Quantum Field Theory

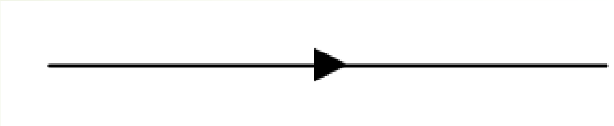
Renormalization

Feynman Rules

Fotón

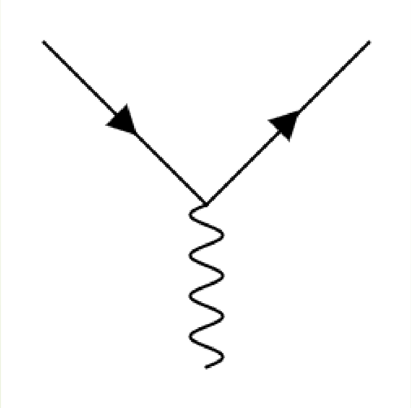
$$i\Pi^{\mu\nu}(p) = \frac{-i}{p^2 + i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

Elétron



$$iS(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

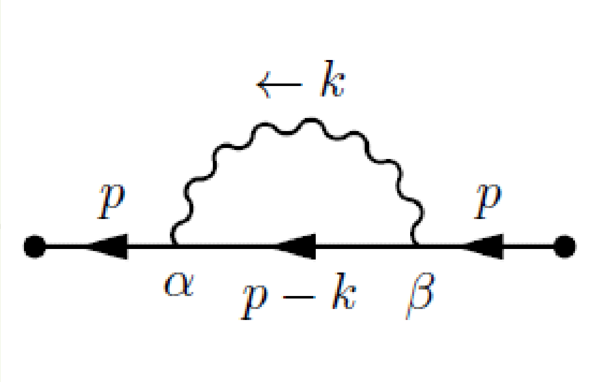
Vértice



$$i\Gamma^\mu = -ie\gamma^\mu$$

Expansion

Electron Self-Energy



$$iS_F(p)\int \frac{d^4}{(4\pi)^4}\left[ie\gamma^{\alpha} iS_F(p-k) ie\gamma^{\beta} iD^{F}_{\alpha\beta}(k)\right] iS_F(p)$$

Onde

$$-i\Sigma_2(p) = (ie)^2 \int \frac{d^4}{(2\pi)^4}\frac{\gamma^{\alpha}(\not{p}-\not{k}+m)\gamma_{\alpha}}{[(p-k)^2-m^2]k^2}$$

Essa integral diverge linearmente. Onde está uma divergência ultravioleta, mas também tem uma divergência no infravermelho.

o propagador modificado

$$iS_F(p) \rightarrow iS_F(p) + iS_F(p)\big(-i\Sigma(p)\big)iS_F(p)$$

Feynman Parameters

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B - A)x^2]^2} = \int_0^1 dxdy\, \delta(x + y - 1) \frac{1}{[xA + yB]^2}$$

$$\frac{1}{AB^n} = \int_0^1 dxdy\, \delta(x + y - 1) \frac{ny^{n-1}}{[xA + yB]^{n+1}}$$

$$\frac{1}{ABC} = \int_0^1 dxdydz\, \delta(x + y + z - 1) \frac{2}{[xA + yB + zC]^3}$$

Regularização Dimensional

A regularização dimensional consiste em modificar a dimensionalidade dessas integrais para que elas se tornem finitas.

$$\int \frac{d^d p}{(p^2 + 2pq - m^2)^\alpha} = (-1)^\alpha i \pi^{d/2} \frac{\Gamma\left(\alpha - \frac{1}{2}d\right)}{\Gamma(\alpha)} \frac{1}{(-q^2 - m^2)^{\alpha - d/2}} \quad , d = 4 - \varepsilon$$

Onde $p = (p_0, \vec{r})$. Tendo em conta

$$(p_0, r, \phi, \theta_1, \ldots, \theta_{d-3})$$

Assim que

$$d^d p = dp_0 r^{d-2} dr d\phi \prod_{k=1}^{d-3} sin^k \theta_k d\theta_k$$

itens a considerar,

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2n-1}(\cos\theta)^{2m-1} d\theta = \frac{1}{2}\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$$

Função beta de Euler

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\infty} dt\, t^{2x-1}(1+t^2)^{-x-y}$$

Para $n \to 0$ temos que,

$$s^{-\frac{n}{2}} = 1 - \frac{n}{2}\ln s$$

E

$$\Gamma\left(\frac{n}{2}\right) = \frac{n}{2} - \gamma + \cdots$$

Onde $\gamma = 0.5772$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell^2)^2}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu \ell^\rho \ell^\sigma}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2}$$

$$\times \frac{1}{4}\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

Alguns relacionamentos importantes:

$$g_{\mu\nu}g^{\mu\nu} = d$$

$$\gamma_\lambda\gamma^\lambda = dI$$

$$\gamma_\lambda\gamma^\alpha\gamma^\lambda = -(d-2)\gamma^\alpha$$

$$\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda = -(d-2)\gamma^\alpha\gamma^\beta + 4g^{\alpha\beta}$$

Também,

$$tr(\gamma^\alpha\gamma^\beta) = f(d)g^{\alpha\beta}$$

$$tr(\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\delta) = f(d)[g^{\alpha\beta}g^{\lambda\delta} - g^{\alpha\lambda}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\lambda}]$$

$$tr(\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu) = 0$$

Electron Self-Energy

$$-i\Sigma_2(p) = (ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \frac{-ig_{\mu\nu}}{k^2} \gamma^\nu$$

$$\Sigma_2(p) = -i\mu^{4-d} e^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{[(p-k)^2 - m^2]k^2}$$

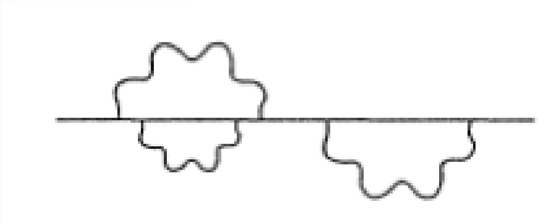
Resultado

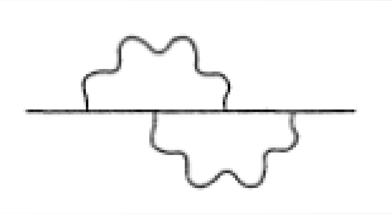
$$\Sigma_2(p) = -2\, i\mu^{4-d}e^2 \int_0^1 dx\,(2m - xp) \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 - \Delta]^2} \quad \forall \qquad \Delta = -x(1-x)p^2 + (1-x)m^2$$

$$\Sigma_2(p) = \frac{e^2}{8\pi^2\varepsilon}(-\not{p} + 4m) + finite$$

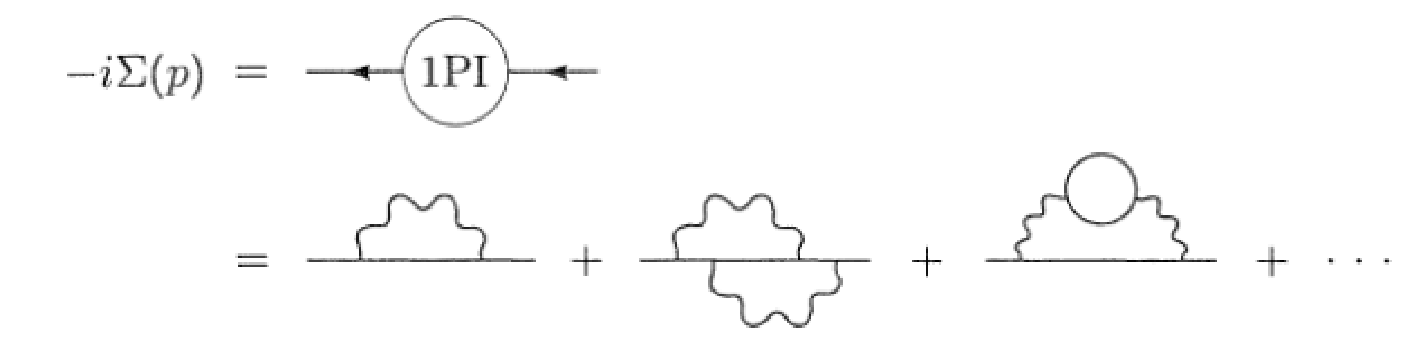
1PI

Um 1PI é qualquer diagrama que não pode ser dividido em dois removidos uma única linha

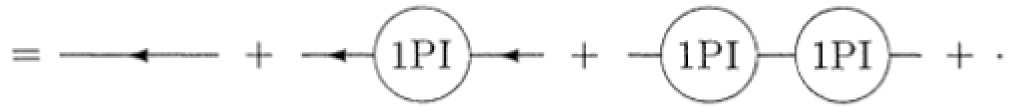




Exemplo:



O propagador modificado para um férmion



$$= \frac{i(\not{p} + m_0)}{p^2 - m_0^2} + \frac{i(\not{p} + m_0)}{p^2 - m_0^2}(-i\Sigma)\frac{i(\not{p} + m_0)}{p^2 - m_0^2} + \cdots .$$

Onde

$$S_F'(p) = S_F^0(p) + S_F^0(p)\frac{\Sigma(p)}{i}S_F^0(p) + S_F^0(p)\frac{\Sigma(p)}{i}S_F^0(p)\frac{\Sigma(p)}{i}S_F^0(p) + \cdots$$

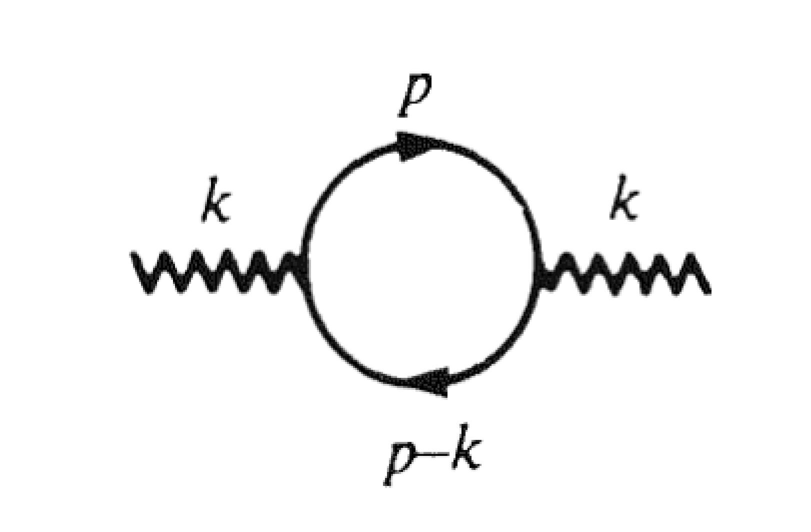
$$= S_F^0(p)\left(1 + \frac{\Sigma(p)}{i}S_F^0(p) + \frac{\Sigma(p)}{i}S_F^0(p)\frac{\Sigma(p)}{i}S_F^0(p) + \cdots\right)$$

$$= S_F^0(p)\left[1 - \frac{\Sigma(p)}{i}S_F^0(p)\right]^{-1}$$

$$\Rightarrow S_F'(p)^{-1} = S_F(p)^{-1} - \Sigma(p), \qquad (\Sigma(\not{p}) = \Sigma_2(\not{p}) + \cdots)$$

$$\Rightarrow S_F'(p)^{-1} = S_F(p)^{-1} - \Sigma_2(p) + \cdots$$

The Photon Self-Energy



$$iD_{F\alpha\beta}(k) \rightarrow iD_{F\alpha\mu}(k) ie_0^2 \Pi^{\mu\nu}(k) iD_{F\nu\beta}(k)$$

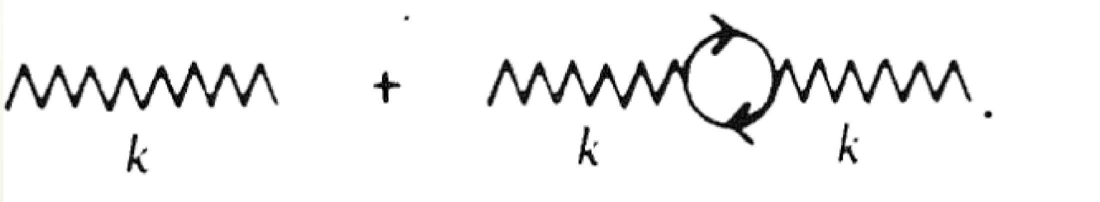
Onde

$$i\Pi^{\alpha\beta}(k) = -(ie)^2 \int \frac{d^4p}{(2\pi)^4} tr\left(\gamma^\alpha \frac{1}{\not{p} - m} \gamma^\beta \frac{1}{\not{p} + \not{k} - m}\right)$$

É integral e quadraticamente divergente, onde é uma divergência ultravioleta.

o propagador modificado

$$iD_{F\alpha\beta}(k) \rightarrow iD_{F\alpha\beta}(k) + iD_{F\alpha\mu}(k) ie_0^2 \Pi^{\mu\nu} iD_{F\nu\beta}(k)$$



The photon Self-Energy

$$i\Pi^{\mu\nu}(k) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} tr\left(\gamma^\mu \frac{i}{\not{p} - m} \gamma^\nu \frac{i}{\not{p} - \not{k} - m}\right)$$

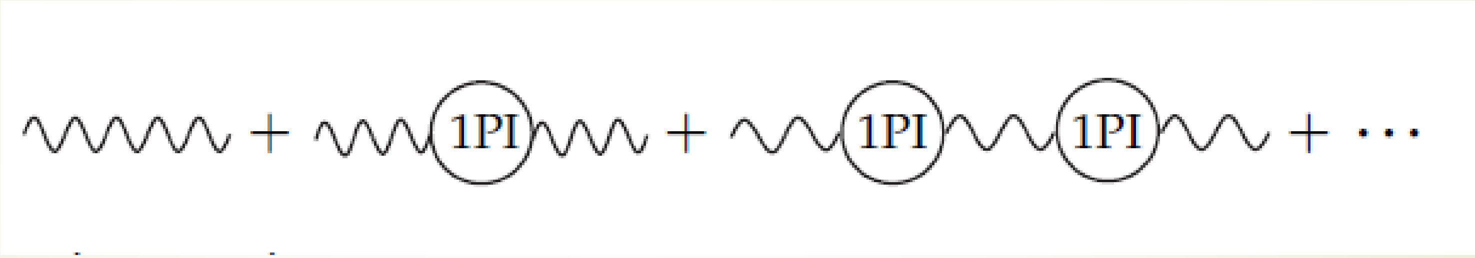
$$\Pi_{\mu\nu}(k) = i\mu^{4-d} e^2 \int \frac{d^dk}{(2\pi)^d} tr\left(\gamma^\mu \frac{i}{\not{p} - m} \gamma^\nu \frac{i}{\not{p} - \not{k} - m}\right)$$

Resultado

$$\Pi_{\mu\nu}(k) = -i\mu^{4-d}e^2 \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{\left(\frac{2}{d} - 1\right) g^{\mu\nu} l^2 - 2x(1-x)p^\mu p^\nu + g^{\mu\nu}(m^2 + x(1-x)p^2)}{[l^2 - \Delta]^2} \quad \forall$$

$$\Delta = m^2 - x(1-x)p^2$$

$$\Pi_{\mu\nu}(k) = \frac{e^2}{6\pi^2\varepsilon}\left(k_\mu k_\nu - g_{\mu\nu}k^2\right) + finite$$



$$D'_{\mu\nu}(k) = -\frac{-ig_{\mu\nu}}{k^2} + \frac{ig_{\mu\rho}}{k^2}\left[i(k^2 g^{\rho\sigma} - k^\rho k^\sigma)\Pi(k^2)\right]\frac{ig_{\sigma\nu}}{k^2} + \cdots$$

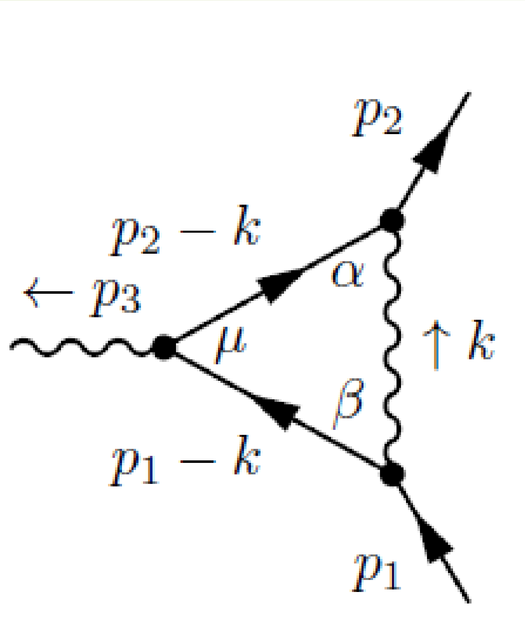
$$D'_{\mu\nu}(k) = -\frac{-ig_{\mu\nu}}{k^2} + \frac{ig_{\mu\rho}}{k^2}\left(\delta^\rho_\nu - \frac{k^\rho k_\nu}{k^2}\right)\Pi(k^2) + \frac{ig_{\mu\rho}}{k^2}\left(\delta^\rho_\nu - \frac{k^\rho k_\nu}{k^2}\right)\Pi^2(k^2) + \cdots$$

$$D'_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2}\frac{1}{1-\Pi(k^2)} + \frac{ik_\mu k_\nu}{k^4}\frac{\Pi(k^2)}{1-\Pi(k^2)}$$

Onde

$$z_3 = \frac{1}{1-\Pi(0)}$$

The Vertex Modification

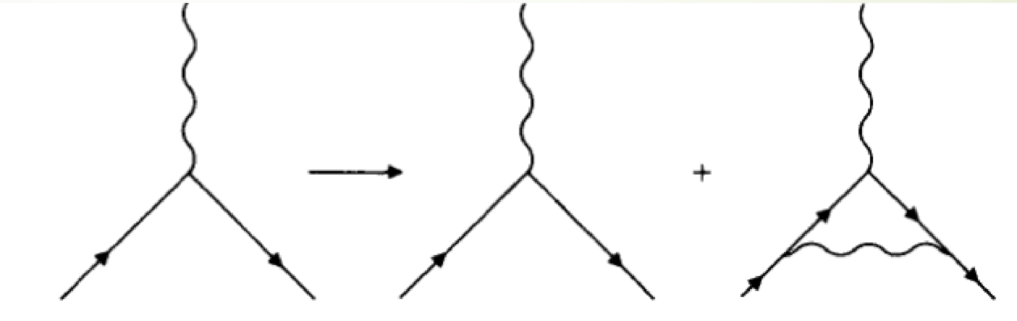


$$\Lambda^{\mu}(p_1, p_2) = -e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^{\alpha}(\not{p}_2 - \not{k} + m)\gamma^{\mu}(\not{p}_1 - \not{k} + m)\gamma_{\alpha}}{[(p_2 - k)^2 - m^2][(p_1 - k)^2 - m^2]k^2}$$

Essa integral diverge logaritmicamente. Onde está uma divergência ultravioleta, mas também tem uma divergência no infravermelho.

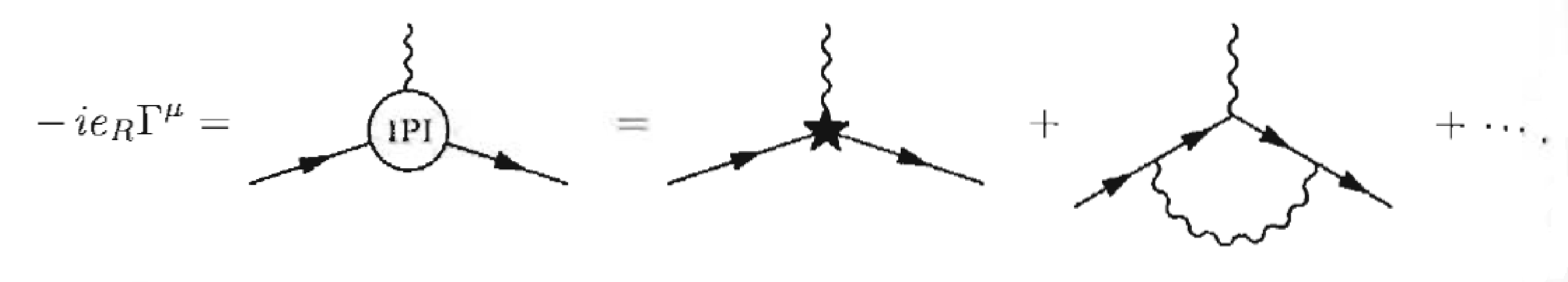
O vértice modificado

$$ie\gamma^{\mu} \rightarrow i\Gamma^{\mu}(p', p) = ie[\gamma^{\mu} + e^2\Lambda^{\mu}(p', p)]$$



Resultado

$$\Lambda_{\mu}^{(1)}(p,q,p') = \frac{e^2}{8\pi^2\varepsilon}\gamma_{\mu} + finite$$



The Callan-Symanzik Equation

Vamos relembrar a relação entre as funções de correlação renormalizadas e não renormalizadas dada por.

$$\langle T\phi(x_1)\ldots\phi(x_n)\rangle = Z^{-n/2}\,\langle T\phi_0(x_1)\ldots\phi_0(x_n)\rangle$$

$$\phi(x) = Z^{-1/2}\,\phi_0(x)$$

Se considerarmos uma variação infinitesimal da escala de renormalização

$$\frac{dG_0^{(n)}}{d\mu} = 0$$

The Callan-Symanzik Equation

$$\begin{aligned} \lambda &\rightarrow \lambda + \delta\lambda \\ \phi &\rightarrow \phi + \delta\phi \equiv (1 + \delta\eta)\phi, \end{aligned}$$

$$\delta\eta = \delta\phi/\phi$$

Então,

$$\frac{d}{d\mu} Z^{n/2} G^{(n)} = \frac{\partial G^{(n)}}{\partial \mu} + \frac{\partial G^{(n)}}{\partial \lambda} \frac{\partial \lambda}{\partial \mu} - n \frac{\partial \eta}{\partial \mu} G^{(n)} = 0$$

$$Z^{1/2} = 1 - \delta\eta$$

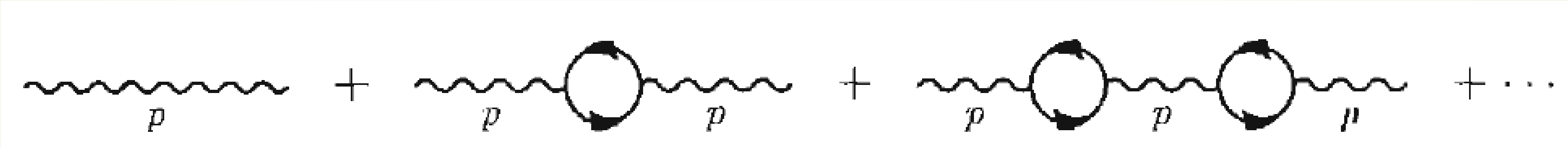
The Callan-Symanzik Equation

$$\left(\mu\,\frac{\partial}{\partial\mu}+\mu\,\frac{\partial\lambda}{\partial\mu}\,\frac{\partial}{\partial\lambda}-n\,\mu\,\frac{\partial\eta}{\partial\mu}\right)\,G^{(n)}=0$$

$$\begin{aligned}\beta &\equiv \mu\,\frac{\partial\lambda}{\partial\mu}\\ \gamma &\equiv -\mu\,\frac{\partial\eta}{\partial\mu}\end{aligned}$$

$$\boxed{\left(\mu\,\frac{\partial}{\partial\mu}+\beta\,\frac{\partial}{\partial\lambda}+n\,\gamma\right)\,G^{(n)}(x_1,\ldots,x_n;\mu,\lambda)=0\;.}$$

The Callan-Symanzik Equation



$$V(p^2) = \frac{e^2}{p^2}\left(1 + \frac{e^2}{12\pi^2}\ln\frac{p^2}{{p_0}^2} + \left(\frac{e^2}{12\pi^2}\ln\frac{p^2}{{p_0}^2}\right)^2 + \cdots\right)$$

Portanto,

$$0 = {p_0}^2\frac{dV(p^2)}{d{p_0}^2}$$

$$\Rightarrow \quad \frac{e^3}{24\pi^2} = {p_0}^2\frac{de}{d{p_0}^2}$$

$$\Rightarrow \quad e^2(p^2) = \frac{{e_0}^2}{1 - \frac{{e_0}^2}{12\pi^2}\ln\frac{p^2}{{p_0}^2}}$$