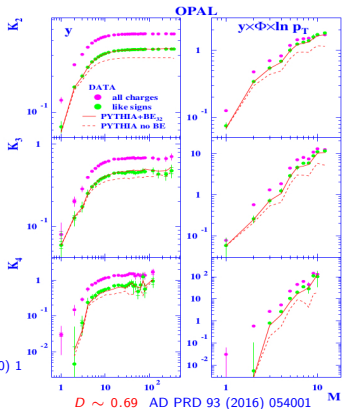
fractal dimension in multiparticle production



G. Wilk and Z. Włodarczyk, PLB 727 (2013) 163

E. Sarkisyan-Grinbaum, PLB 477 (2000) 1

fractal dimension - from intermittency analysis





# Multiplicity and energy

Multiplicity as a manifestation of fractal aspects: [AD, PRD 93 \(2016\) 054001](#)

$r$  is the scale in which energies are measured.

$\varepsilon \sim r^{-D}$  is the scaling behavior of the individual parton energy.

$E \sim r^{-1}$  is the scaling behavior of the total energy.

$\mathcal{N}$  is the multiplicity.

$R$  is the ratio between parton energy  $\varepsilon$  and its immediate parent energy

$$\mathcal{N}r^{-D} \propto r^{-1} \rightarrow \mathcal{N} \propto r^{-1+D}$$

$$\mathcal{N} \propto E^{1-D}$$

$D \sim 0.69$  from fractal dimension analysis and intermittence analysis

Theory:  $1 - D \sim 0.31$

Experiment:  $1 - D \sim 0.302$

[E. Sarkisyan-Grinbaum et al. PRD 93 \(2016\) 054046](#)

# Generalization of Hagedorn's Thermodynamics

## Hagedorn's Self-Consistent Thermodynamics

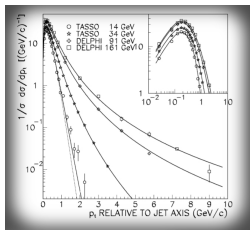
Limiting temperature  $T_H$ .

$$\text{Hadron mass spectrum } \rho(m) = \frac{a}{m^{5/2}} \exp\{m/T_H\}$$

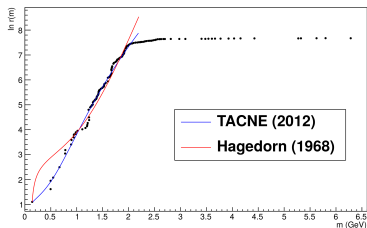
## Nonextensive Self-Consistent Thermodynamics

Limiting Temperature

$$\text{Hadron mass spectrum } \rho(m) = \frac{a}{m^{5/2}} \exp_q\{m/T_H\}$$



power-law distributions



non extensive mass spectrum

## Generalized Hagedorn Self-Consistent Thermodynamics

# Fractal structure of hadrons (in development)

Thermofractal structure of hadrons:

$$\frac{d^4\sigma}{dq^4} = |\langle \varphi\phi | g(p) | \phi_o \rangle|^2 \delta(q^2 - m_q)$$

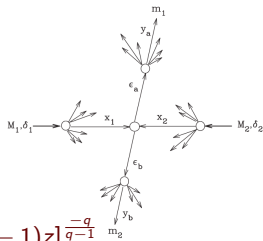
$$E \frac{d^2\sigma}{2\pi q_t dq_t dq_z} = \frac{1}{2} |\langle \varphi\phi | g(E) | \phi_o \rangle|^2 \propto e_q(\varepsilon/\lambda)^2$$

M. Tokarev, I. Zborovsky: PRD 56 (1996) 5548

$$E \frac{d^2\sigma}{dq^2} = \frac{\sigma_0}{\pi s} \frac{d^2\sigma}{dx_1 dx_2}$$

z-scaling:  $\psi(z) = \frac{d\sigma(z)}{dz}$

$$z = \frac{s_x}{Q} \Rightarrow \psi(z) = [1 + (q-1)z]^{\frac{-q}{q-1}}$$



# Dynamics of quarks in the medium

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Boltzmann Equation:

$$\frac{\partial f(\mathbf{p}, t)}{\partial t} = -\nabla_{\mathbf{p}} f(\mathbf{p}, t) \cdot \mathbf{F} + C[f]$$

$$h[f(\mathbf{p}), f(\mathbf{q})] = f(\mathbf{p})f(\mathbf{q})$$

Fokker-Planck Equation

$$\frac{\partial f}{\partial t} - \frac{\bar{\partial}}{\partial p_i} \left[ A_i f + \frac{\bar{\partial}(B_{ij} f)}{\partial p_j} \right] = 0$$

Non-Additive Boltzmann Equation:

Correlation functional

$$h[f(\mathbf{p}), f(\mathbf{q})] = [f(\mathbf{p})^{1-q} + f(\mathbf{q})^{1-q} - 1]^{\frac{1}{1-q}}$$

Plastino-Plastino Equation

$$\frac{\partial f}{\partial t} - \frac{\bar{\partial}}{\partial p_i} \left[ A_i f + \frac{\bar{\partial}(B_{ij} f^{2-q})}{\partial p_j} \right] = 0$$

Plastino-Plastino Equation was proved for the 1st time in

A.D., E. Megias, A. Golmankhaneh and R. Pasechnik in

PLB 839 (2023) 137752

*Special Derivative:*  $\partial^2 f / \partial x^2 \rightarrow \partial^2 f^{2-q} / \partial x^2$

comes from q-deformed derivative:  $\partial f / \partial x \rightarrow f^{1-q} \partial f / \partial x$

# Fractal Derivatives

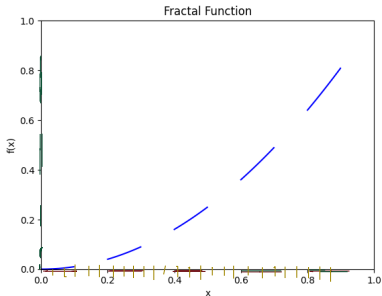
## Hausdorff Geometry:

$\delta$ -measure -  $\mathcal{H}_\delta^\alpha(\mathbb{F}) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^\alpha : \{U_i\} \text{ is a cover of } \mathbb{F} \right\}$

Hausdorff measure -  $\mathcal{H}^\alpha(\mathbb{F}) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^\alpha(\mathbb{F})$

If  $\mathbb{F}$  is a Borel set in  $\mathbb{R}^\alpha \Rightarrow \mathcal{H}^\alpha(\mathbb{F}) = c_\alpha^{-1} \text{vol}^\alpha(\mathbb{F})$

Mass distribution  $\gamma_{\mathbb{F}}^\alpha(a, b) = \mathcal{H}^\alpha([a, b])$  if  $0 < \mathcal{H}^\alpha < \infty$



# Fractal Derivatives

## Hausdorff Geometry:

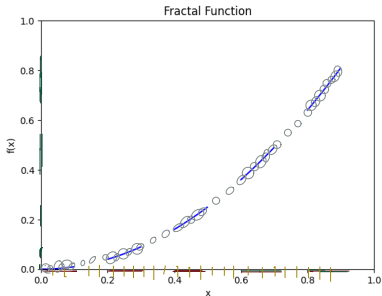
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Staircase function  $S_{\mathbb{F}, a}^\alpha(x) = \begin{cases} \gamma_{\mathbb{F}}^\alpha(a, x) & \text{if } x > a \\ -\gamma_{\mathbb{F}}^\alpha(x, a) & \text{if } x < a \end{cases}$



## FractalDerivative

$$D_{\mathbb{F}, a}^\alpha f(x_0) = \begin{cases} F \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{S_{\mathbb{F}, a}^\alpha(x) - S_{\mathbb{F}, a}^\alpha(x_0)} & \\ 0 & \text{otherwise} \end{cases}$$

# Fractal Derivatives

## Hausdorff Geometry:

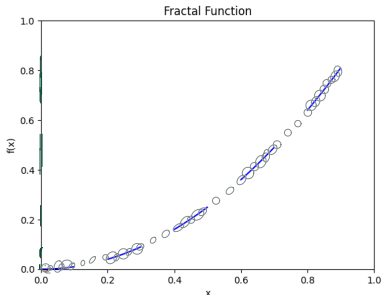
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$$D_{\mathbb{F}, \varphi}^\alpha f(x_0) = \begin{cases} F \lim_{x \rightarrow x_0} \frac{S_{\mathbb{F}, \varphi}^\alpha(f_x) - S_{\mathbb{F}, \varphi}^\alpha(f_{x_0})}{x - x_0} & \\ 0 & \text{otherwise} \end{cases}$$

# Continuous Approximations and Fractional Derivatives

Continuous Approximation:

$$dS_{F,\varphi}^\alpha(x) = S_{F,\varphi}^\alpha(x + dx) - S_{F,\varphi}^\alpha(x)$$

$dS_{F,\varphi}^\alpha(x)$  is the volume of the ball at  $x + dx$ .

$$dS_{F,\varphi}^\alpha(x) = \begin{cases} A(\alpha)x^{\alpha-1}dx & \text{if } x, x + dx \in \mathbb{F} \\ 0 & \text{otherwise} \end{cases} .$$

## Fractal Derivatives

In domain space

$$D_{conf}f(x) = x^{1-\alpha} \frac{df}{dx}(x)$$

$$D_Cf(x) = \int_{x_0}^x (x-t)^{1-\alpha} \frac{df}{dt}(t)dt$$

In image space

$$D_qf(x) = f^{1-q} \frac{df}{dx}(x), \quad \alpha = 2 - q$$

$$D_{CI}f(x) = \int_{x_0}^x [f(x) - f(t)]^{1-q} \frac{df}{dt} dt$$



# Continuous Approximations and

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Continuous Approximation:

$$dS_{F,\varphi}^\alpha(x) = S_{F,\varphi}^\alpha(x + dx) - S_{F,\varphi}^\alpha(x)$$

$dS_{F,\varphi}^\alpha(x)$  is the volume of t

$$dS_{F,\varphi}^\alpha(x) = \begin{cases} A(\alpha)x^{\alpha-1}dx & \text{if } x < x_0 \\ 0 & \text{otherwise} \end{cases}$$

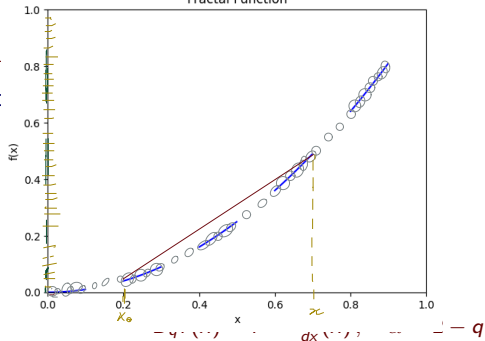
Fractal

In domain space

$$D_{conf}f(x) = x^{1-\alpha} \frac{df}{dx}(x)$$

$$D_C f(x) = \int_{x_0}^x (x-t)^{1-\alpha} \frac{df}{dt}(t) dt$$

Fractal Function



$$D_{CI}f(x) = \int_{x_0}^x [f(x) - f(t)]^{1-q} \frac{df}{dt} dt$$

## Conclusions:

Thermofractal in QCD (and any YMF theory)

Thermofractals and Tsallis Statistics

Emergence of Tsallis Statistics in QGP

Determination of  $q$  in terms of  $N_c$  and  $N_f$ .

Agreement with experimental data

Possibility of thermofractal structures in hadrons.

Dynamics of non-additive systems: heavy-quark in the medium

Fractal derivative and fractional derivatives.

$q$ -Deformed derivative as continuous approximation of fractal derivative.

Thank you