The Black-Scholes Option Pricing Model

A Comprehensive Analysis

Renoir Vieira

April 2025

Table of Contents

Origins of Option Pricing Theory

- The quest for reliable option pricing dates back centuries
- Louis Bachelier's 1900 thesis "Théorie de la Spéculation"
  - First to introduce Brownian motion to model stock prices
  - Work remained largely unrecognized for decades
- Early attempts at option pricing were primarily based on:
  - Empirical observations
  - Rules of thumb
  - Simple statistical models

Development of the Black-Scholes Model

- Early 1970s: Fischer Black and Myron Scholes, later joined by Robert Merton
- Two seminal papers published in 1973:
  - Black, F. and Scholes, M. (1973) "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*
  - Merton, R. (1973) "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*
- Key innovation: Using no-arbitrage principle and continuous-time stochastic calculus
- Coincided with the founding of the Chicago Board Options Exchange (CBOE) in 1973

Nobel Prize and Recognition

- 1997: Nobel Prize in Economic Sciences awarded to:
  - Myron Scholes
  - Robert Merton
- Fischer Black had passed away in 1995
  - Nobel Committee acknowledged his crucial contributions
- The model revolutionized financial markets:
  - Provided theoretical framework for options pricing
  - Catalyzed growth of derivatives markets
  - Led to development of sophisticated trading strategies
  - Transformed risk management practices

Key Assumptions

The Black-Scholes model is built on several key assumptions:

- Stock price follows a geometric Brownian motion with constant drift and volatility
- No transaction costs or taxes
- No dividends during the life of the option
- Markets are efficient (no arbitrage opportunities)
- Risk-free interest rate is constant and known
- Options are European-style (can only be exercised at expiration)
- Trading can occur continuously in any amount

The Black-Scholes Partial Differential Equation

- Derived by constructing a risk-free portfolio:
    - Long position in the option
    - Short position in the underlying stock
- The Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

Where:

- V is the option price as a function of stock price S and time t
- σ is the volatility of the stock
- r is the risk-free interest rate

The Black-Scholes Formula

The solution to the PDE gives the famous Black-Scholes formula:
For a European call option:

$$C(S, t) = S \cdot N(d_1) - Ke^{-rT} \cdot N(d_2) \tag{2}$$

For a European put option:

$$P(S, t) = Ke^{-rT} \cdot N(-d_2) - S \cdot N(-d_1) \tag{3}$$

Where:

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{4}$$

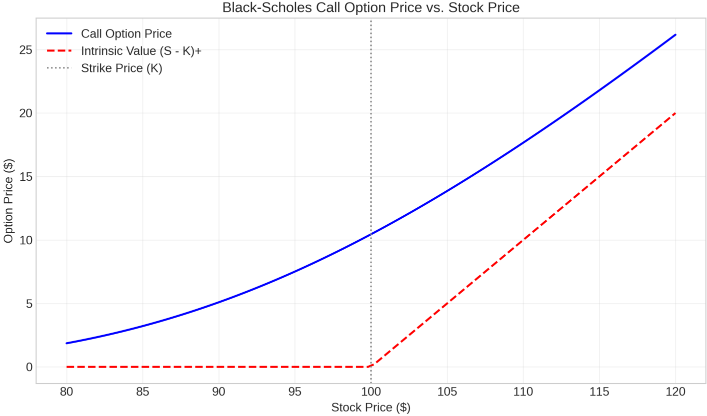
$$d_2 = d_1 - \sigma\sqrt{T} \tag{5}$$

The Black-Scholes Formula (continued)

Parameters in the Black-Scholes formula:

- C and P are the call and put option prices
- S is the current stock price
- K is the strike price
- r is the risk-free interest rate
- T is the time to expiration
- σ is the volatility of the underlying asset
- $N(\cdot)$ is the cumulative distribution function of the standard normal distribution

Call Option Price vs. Stock Price



Put Option Price vs. Stock Price

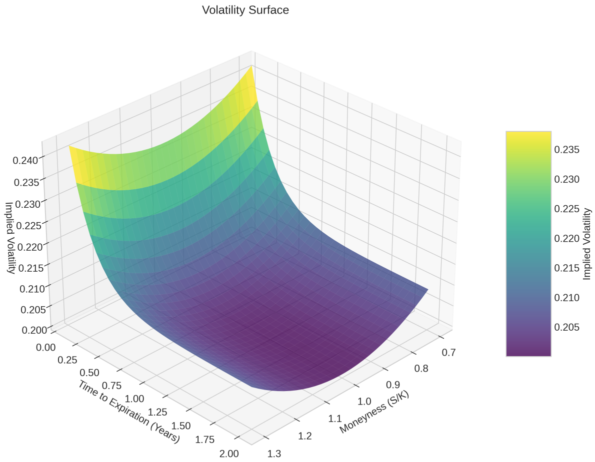
Implied Volatility

- Implied volatility is the volatility parameter that:
  - When input into the Black-Scholes formula
  - Yields the market price of the option
- Represents the market's expectation of future volatility
- In the original Black-Scholes model:
  - Implied volatility should be constant across all strike prices
  - Implied volatility should be constant across all expiration dates
- In reality, implied volatility varies with:
  - Strike price (moneyness)
  - Time to expiration

The Volatility Surface Concept

- The volatility surface is a three-dimensional representation:
  - X-axis: Strike prices (or moneyness)
  - Y-axis: Time to expiration
  - Z-axis: Implied volatility
- Provides a visual representation of market expectations
- Reveals patterns that contradict Black-Scholes assumptions
- Used for:
  - Pricing exotic options
  - Risk management
  - Calibrating more advanced models

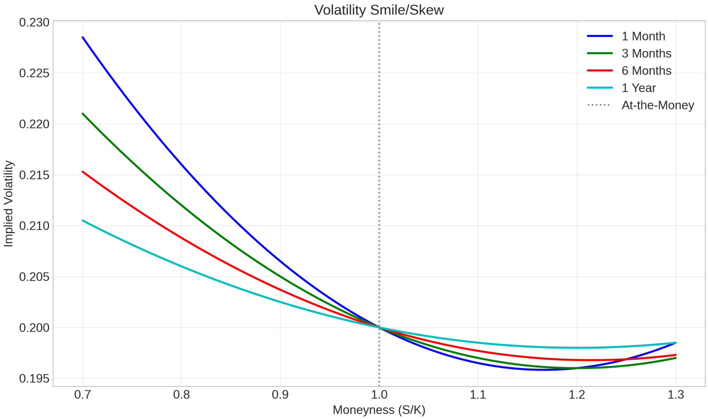
Volatility Surface Visualization



Volatility Smile and Skew

- **Volatility Smile:**
  - U-shaped pattern of implied volatility across strike prices
  - Out-of-the-money options have higher implied volatilities than at-the-money options
  - Common in currency and interest rate options
- **Volatility Skew:**
  - Downward sloping pattern of implied volatility
  - Lower strike prices (out-of-the-money puts) have higher implied volatilities
  - Common in equity options, especially after the 1987 market crash
  - Reflects market's fear of downside moves

Volatility Smile/Skew Visualization



Interpreting the Volatility Surface

- The volatility surface provides valuable insights:
  - Overall level indicates expected market volatility
  - Steepness of the skew reflects concerns about market downturns
  - Term structure indicates expectations about future volatility regimes
- Historical note:
  - Volatility skew became more pronounced after the 1987 stock market crash
  - Reflects increased market awareness of tail risk
  - Shows possibility of extreme downside movements

Option Price vs. Volatility

Introduction to the Greeks

- "Greeks" are risk measures that indicate sensitivity of option prices
- Essential tools for options traders and risk managers
- Provide insights into how option prices change with different factors
- Named after Greek letters:
  - Delta (Δ): Sensitivity to underlying price
  - Gamma (Γ): Rate of change of Delta
  - Theta (Θ): Sensitivity to time decay
  - Vega (ν): Sensitivity to volatility
  - Rho (ρ): Sensitivity to interest rates

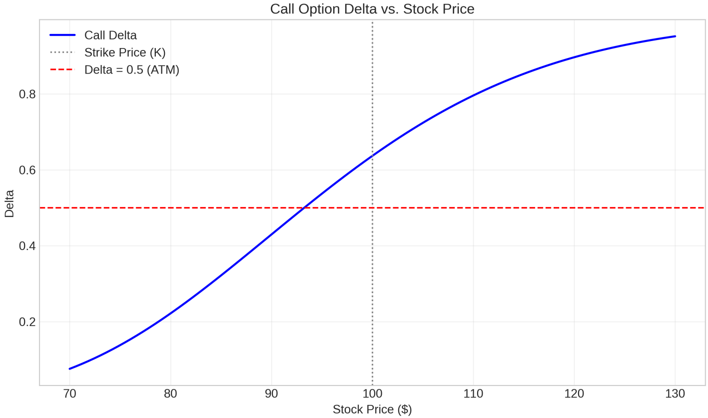
Delta (Δ)

- Measures the rate of change in option price with respect to changes in the underlying asset's price
- Mathematical definition for call options:

$$\Delta_{\text{call}} = \frac{\partial C}{\partial S} = N(d_1) \tag{6}$$

- Call option delta ranges from 0 to 1
- Approximate probability of option expiring in-the-money
- Used as hedge ratio in delta hedging strategies
- At-the-money options have delta around 0.5
- Deep in-the-money calls approach delta of 1
- Deep out-of-the-money calls approach delta of 0

Call Option Delta vs. Stock Price



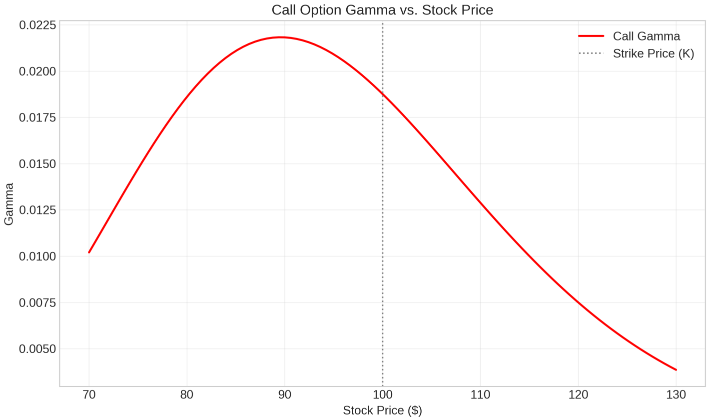
Gamma (Γ)

- Measures the rate of change in delta with respect to changes in the underlying asset's price
- Second derivative of option price with respect to underlying price
- Mathematical definition for call options:

$$\Gamma_{\text{call}} = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_{\text{call}}}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T}} \tag{7}$$

- Highest for at-the-money options
- Increases as expiration approaches for at-the-money options
- Long call options have positive gamma
- Short call options have negative gamma
- Same formula for calls and puts (but we focus on calls as requested)

Call Option Gamma vs. Stock Price



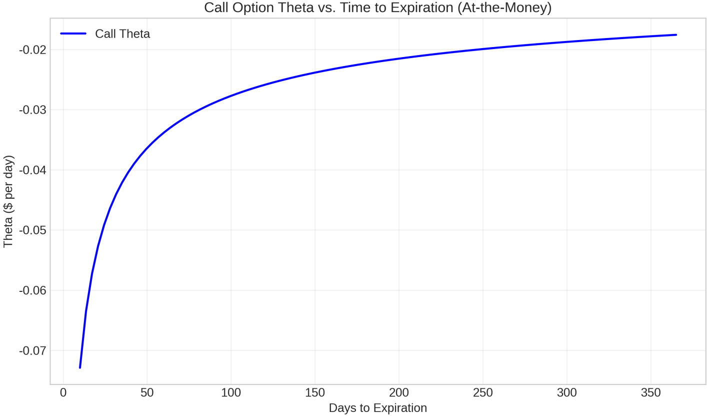
Theta (Θ)

- Measures the rate of change in option price with respect to the passage of time (time decay)
- Mathematical definition for call options:

$$\Theta_{\text{call}} = \frac{\partial C}{\partial t} = -\frac{S\sigma N'(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2) \tag{8}$$

- Typically negative for long call options positions
- Accelerates as expiration approaches
- At-the-money call options generally have the highest theta
- Expressed as the dollar value an option will lose per day
- Time decay works against the call option buyer

Call Option Theta vs. Time to Expiration



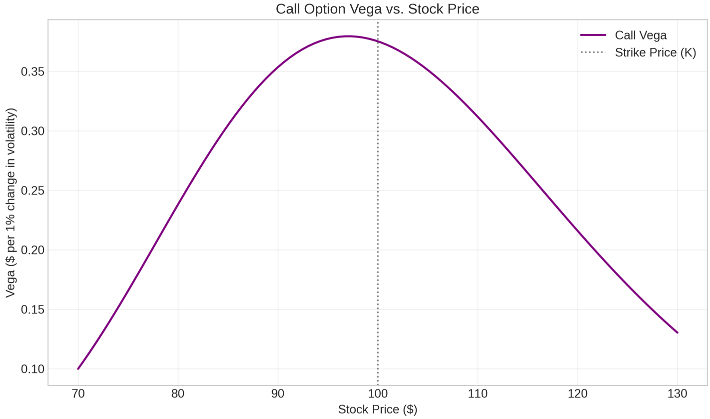
Vega (ν)

- Measures the rate of change in option price with respect to changes in implied volatility
- Mathematical definition for call options:

$$\text{Vega}_{\text{call}} = \frac{\partial C}{\partial \sigma} = S\sqrt{T}N'(d_1) \tag{9}$$

- Highest for at-the-money call options
- Decreases as expiration approaches
- Long call options have positive vega
- Short call options have negative vega
- Not actually a Greek letter, but conventionally included among "the Greeks"
- Same formula for calls and puts (but we focus on calls as requested)

Call Option Vega vs. Stock Price



Rho (ρ)

- Measures the rate of change in option price with respect to changes in the risk-free interest rate
- Mathematical definition for call options:

$$\rho_{\text{call}} = \frac{\partial C}{\partial r} = KTe^{-rT} N(d_2) \tag{10}$$

- Call options have positive rho (increase in value when rates rise)
- Generally has less impact than other Greeks for short-term options
- More significant for long-term call options
- Higher for in-the-money call options
- Represents sensitivity to interest rate changes

Call Option Rho vs. Stock Price

Applications in Risk Management

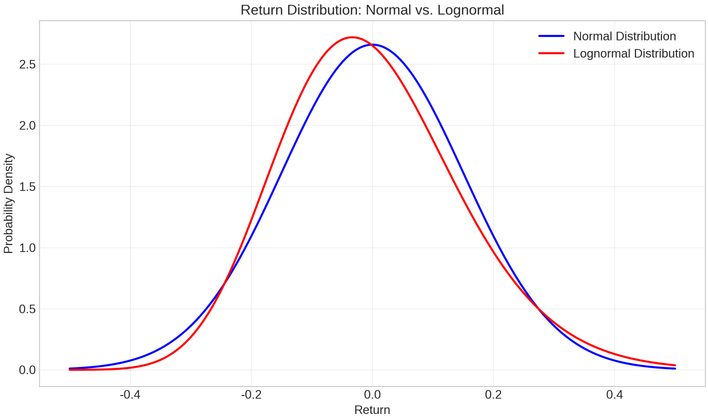
- The Greeks allow risk managers to:
  - Quantify exposure to different market factors
  - Stress test portfolios under various scenarios
  - Set risk limits and monitor compliance
  - Develop sophisticated hedging strategies
- Common hedging strategies:
  - Delta hedging: Neutralize exposure to small price movements
  - Delta-gamma hedging: Protect against larger price movements
  - Vega hedging: Reduce exposure to volatility changes
  - Portfolio immunization: Balance Greeks across entire portfolio

Unrealistic Assumptions

Despite its elegance, the Black-Scholes model relies on several assumptions that don't hold in real markets:

- **Constant Volatility:** Volatility varies over time and across strike prices
- **Lognormal Distribution:** Actual market returns exhibit fatter tails and negative skewness
- **Continuous Trading:** Markets have transaction costs, liquidity constraints, and trading hours
- **Constant Interest Rates:** Interest rates fluctuate over time
- **No Dividends:** Many stocks pay dividends
- **European Exercise:** Many options can be exercised early (American options)

Return Distribution: Normal vs. Lognormal



Empirical Evidence of Model Failures

Market data consistently shows patterns that contradict Black-Scholes assumptions:

- **Volatility smiles and skews** across strike prices
- **Extreme market movements** occur more frequently than predicted
  - 1987 stock market crash: S&P 500 fell over 20% in a single day
  - 2008 financial crisis: Unprecedented volatility levels
- **Deep out-of-the-money options** are typically underpriced by the model
- **Volatility clustering** in time series data
- **Correlation breakdowns** during market stress

Practical Implications

These limitations have significant practical consequences:

- **Risk management failures** during market stress
  - Underestimation of tail risks
  - Inadequate hedging during extreme events
- **Arbitrage opportunities** for sophisticated traders
- **Model risk** for financial institutions
- **Regulatory concerns** about systemic risk
- **Mispricing of complex derivatives**

Stochastic Volatility Models

These models allow volatility to vary over time according to a stochastic process:

- **Heston Model (1993):** Assumes volatility follows a mean-reverting process

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^S \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v \\
dW_t^S dW_t^v &= \rho dt
\end{aligned}
\tag{11}
$$

- **SABR Model:** Popular in interest rate derivatives markets
- **GARCH Option Pricing:** Captures volatility clustering

Jump-Diffusion Models

These models incorporate discrete jumps in the price process:

- **Merton Jump-Diffusion (1976):** Adds Poisson jumps to geometric Brownian motion

$$\frac{dS}{S} = (\mu - \lambda k)dt + \sigma dW + (J - 1)dq \tag{12}$$

- **Kou Model (2002):** Uses double exponential jumps
- **Bates Model:** Combines stochastic volatility with jumps

Local Volatility Models

These models make volatility a function of the current price and time:

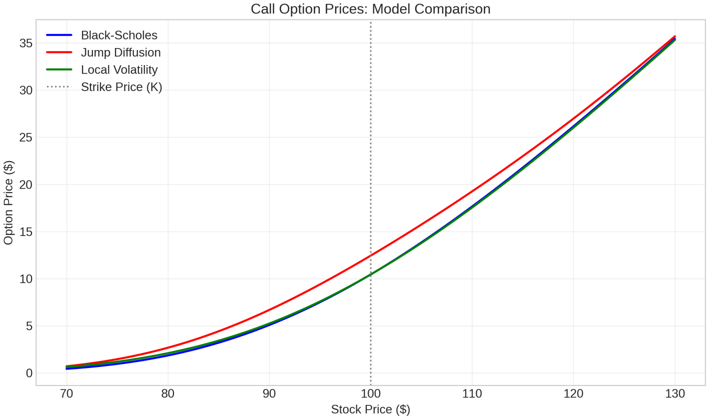
- **Dupire Model (1994):** Perfectly calibrates to the volatility surface

$$\sigma^2(K, T) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}} \tag{13}$$

- **CEV Model:** Makes volatility a function of the asset price

$$dS_t = \mu S_t dt + \sigma S_t^{\beta} dW_t \tag{14}$$

Model Comparison



Numerical Methods

Advanced computational techniques have expanded the applicability of option pricing models:

- **Finite difference methods**
  - Solve the option pricing PDE numerically
  - Handle complex boundary conditions
- **Monte Carlo simulation**
  - Simulate thousands of possible price paths
  - Handle path-dependent options
- **Fast Fourier Transform (FFT) methods**
  - Efficiently price options under various models
  - Particularly useful for Lévy processes
- **Machine learning approaches**

The Legacy of Black-Scholes

The Black-Scholes model represents one of the most significant achievements in financial economics. Despite its limitations, it continues to serve as:

- A theoretical foundation for understanding option pricing
- A benchmark for evaluating more complex models
- A practical tool for quick approximations
- A framework that has inspired decades of research

Current State of Option Pricing

Modern option pricing reflects a balance between:

- Theoretical elegance and practical applicability
- Simplicity and accuracy
- Computational efficiency and model realism
- Model risk and model complexity

Modern practice often employs a pragmatic, multi-model approach, using different models for different market conditions and instrument types.

Future Directions

The field continues to evolve with:

- Integration of machine learning and artificial intelligence
- Models that better capture market microstructure
- Approaches that account for liquidity risk and funding costs
- Techniques that incorporate behavioral finance insights
- High-frequency and algorithmic trading considerations

References

- Black, F. and Scholes, M. (1973) "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81(3), pp. 637-654.
- Merton, R. (1973) "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4(1), pp. 141-183.
- Hull, J. (2018) *Options, Futures, and Other Derivatives*, 10th Edition, Pearson.
- Gatheral, J. (2006) *The Volatility Surface: A Practitioner's Guide*, Wiley Finance.

References (continued)

- Heston, S. (1993) "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6(2), pp. 327-343.
- Dupire, B. (1994) "Pricing with a Smile," *Risk*, 7(1), pp. 18-20.
- Merton, R. (1976) "Option Pricing When Underlying Stock Returns Are Discontinuous," *Journal of Financial Economics*, 3(1-2), pp. 125-144.
- Carr, P. and Madan, D. (1999) "Option Valuation Using the Fast Fourier Transform," *Journal of Computational Finance*, 2(4), pp. 61-73.