Fusion-Fission Interactions
+
A Correction on CRISP

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Fusion cross section

Delagrange, 1977

$$\sigma_{CN} \equiv \sigma_{fus} = \pi \lambda\!\!\!^{-}{}^2 \sum_{l=0}^{l_{crit}} (2l+1) T_l$$

$\lambda\!\!\!^{-}$: reduced wave length ($\lambda\!\!\!^{-} = \hbar^2/2\mu E$)

T_l: transmission coefficient for the lth partial wave

In CRISP: geometrical cross section is used to calculate σ_{CN}

$$\sigma_{CN} = \frac{\sigma_{Geom} \times N_{casc}}{N_{attempts}}$$

Fusion cross section

$$\sigma_{CN} \equiv \sigma_{fus} = \pi \lambda\!\!\!^{-}{}^2 \sum_{l=0}^{l_{crit}} (2l+1) T_l$$

For a parabolic potential (Hill and Wheeler):

$$T = \frac{1}{1 + exp[2\pi(B - E)/\hbar\omega]}$$

B: height of the barrier, E: energy of the system

ω: vibrational frequency of the harmonic oscillator

A sharp cut-off approximation:

$$T_l = \begin{cases} 1 & \texttt{for} \quad l \leq l_{max} \\ 0 & \texttt{for} \quad l > l_{max} \end{cases}$$

Fusion cross section

Wilzinski, 1973

$$\sigma_{CN} \equiv \sigma_{fus} = \pi \lambda\!\!\!^{-2} \sum_{l=0}^{l_{crit}} (2l+1) T_l$$

Critical angular momentum of the colliding system:

$$\frac{2\pi(\gamma_1 + \gamma_2) R_1 R_2}{R_1 + R_2} = \frac{Z_1 Z_2 e^2}{(R_1 + R_2)^2} + \frac{\hbar^2 l_{crit}(l_{crit} + 1)}{\mu (R_1 + R_2)^2}$$

μ: reduced mass

γ_1 and γ_2: surface tension coefficients for nuclei 1 and 2

Maximum angular momentum

Karapetyan, 2013

$$T_l = \begin{cases} 1 & \text{for} \quad l \leq l_{max} \\ 0 & \text{for} \quad l > l_{max} \end{cases}$$

Maximum angular momentum:

$$l_{max} = \sqrt{\frac{2\mu R^2 (E_{c.m.} - V_{CB})}{\hbar^2}}$$

R: maximum distance between two nuclei at which the collision leads to a reaction

V_{CB}: Coulomb energy of the system at distance R

$E_{c.m.}$: at the center-of-mass bombarding energy (the initial energy of the projectile)

$$V_{CB} = \frac{Z_1 Z_2 e^2}{R}$$

Average angular momentum

Karapetyan, 2013

Maximum angular momentum:

$$l_{max} = \sqrt{\frac{2\mu R^2 (E_{c.m.} - V_{CB})}{\hbar^2}}$$

Average angular momentum:

$$\langle l \rangle = \frac{2}{3}\sqrt{\frac{2\mu R^2 (E_{c.m.} - V_{CB})}{\hbar^2}}$$

R: maximum distance between two nuclei at which the collision leads to a reaction

Fusion-Fission

Fusion
cross section
L_max
L_average

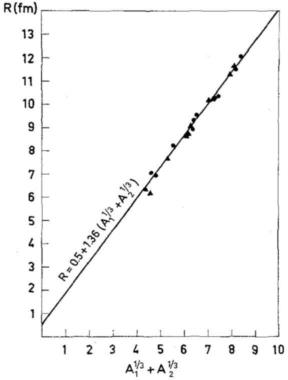
Fission
cross section
fission width
E_rotational
fission barrier
fission width

Correction
old formula
new formula
Test on CRISP

Distance R (Method 1)

Wilzinski, 1973

$$R = R_1 + R_2 + 0.5 = 1.36(A_1^{1/3} + A_2^{1/3}) + 0.5 \quad (fm)$$



Distance R (Method 2)

Heckman, 1978

$R \equiv b$ (impact parameter of the collision)

$$R \equiv b = r_0(A_T^{1/3} + A_p^{1/3} - b_{Tp}) \quad (fm)$$

A_T and A_p: mass numbers of target and projectile nuclei, respectively

b_{Tp}: overlap parameter

where b_{Tp} is calculated using the experimental determined total reaction cross section data:

$$\sigma_{tot} = \pi r_0^2(A_T^{1/3} + A_p^{1/3} - b_{Tp})^2 \quad (fm^2)$$

Average angular momentum

Karapetyan, 2015

- Both methods give the same $\langle l \rangle$.
- Average angular momentum calculated theoretically for interaction $^{7}Li +^{208} Pb \ \ (245MeV)$:

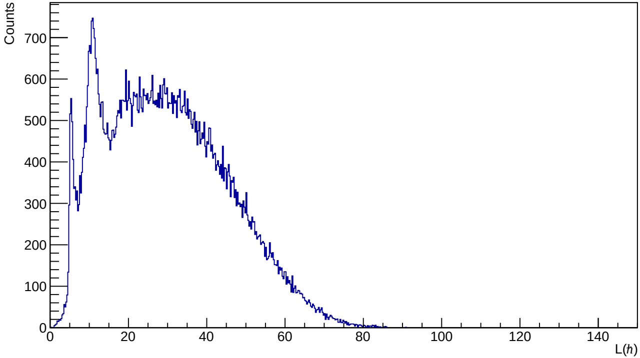
$$\langle l \rangle = 55 - 60\hbar$$

Average angular momentum

Angular momentum in CRISP (for interaction $^{7}Li +^{208} Pb$):

$$l = b \times p$$

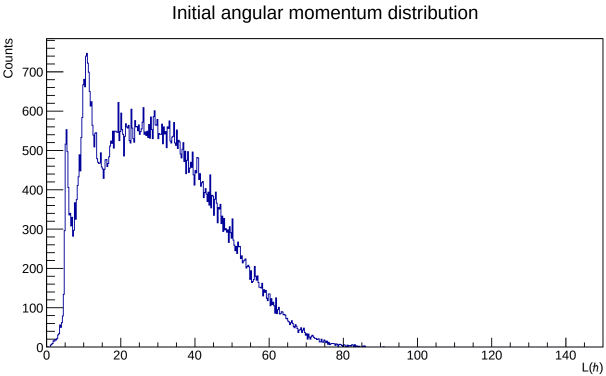
Initial angular momentum distribution



Average angular momentum

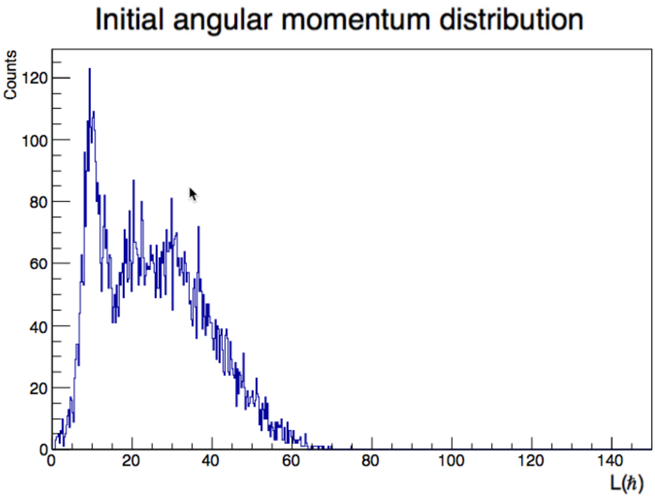
- Average angular momentum calculated by CRISP for interaction $^{7}Li + ^{208}Pb$:

$$\langle l \rangle = 29.7619\hbar$$



Average angular momentum

- Not considering the escaped particles for $^{7}Li +^{208} Pb$:



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Fission cross section

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$$\sigma_f = \sigma_{CN} \times \frac{\Gamma_f(E,J)}{\Gamma_f(E,J) + \Gamma_n(E,J) + \Gamma_p(E,J) + \Gamma_\alpha(E,J)}$$

Fission width

Baba, 1997

$$\frac{\Gamma_f(E,J)}{\Gamma_n(E,J)} = K_f \cdot exp\{2[(a_f E_f)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

with:

$$K_f = K_0 a_n \frac{2(a_f E_f)^{1/2} - 1}{(4A^(2/3) a_f E_n)}$$

$$E_f = E - E_{rot} - B_f(J)$$

$$E_n = E - E_{rot} - B_n - T$$

E_{rot}: rotational energy at the equilibrium deformation

$B_f(J)$: angular-momentum-dependent fission barrier

B_n: neutron separation energy

T: kinetic energy of the particle being ejected

Rotational energy

Cohen, 1974

- Rotational energy of the spherical configuration:

$$E_{rot}^{0} = \frac{\hbar^2 l^2}{2I_0} = \frac{1}{2}\frac{\hbar^2 l^2}{(\frac{2}{5})MR^2} = \frac{5}{4}\frac{\hbar^2}{m_0 r_0^2}\frac{l^2}{A^{5/3}}$$

$L = \hbar l$: angular momentum of the nucleus

M: mass of the nucleus; $M = m_0 A$, $m_0 = 939.15 MeV$

$R = r_0 A^{1/3}$

Rotational energy

D'Arrigo, 1992

- For non-spherical nuclei (with deformation parameter $\beta > 0.05$):
  - excitation energy E is replaced by $(E - E_{rot})$

$$E_{rot} = \frac{\hbar^2 J(J+1)}{2I_{eff}}$$

I_{eff}: effective moment of inertia

$$\frac{1}{I_{eff}} = \frac{1}{I_{\|}} - \frac{1}{I_{\perp}}$$

$I_{\|}$: moment of inertia parallel to nuclear symmetry axis of fissioning nucleus

$I_{\perp}$: moment of inertia perpendicular to nuclear symmetry axis of fissioning nucleus

Effective moment of inertia

Soheyli, 2011; D'Arrigo, 1991

$$I_{\parallel} = \tfrac{2}{5} A m_0 R^2 (1 - 0.63\ \beta_2)$$

$$I_{\perp} = \tfrac{2}{5} A m_0 R^2 (1 + 0.32\ \beta_2)$$

$$I_{eff} = \frac{I_{\parallel}\ I_{\perp}}{I_{\perp} - I_{\parallel}} \approx \tfrac{2}{5} A m_0 R^2 \left(\frac{1.05 - 0.33\ \beta_2}{\beta_2} \right)$$

A: mass number of deformed nucleus with ellipsoid shape

m_0: mass of a nucleon

R: spherical equivalent radius of the nucleus

β_2: quadrupole deformation parameter

Quadrupole deformation parameter

Soheyli, 2011; Shneidman, 1999

- Dinuclear system (DNS)

  frozen density approximation:

  1. charge and mass density can be written as a sum of the densities of individual clusters,
  2. assuming axial symmetry of the nuclear shapes, quadrupole deformation parameter of the DNS nuclei can be calculated:

$$\beta_2 = \sqrt{\frac{5\pi}{4}\frac{4\pi}{3}\frac{A_1 A_2}{A^2}\frac{R^2}{R_0^2}}$$

Quadrupole deformation parameter

Soheyli, 2011; Shneidman, 1999

$$
\begin{aligned}
I_{\parallel} &= \tfrac{2}{5} A m_0 R^2 (1 - 0.63\ \beta_2) \\
I_{\perp} &= \tfrac{2}{5} A m_0 R^2 (1 + 0.32\ \beta_2) \\
I_{eff} &= \frac{I_{\parallel}\ I_{\perp}}{I_{\perp} - I_{\parallel}} \approx \tfrac{2}{5} A m_0 R^2 \left(\frac{1.05 - 0.33\ \beta_2}{\beta_2} \right)
\end{aligned}
$$

$$
\beta_2 = \sqrt{\frac{5\pi}{4}} \frac{4\pi}{3} \frac{A_1 A_2}{A^2} \frac{R^2}{R_0^2}
$$

$A = A_1 + A_2$: mass number of the system

R: maximum distance between two nuclei at which the collision leads to a reaction $(R \approx R_1 + R_2)$

R_0: spherical equivalent radius of the nucleus

Quadrupole deformation parameter

Soheyli, 2011; Shneidman, 1999

$$I_{eff} > I_0$$

$$E_{rot} = \frac{\hbar^2 l^2}{2I_{eff}} < E^0_{rot} = \frac{\hbar^2 l^2}{2I_0}$$

$$\begin{aligned} E_f &= E - E_{rot} - B_f(J) \\ E_n &= E - E_{rot} - B_n - T \\ E_p &= E - E_{rot} - B_p - V_p \\ E_\alpha &= E - E_{rot} - B_\alpha - V_\alpha \end{aligned}$$

Rotational energy

Vaz, 1983; Kapoor, 1989

$$E_{rot} = \frac{\hbar^2 R^2}{2I_\perp} + \frac{\hbar^2 K^2}{2I_\parallel} = \frac{\hbar^2 J^2}{2I_\perp} + \frac{\hbar^2 K^2}{2I_{eff}}$$

Second term: rotation about symmetry axis

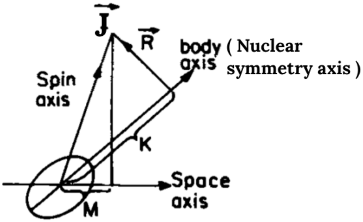
First term: rotation about axis perpendicular to symmetry axis

J: total angular momentum of fissioning nucleus

M: projection of J on space axis

K: projection of J on nuclear symmetry axis

R: component of J perpendicular to K



Fission width

Baba, 1997

$$\frac{\Gamma_f(E,J)}{\Gamma_n(E,J)} = K_f \cdot exp\{2[(a_f E_f)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

with:

$$K_f = K_0 a_n \frac{2(a_f E_f)^{1/2} - 1}{(4A^(2/3) a_f E_n)}$$

$$E_f = E - E_{rot} - B_f(J)$$

$$E_n = E - E_{rot} - B_n - T$$

E_{rot}: rotational energy at the equilibrium deformation

$B_f(J)$: angular-momentum-dependent fission barrier

B_n: neutron separation energy

T: kinetic energies of the particle being ejected

Binding energies

Lestone, 1993

- B_n:

  Variation of J due to neutron emission is negligible.

  Liquid drop mass formula gives almost the same B_n ($\pm 0.1 MeV$).

- B_p and B_α:

  Change considerably as the nuclear deformation increases.

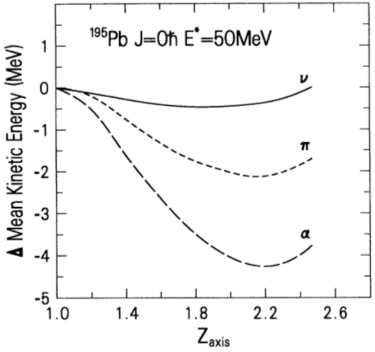
$$
\begin{aligned}
B_p &= m_p + m(A-1, Z-1) - m(A, Z) + D_d(\alpha_i) - D_p(\alpha_i) \\
B_\alpha &= m_\alpha + m(A-4, Z-2) - m(A, Z) + D_d(\alpha_i) - D_p(\alpha_i)
\end{aligned}
$$

D_d: deformation energy of the daughter nuclei

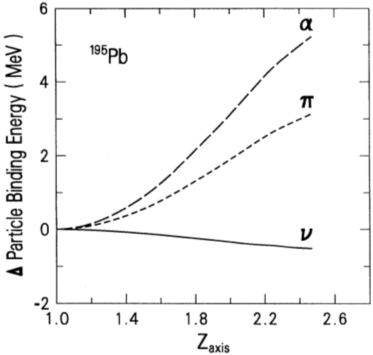
D_p: deformation energy of the parent nuclei

Binding energies

Lestone, 1993



Change in mean kinetic energies of n, p, and α particles as a function of elongation of symmetry axis Z_{axis} relative to emission from spherical nuclei



Change in n, p, and α particle binding energies as a function of symmetry axis Z_{axis} relative to spherical binding energies

Binding energies

Lestone, 1993

Transmission coefficient for a parabolic potential (Hill & Wheeler):

$$T_l = \frac{1}{1 + exp[2\pi(B - E)/\hbar\omega]}$$

B: height of the barrier, E: energy of the system

ω: vibrational frequency of the harmonic oscillator

- nuclear optical potentials
- Coulomb potentials

$\Rightarrow$ change in particle transmission coefficients

$\Rightarrow$ proton and α-particle emission barriers decrease with increasing deformation.

Fission barrier

- Liquid-drop model:

Nucleus-nucleus interactions

↓

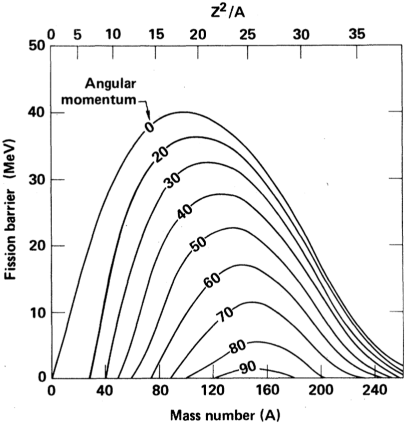
Angular momentum

↓

Decrease of fission barrier

↓

Increase of fission X-section



Fission barrier

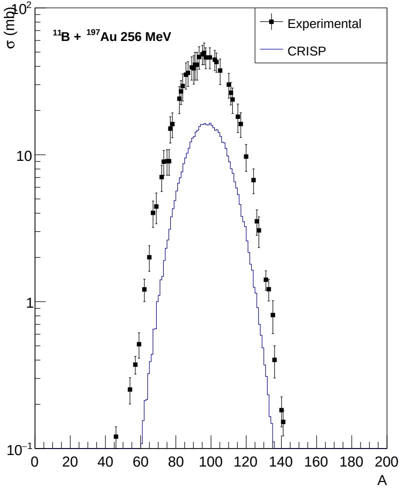
Angular-momentum-dependent fission barrier $B_f(J)$

- Cohen, Plasil, Swiatecki, 1974
  - Rotating liquid drop model (RLDM)
  - Problem:

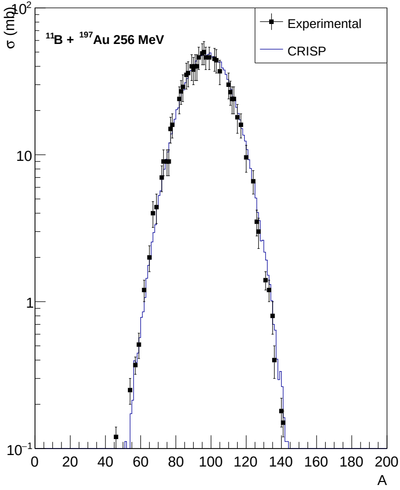
    RLDM barriers do not reproduce the experimental data for heavy-ion-induced fission (in most cases)
  - Solution:

    Fission-barrier heights must be reduced by a factor between about 0.5 to 0.9, for nuclei with mass numbers less than about 200.

Fission barrier



Fission barrier



Reduced factor = 0.55

Fission barrier

Angular-momentum-dependent Fission barrier $B_f(J)$

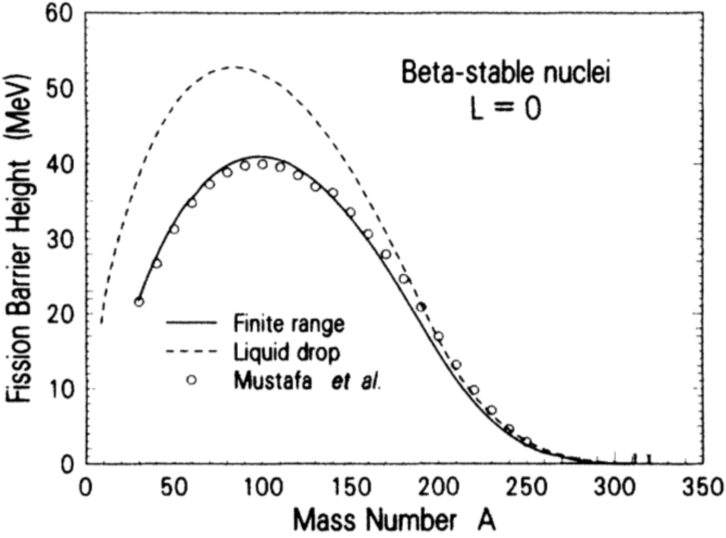
- Mustafa, 1982; Sierk, 1986
  - Rotating finite range model (RFRM)
  - Advantage:

    RFRM barrier gives the exact fission barrier, no scaling factor is needed

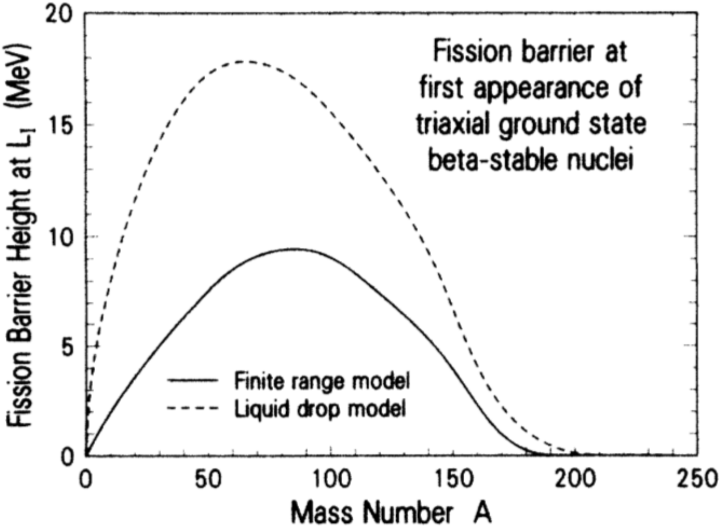
$\Longrightarrow$ Fission cross section is well reproduced in

the rotating liquid drop framework

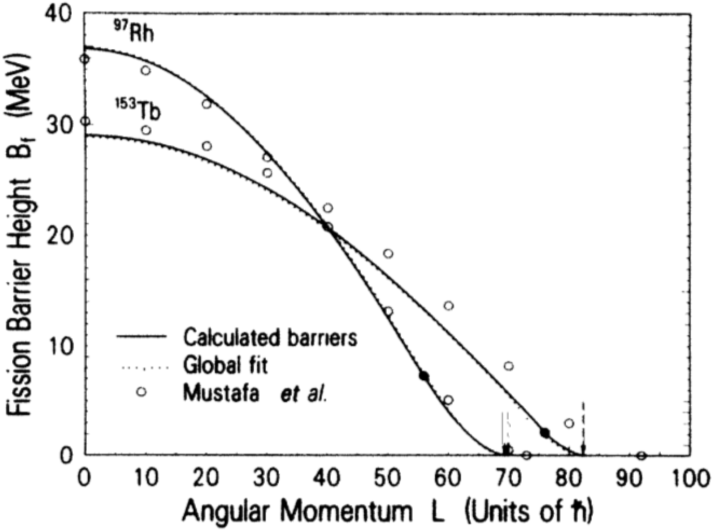
Fission barrier



Fission barrier



Fission barrier



Fission barrier (Sierk, 1986)

- **Fission barrier heights:**

$$B_f(A,Z,L) = B_f^0(A,Z) \times h(A,Z,L)$$

B_f^0: fission barrier at $L = 0$

$$h(A,Z,L) = \begin{cases} 1 + \delta_2 L^2 + \delta_3 L^3 & : L \leq L_{80} \\ 1 + \gamma_2 l^2 + \gamma_3 l^3 + \gamma_4 l^4 + \gamma_5 l^5 & : L \geq L_{20} \end{cases}$$

$l = \frac{L}{L_{max}}$

L_{max}: value of L where the barrier disappears

L_{20}: value of L where calculated barrier height is 20% of B_f^0

L_{80}: value of L where calculated barrier height is 80% of B_f^0

Fission barrier (Sierk, 1986)

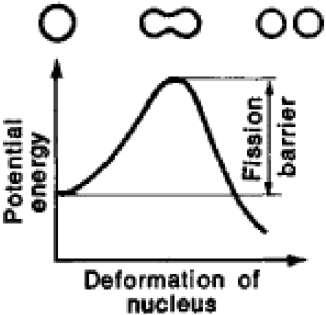
- Total energy of the rotating nuclei:

$$E_{tot} = E_S + E_C + E_R$$

E_S : surface energy

E_C : Coulomb energy

E_R : rotational energy



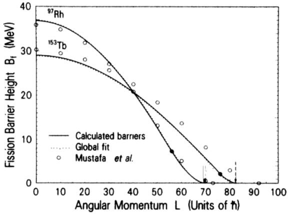
Fission barrier (Sierk, 1986)

The functional form of the global fit is defined by eqs.:

$$B_f(A,Z,L) = B_f^0(A,Z) \times h(A,Z,L)$$

$$h(A,Z,L) = \begin{cases} 1 + \delta_2 L^2 + \delta_3 L^3 & : L \leq L_{80} \\ 1 + \gamma_2 l^2 + \gamma_3 l^3 + \gamma_4 l^4 + \gamma_5 l^5 & : L \geq L_{20} \end{cases}$$

$$p_m(A,Z) = \sum_{i=0}^{N_A-1} \sum_{j=0}^{N_Z-1} C_{ij}^{(m)} P_i(\eta) P_j(\mu)$$



Fission barrier (Sierk, 1986)

- BARFIT (Fortran 77 computer subroutine):

  To calculate the height of barrier and the energy of ground state for given values of Z, A, and L

- MOMFIT (Fortran 77 computer subroutine):

  To calculate the three principal axis moments of inertia of the saddle-point shape

- BARMOM (Fission Barrier & Moment of Inertia):

  a collection of three FORTRAN 77 subroutines: BARFIT, MOMFIT, and LPOLY (LPOLY calculates Legendre polynomials and is called by BARFIT and MOMFIT.)

Fission barrier (Mustafa, 1982)

- Total energy of the rotating nuclei:

$$E_{tot}(\alpha, J) = E_S(\alpha) + E_C(\alpha) + E_R(\alpha, J)$$

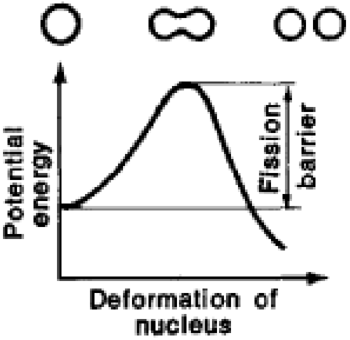
E_S: surface energy

E_C: Coulomb energy

E_R: rotational energy

α: all deformation variables necessary to describe shape variations under rotation

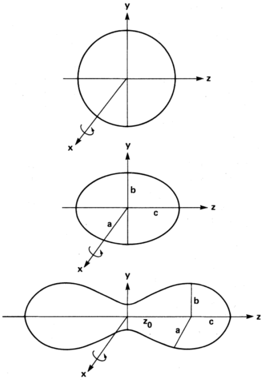
J: angular momentum



Fission barrier (Mustafa, 1982)

- **Equation governing nuclear shapes under rotation:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(|z| - z_0)^2 + \lambda(|z| - z_0)^4 \theta(|z| - z_0)}{c^2} = 1$$



Fission width (D'Arrigo, 1992)

- Partial width for neutron emission channel:

$$\Gamma_n(A,E,J) = \hbar \int_0^{E-B_n} \frac{\sigma(E,\epsilon) g m_0 \epsilon}{\pi^2 \hbar^2} \frac{\rho(E-B_n-\epsilon,J)}{\rho_{CN}(E,J)} d\epsilon$$

- Partial width for fission and charged particle emission channels:

$$\Gamma_\mu(A,E,J) = \hbar \frac{1}{2\pi\rho_{CN}(E,J)} \int_0^{E-B_\mu} \rho(E-B_\mu-\epsilon,J) d\epsilon$$

- For non-spherical nuclei (with deformation parameter $\beta > 0.05$): excitation energy E is replaced by $(E - E_{rot})$ and:

$$E_{rot} = \frac{\hbar^2 J(J+1)}{2I_{eff}}$$

Fission width (D'Arrigo, 1992)

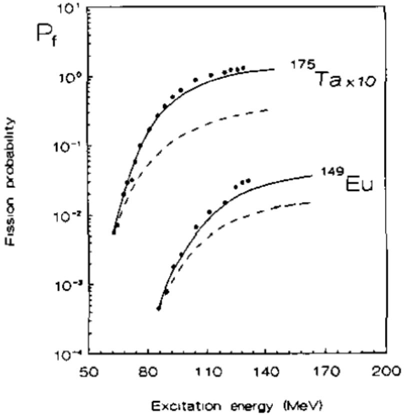


Figure: Fission probability $P_f(E)$ of ^{149}Eu and ^{175}Eu (full lines). The points are the experimental data obtained in $^{16}O+^{133}Cs$ and $^{16}O+^{159}Tb$ reactions. The dashed lines represent fission probability calculated within the spin-independent approach.

Fission width (Choudhury, 2001)

- Fission width:

$$\Gamma_f^J = \frac{2J+1}{2I\perp t} exp(-\frac{J(J+1)}{2I\perp t}) \int_{\epsilon_f=0}^{E-B_f} \rho_f(E-B_f-\epsilon_f)T(\epsilon_f)d\epsilon_f$$

- Particle emissions:

$$\Gamma_\lambda^J = \frac{2J+1}{2\sigma_\lambda^2} exp(-\frac{J(J+1)}{2\sigma_\lambda^2}) \int_{\epsilon_\lambda=0}^{E-B_\lambda} \rho_\lambda(E-B_\lambda-\epsilon_\lambda)T_J(\epsilon_\lambda)d\epsilon_\lambda$$

$I\perp$ and t: moment of inertia and temperature at fission saddle point

$\lambda \equiv n, p, \alpha$

B_λ and ϵ_λ: binding energy and kinetic energy of particle λ

$$\sigma_\lambda^2 = \frac{I_\lambda t}{\hbar^2}$$

Fission width (Baba, 1997)

$$\frac{\Gamma_f(E,J)}{\Gamma_n(E,J)} = K_f \cdot exp\{2[(a_f E_f)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

with:

$$\begin{aligned} K_f &= K_0 a_n \frac{2(a_f E_f)^{1/2} - 1}{(4A^(2/3) a_f E_n)} \\ E_f &= E - E_{rot} - B_f(J) \\ E_n &= E - E_{rot} - B_n - T \end{aligned}$$

E_{rot}: rotational energy at the equilibrium deformation

$B_f(J)$: angular-momentum-dependent fission barrier

B_n: neutron separation energy

T: kinetic energies of the particle being ejected

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Level density parameters in the old formula

- Level density parameter for neutron emission:

$$a_n = (0.134A - 1.21) \times 10^{-4} A^2 MeV^{-1}$$

- Level density parameter for proton and α-particle emission:

$$a_j = r_j a_n$$

- where $r_p = r_\alpha = 1$, hence:

$$a_p = a_\alpha = a_n$$

$$\Downarrow$$

$$\frac{a_n}{a_p} = \frac{a_n}{a_\alpha} = 1$$

Level density parameters in the old formula

$$\frac{\Gamma_p}{\Gamma_n} = (\frac{E_p}{E_n})(\frac{a_n}{a_p})exp\{2[(a_p E_p)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

$$\frac{\Gamma_\alpha}{\Gamma_n} = (\frac{2E_\alpha}{E_n})(\frac{a_n}{a_\alpha})exp\{2[(a_\alpha E_\alpha)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

$$\frac{a_n}{a_p} = \frac{a_n}{a_\alpha} = 1$$

$$\Downarrow$$

$$\frac{\Gamma_p}{\Gamma_n} = (\frac{E_p}{E_n})exp\{2[(a_p E_p)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

$$\frac{\Gamma_\alpha}{\Gamma_n} = (\frac{2E_\alpha}{E_n})exp\{2[(a_\alpha E_\alpha)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

Level density parameters in the new formula

$$a_n = \frac{A}{a_1}(1 - a_2 \frac{A-2Z}{A^2})^2$$

$$a_p = \frac{A}{a_3}(1 + a_4 \frac{A-2Z}{A^2})^2$$

$$a_\alpha = \frac{A}{a_5}(1 - \frac{a_6}{A})^2$$

$$\Downarrow$$

$$a_n \neq a_p \neq a_\alpha$$

$$\Downarrow$$

$\frac{a_n}{a_p}$, $\frac{a_n}{a_\alpha}$ `should not be omitted!`

Level density parameters in the new formula

$$a_n \neq a_p \neq a_\alpha$$

$$\Downarrow$$

- For proton emission:

$$\frac{\Gamma_p}{\Gamma_n} = (\frac{E_p}{E_n})(\frac{a_n}{a_p})exp\{2[(a_p E_p)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

- For α-particle emission:

$$\frac{\Gamma_\alpha}{\Gamma_n} = (\frac{2E_\alpha}{E_n})(\frac{a_n}{a_\alpha})exp\{2[(a_\alpha E_\alpha)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

Test done on CRISP

- Comparison between:

$$\frac{\Gamma_p}{\Gamma_n} = (\frac{E_p}{E_n}) exp\{2[(a_p E_p)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

$$\frac{\Gamma_\alpha}{\Gamma_n} = (\frac{2E_\alpha}{E_n}) exp\{2[(a_\alpha E_\alpha)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

- and the corrected formulae:

$$\frac{\Gamma_p}{\Gamma_n} = (\frac{E_p}{E_n})(\frac{a_n}{a_p}) exp\{2[(a_p E_p)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

$$\frac{\Gamma_\alpha}{\Gamma_n} = (\frac{2E_\alpha}{E_n})(\frac{a_n}{a_\alpha}) exp\{2[(a_\alpha E_\alpha)^{\frac{1}{2}} - (a_n E_n)^{\frac{1}{2}}]\}$$

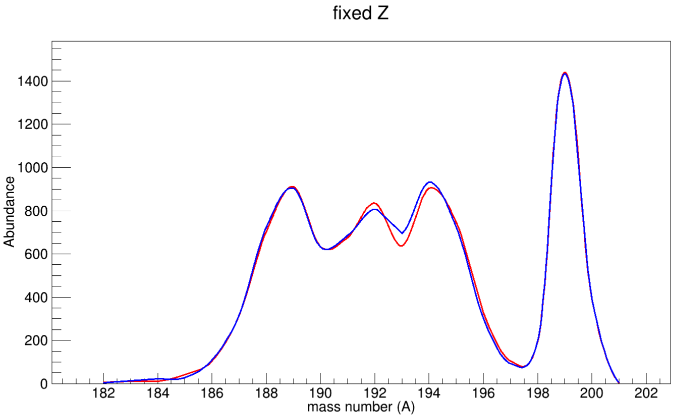
Test done on CRISP

- Target Nucleus : Pb208
- Projectile : proton
- Initial Energy : 1000 MeV

Spallation, Fixed $Z = 80$

old formula

new formula



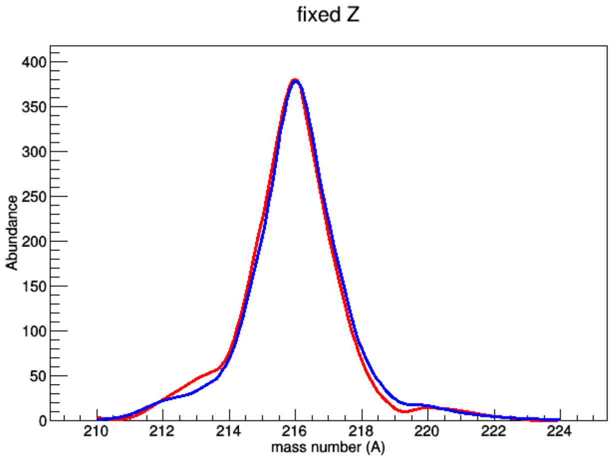
Test done on CRISP

- Target Nucleus : U238
- Projectile : proton
- Initial Energy : 1000 MeV

Spallation, Fixed $Z = 88$

— old formula

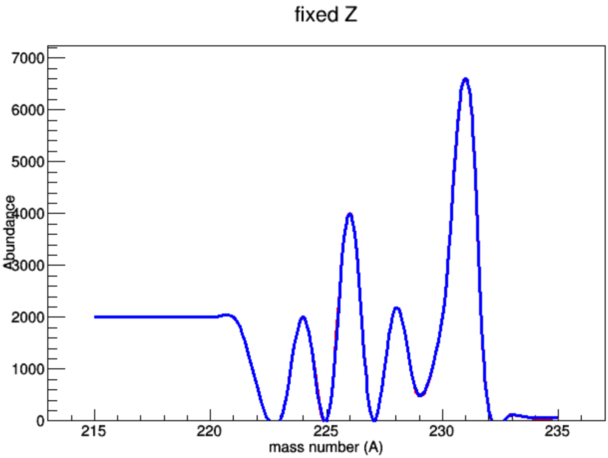
— new formula



Fission, Fixed $Z = 90$

— old formula

— new formula



Thank you for your attention!

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